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The structure of generic subintegrality

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Abstract. In order to give an elementwise characterization of a subintegral extension of $\mathbb{Q}$-algebras, a family of generic $\mathbb{Q}$-algebras was introduced in [3]. This family is parametrized by two integral parameters $p \geqslant 0$, $N \geqslant 1$, the member corresponding to p, N being the subalgebra $R = \mathbb{Q}[\{\gamma_n \mid n \geqslant N\}]$ of the polynomial algebra $\mathbb{Q}[x_1, \ldots, x_p, z]$ in $p+1$ variables, where $\gamma_n = z^n + \sum_{i=1}^{p} \binom{n}{i} x_i z^{n-i}$. This is graded by weight $(z) = 1$, weight $(x_i) = i$, and it is shown in [2] to be finitely generated. So these algebras provide examples of geometric objects. In this paper we study the structure of these algebras. It is shown first that the ideal of relations among all the γ_n's is generated by quadratic relations. This is used to determine an explicit monomial basis for each homogeneous component of R, thereby obtaining an expression for the Poincaré series of R. It is then proved that R has Krull dimension $p+1$ and embedding dimension $N+2p$, and that in a presentation of R as a graded quotient of the polynomial algebra in $N+2p$ variables the ideal of relations is generated minimally by $\binom{N+p}{2}$ elements. Such a minimal presentation is found explicitly. As corollaries, it is shown that R is always Cohen–Macaulay and that it is Gorenstein if and only if it is a complete intersection if and only if $N + p \leqslant 2$. It is also shown that R is Hilbertian in the sense that for every $n \geqslant 0$ the value of its Hilbert function at n coincides with the value of the Hilbert polynomial corresponding to the congruence class of n.

Keywords. Subintegral extensions; subrings of polynomial rings.

Introduction

Let $A \subseteq B$ be an extension of commutative rings containing the rational numbers $\mathbb{Q}$. In [3] an element $b \in B$ is defined to be subintegral over A if there exist integers $p \geqslant 0$, $N \geqslant 1$ and $c_1, \ldots, c_p \in B$ such that $g_n := b^n + \sum_{i=1}^{p} \binom{n}{i} c_i b^{n-i} \in A$ for all integers $n \geqslant N$. With this definition the extension $A \subseteq B$ is subintegral in the sense of Swan [7] if and only if every element of B is subintegral over A [3, §4].

In [3] the tuple $(0, p, N; 1, c_1, \ldots, c_p)$ with the above properties was called a system of subintegrality for b over A. There was an extra parameter s which we can take to be 0 in the present discussion, and the 1 represents c_0. In [3] we assumed that

$N \geqslant s+p$. Here (as in [4]) we adopt the conventions that for any element b in a ring, $b^0 = 1$ and $\binom{n}{i} b^{n-i} = 0$ if $i > n$. Then it suffices to assume that $N \geqslant 1$. By [3, proof of (4.2) (iv)$\Rightarrow$(i)] (note also [4, (1.1)]) if b has a system of subintegrality for some $N \geqslant 1$, then b has a system of subintegrality with $N = 1$. Systems with $N > 1$ are still of interest, however, since freedom in the choice of N may result in a simpler system of subintegrality.

Let $x_1, \ldots, x_p, z$ be independent indeterminates over $\mathbb{Q}$, and let $x_0 = 1$. For $n \geqslant 0$ let $\gamma_n = \sum_{i=0}^{p} \binom{n}{i} x_i z^{n-i}$ and let $R^{(N)} := \mathbb{Q}[\{\gamma_n | n \geqslant N\}] \subseteq S := \mathbb{Q}[x_1, \ldots, x_p, z]$. Then z is subintegral over $R^{(N)}$ with system of subintegrality $(0, p, N; 1, x_1, \ldots, x_p)$. Furthermore this setup is universal for subintegral elements together with their systems of subintegrality, in the sense that given any extension of commutative $\mathbb{Q}$-algebras $A \subseteq B$ with $b \in B$ having a system of subintegrality $(0, p, N; 1, c_1, \ldots, c_p)$, the homomorphism $\varphi: S \to B$ given by $\varphi(x_i) = c_i$ and $\varphi(z) = b$ satisfies $\varphi(\gamma_n) = g_n$ and $\varphi(R^{(N)}) \subseteq A$. Such universal extensions played a crucial role in [3].

The rings $R^{(N)}$ have an interesting algebraic structure, which we discuss in the present paper. First of all $R^{(N)}$ and S are graded by weight $(x_i) = i$, weight $(z) = 1$, which imply that weight $(\gamma_n) = n$. In §1 we find relations (1.2) of degree two (but not necessarily homogeneous) among the γ_n, where degree means $\deg(\gamma_n) = 1$ for all $n \geqslant 1$, and is to be distinguished from weight. We show in (2.2) that these quadratic relations generate the ideal of all relations. These quadratic relations include those used in [2] to prove that $R^{(N)}$ is a $\mathbb{Q}$-algebra of finite type, although in [2] we did not find a complete set of relations. In (2.1) we use the quadratic relations to obtain an explicit monomial basis for $R_k^{(N)}$, the weight k part of $R^{(N)}$, from which we obtain in (2.8) the Poincaré series of $R^{(N)}$ for arbitrary p and N (generalizing both [4, (4.4)], which handles the case $N = 1$, and [4, (4.7)], which is the case $p = 1$, N arbitrary).

In §3 we use the quadratic relations to eliminate all but a finite number of the γ_n, obtaining thereby our main result (3.2) which gives a minimal presentation of $R^{(N)}$ as a graded $\mathbb{Q}$-algebra of finite type. Of course, after eliminating these variables, the relations among the remaining variables are no longer all quadratic. From (3.2) we derive several corollaries ((3.3)–(3.7)) on the nature of $R^{(N)}$: (3.5) says that $R^{(N)}$ is always Cohen–Macaulay, which was a surprise to us; (3.6) says that $R^{(N)}$ is Gorenstein if and only if it is a complete intersection if and only if $N + p \leqslant 2$.

In §4 we give an alternative proof of the linear independence of our basis for $R_k^{(N)}$. This method is more complicated but also more precise than the argument of §2.

We conclude the paper by studying in §5 the Hilbert function of $R^{(N)}$. We find the minimal number d of Hilbert polynomials needed to express the Hilbert function of $R^{(N)}$, and show that if $p \geqslant 2$ then $R^{(N)}$ is Hilbertian, meaning that the value of its Hilbert function at n coincides with the value of the Hilbert polynomial corresponding to the congruence class of n modulo d, for every $n \geqslant 0$ (rather than just for $n \gg 0$).

The non-negative integers are denoted by $\mathbb{Z}^+$, and $\lfloor a \rfloor$ is the integral part of the real number a (i.e. the largest integer $\leqslant a$).

1. The quadratic relations

Let $R^{(N)} \subseteq S$ be the universal extension as defined above. Let T be an indeterminate

over S, and let $F(T)=1+\sum_{i=1}^{p}\binom{T}{i}x_i z^{-i}$ (so that $\gamma_n = z^n F(n)$). Then we have the following (generalizing [2, (1.2)]).

Theorem 1.1. *Let k be an integer $\geqslant 2p$, and let $0 \leqslant d_1 < d_2 < \cdots < d_{p+1} \leqslant k/2$ be any $p+1$ distinct integers. Let d be any integer $0 \leqslant d \leqslant k/2$, distinct from the d_i. Then*

$$\gamma_d \gamma_{k-d} = \sum_{i=1}^{p+1} a_i \gamma_{d_i} \gamma_{k-d_i} \tag{1.2}$$

for some rational numbers a_i.

Proof. Note that we have $d_i < k-d_i$ $(1 \leqslant i \leqslant p)$, $d_{p+1} \leqslant k-d_{p+1}$, and the $p+1$ pairs $(d_i, k-d_i)$ are distinct (as unordered pairs). First consider the case $d_{p+1} < k-d_{p+1}$ so that each pair $(d_i, k-d_i)$ consists of two distinct integers. Let $I = \{d_1, \ldots, d_{p+1}, k-d_{p+1}, \ldots, k-d_1\}$. For $p+2 \leqslant i \leqslant 2p+2$ define $d_i = k-d_{2p+3-i}$, so that $I = \{d_i\}_{1 \leqslant i \leqslant 2p+2}$. The set I contains $2p+2$ distinct integers. For $1 \leqslant i \leqslant 2p+2$ let π_i be the interpolating polynomial of degree $2p+1$, which is 1 at d_i and 0 at the remaining elements of I. Let $G(x) = \sum_{i=1}^{2p+2} \pi_i(x) F(d_i) F(k-d_i)$ and $H(x) = F(x)F(k-x)$. Then $G(c) = H(c)$ for all $c \in I$. The polynomial $G(x)$ is of degree $\leqslant 2p+1$ in x, whereas $H(x)$ is of degree $2p$ in x. These two polynomials (with coefficients in the integral domain $\mathbb{Q}[x_1, \ldots, x_p, z^{-1}]$) agree at $2p+2$ values of x, hence are equal. Setting $x=d$, $a_i = \pi_i(d) + \pi_{2p+3-i}(d)$ $(1 \leqslant i \leqslant p+1)$ and multiplying by z^k yields (1.2).

Now consider the case $d_{p+1} = k-d_{p+1}$. Let $I = \{d_1, \ldots, d_{p+1}, k-d_p, \ldots, k-d_1\}$. For $p+2 \leqslant i \leqslant 2p+1$ define $d_i \leqslant k-d_{2p+2-i}$, so that $I = \{d_i\}_{1 \leqslant i \leqslant 2p+1}$. The set I contains $2p+1$ distinct integers. For $1 \leqslant i \leqslant 2p+1$ let π_i be the interpolating polynomial of degree $2p$, which is 1 at d_i and 0 at the remaining elements of I. Let $G(x) = \sum_{i=1}^{2p+1} \pi_i(x) F(d_i) F(k-d_i)$ and $H(x) = F(x)F(k-x)$. Then $G(c) = H(c)$ for all $c \in I$. The polynomials $G(x)$ and $H(x)$ are both of degree $\leqslant 2p$ in x. These two polynomials (with coefficients in the integral domain $\mathbb{Q}[x_1, \ldots, x_p, z^{-1}]$) agree at $2p+1$ values of x, hence are equal. Setting $x=d$, $a_i = \pi_i(d) + \pi_{2p+2-i}(d)$ $(1 \leqslant i \leqslant p)$, $a_{p+1} = \pi_{p+1}(d)$, and multiplying by z^k yields (1.2). ∎

COROLLARY 1.3.

(a) *If $k \geqslant 2p$ then the monomials of degree $\leqslant 2$ and weight k in the γ_i span a vector space $V_{k,2}$ of dimension $p+1$, and any set of $p+1$ distinct monomials of degree $\leqslant 2$ is a basis for this vector space.*

(b) *If $k \leqslant 2p+1$ then any set of distinct monomials of degree $\leqslant 2$ and weight k is linearly independent.*

(c) *In any relation* (1.2) *all the a_i are uniquely determined and nonzero.*

Proof. The monomials $\gamma_k, \gamma_1\gamma_{k-1}, \ldots, \gamma_d\gamma_{k-d}$ $(d = \min(\lfloor k/2 \rfloor, p)$ are linearly independent by [4, proof of (4.1)] from which (b) follows. It also follows that if $k \geqslant 2p$ then $V_{k,2}$ is of dimension $\geqslant p+1$, and by (1.2) any $p+1$ elements span. Thus (for $k \geqslant 2p$) dim $V_{k,2} = p+1$, and (a) and (c) follow. (Note that (c) is vacuous unless $k \geqslant 2p+2$.) ∎

Examples 1.4. Here are a few examples of the quadratic relations (obtained using a computer program that we wrote):

for $p=1$:

(1.4.1) $\gamma_4 = 4\gamma_1\gamma_3 - 3\gamma_2^2$

(1.4.2) $\gamma_5 = 3\gamma_1\gamma_4 - 2\gamma_2\gamma_3$

(1.4.3) $\gamma_1\gamma_5 = 4\gamma_2\gamma_4 - 3\gamma_3^2$

and for $p=2$:

(1.4.4) $\gamma_8 = 20\gamma_2\gamma_6 - 64\gamma_3\gamma_5 + 45\gamma_4^2$

(1.4.5) $\gamma_9\gamma_1 = 20\gamma_3\gamma_7 - 64\gamma_4\gamma_6 + 45\gamma_5^2$

(1.4.6) $\gamma_{10} = (63/5)\gamma_2\gamma_8 - (128/5)\gamma_3\gamma_7 + 14\gamma_4\gamma_6$

■

These examples illustrate the following.

Theorem 1.5. (1) *The quadratic relations are translation-invariant, i.e. if*

$$\gamma_d\gamma_{k-d} = \sum_{i=1}^{p+1} a_i\gamma_{d_i}\gamma_{k-d_i}$$

then also

$$\gamma_{d+j}\gamma_{k-d+j} = \sum_{i=1}^{p+1} a_i\gamma_{d_i+j}\gamma_{k-d_i+j}$$

for any integer $j \geqslant 0$ (with the same a_i). (Homogenize by putting in γ_0 if necessary.)

(2) *If the d_i are consecutive integers, then the coefficients a_i in* (1.2) *are integers.*

Proof. (1) In (1.1) replace d_i by $d_i' = d_i + j$ $(1 \leqslant i \leqslant p+1)$, d by $d+j$ and k by $k+2j$. Then also d_i is replaced by $d_i' = d_i + j$ $(p+2 \leqslant i \leqslant 2p+2$ or $p+2 \leqslant i \leqslant 2p+1$ respectively in the two parts of the proof of (1.1)). Formula (1.2) becomes

$$\gamma_{d+j}\gamma_{k-d+j} = \sum_{i=1}^{p+1} a_i'\gamma_{d_i+j}\gamma_{k-d_i+j}$$

where $a_i' = \pi_i'(d+j) + \pi_{2p+3-i}'(d+j)$ for $1 \leqslant i \leqslant p+1$ (respectively $a_i' = \pi_i'(d+j) + \pi_{2p+2-i}'(d+j)$ for $1 \leqslant i \leqslant p$ and $a_{p+1}' = \pi_{p+1}'(d+j)$), π_i' being the interpolating polynomial of degree $2p+1$ (respectively degree $2p$) which is 1 at d_i', and 0 at the remaining d_j'. Obviously $\pi_i'(c+j) = \pi_i(c)$ for all real numbers c, from which it follows that $a_i' = a_i$ for all i, proving (1).

(2) If the d_i $(1 \leqslant i \leqslant 2p+2$, resp. $1 \leqslant i \leqslant 2p+1$ in the two cases) are consecutive integers, then the Lagrange formula for the π_i (when evaluated at any integer) is (up to sign) the product of two binomial coefficients. Thus the $\pi_i(d)$ are integers, hence also the a_i, proving (2). ■

Example (1.4.6) shows that in general the a_i need not be integers. We can arrange to have the d_i consecutive by taking $c = \lfloor k/2 \rfloor$ and $\{\gamma_c\gamma_{k-c}, \gamma_{c-1}\gamma_{k-c+1}, \cdots, \gamma_{c-p}\gamma_{k-c+p}\}$ as the set of quadratic monomials on the right-hand side of (1.2).

2. The Poincaré series of $R^{(N)}$

Determining the Poincaré series of $R^{(N)}$ is essentially the same as determining the dimension of the $\mathbb{Q}$-vector space $R_k^{(N)}$, the weight k part of $R^{(N)}$, for every k. In fact,

we do more. Namely, using a basis interchange technique, we find in the following theorem an explicit monomial basis for $R_k^{(N)}$.

Theorem 2.1. *$R_k^{(N)}$ has $\mathbb{Q}$-basis*

$$\mathscr{B}_{N,k} = \left\{ \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_d} \mid N \leqslant i_1 \leqslant \cdots \leqslant i_{d-1} \leqslant i_d,\ i_{d-1} < N+p \quad \text{if} \quad d>1,\ \sum_{j=1}^{d} i_j = k \right\}.$$

Proof. If $p=0$ the result is trivial. For then $\gamma_i = z^i$ for all i and $R^{(N)} = \mathbb{Q}[z^i \mid i \geqslant N]$. If $k \geqslant N$ then $R_k^{(N)}$ has basis γ_k and $\mathscr{B}_{N,k}$ contains only γ_k, since we must have $d=1$. If $k=0$ then $R_0^{(N)}$ has basis $\gamma_0 = 1$ and $\mathscr{B}_{N,0}$ contains only the empty product 1 since we must have $d=0$. If $0<k<N$ then $R_k^{(N)} = 0$ and $\mathscr{B}_{N,k}$ is empty. Hence assume $p \geqslant 1$. First consider the case $N=1$. In [4, (4.1)] a basis $\{z^k G_t'(k) \mid t \in \mathscr{T}_k\}$ for $R_k^{(1)}$ (there denoted simply as R_k) is obtained. The definition of this basis is quite technical, so we will not recall its definition completely. It suffices to note that $\mathscr{T}_k$ is a set of integers indexing all sequences of the form $a_t = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_k \leqslant p$, $\alpha_{k-1} = \alpha_k$, and $\sum \alpha_i \leqslant k$. Also, in the proof of [4, (4.1)] the above basis is put in one-to-one correspondence with another basis of $R_k^{(1)}$ that consists of monomials in the γ's. Under this bijection, $z^k G_t'(k)$, for $a_t = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, corresponds to $\gamma_{\alpha_1}\cdots\gamma_{\alpha_{k-1}}\gamma_{\beta_k}$, where $\beta_k \geqslant \alpha_k$ is chosen so as to make the weight $\alpha_1 + \cdots + \alpha_{k-1} + \beta_k = k$ (remember that some of the α's can be 0, and that $\gamma_0 = 1$). But (omitting the γ_0's, renumbering the remaining γ's and noting that $N+p-1=p$) this is just the basis $\mathscr{B}_{1,k}$ claimed for $N=1$ in the statement of the theorem.

Now, for general N, if $\gamma_i\gamma_j$ is a factor of a monomial in the γ's of weight k, with i and j both $\geqslant N+p$, then the quadratic relations (1.2) can be used to replace $\gamma_i\gamma_j$ by a linear combination of

$$\gamma_{i+j},\ \gamma_N\gamma_{i+j-N},\ \gamma_{N+1}\gamma_{i+j-N-1},\ \cdots,\ \gamma_{N+p-1}\gamma_{i+j-N-p+1}$$

(note that $i+j-N-p+1 \geqslant N+p-1 \geqslant N$) from which it follows that $\mathscr{B}_{N,k}$ spans $R_k^{(N)}$. Thus it suffices to prove the linear independence of $\mathscr{B}_{N,k}$. This we prove by induction on N. The idea is to produce a basis for $R_k^{(N-1)}$ that contains $\mathscr{B}_{N,k}$ as a subset.

Hence suppose that $N \geqslant 2$ and $\mathscr{B}_{N-1,k}$ is a basis for $R_k^{(N-1)}$. We have $\mathscr{B}_{N-1,k} = \{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_d} \mid N-1 \leqslant i_1 \leqslant \cdots \leqslant i_{d-1} \leqslant i_d,\ i_{d-1} < N+p-1 \text{ if } d>1, \sum_{j=1}^{d} i_j = k\}$. Let $\mathscr{C} = \mathscr{B}_{N,k} \cap \mathscr{B}_{N-1,k}$ (= those elements of $\mathscr{B}_{N-1,k}$ that do not contain any γ_{N-1}'s). Let $\mathscr{E}_0$ be the set of those elements of $\mathscr{B}_{N-1,k}$ which contain a certain number of γ_{N-1}'s, say $e \geqslant 1$ of them, and which have the largest subscript i_d satisfying $i_d - ep \geqslant N+p-1$. Let $\mathscr{E}$ be obtained (elementwise) from $\mathscr{E}_0$ by replacing each γ_{N-1} by γ_{N+p-1} and decreasing the highest subscript accordingly. The theorem follows from

(2.1.1) *Claim*

(2.1.2) $\mathscr{B}_{N,k} = \mathscr{C} \amalg \mathscr{E}$

(2.1.3) $(\mathscr{B}_{N-1,k} - \mathscr{E}_0) \cup \mathscr{E}$ is a basis for $R_k^{(N-1)}$.

Proof of (2.1.2). Obviously $\mathscr{C} \subseteq \mathscr{B}_{N,k}$ and $\mathscr{E} \subseteq \mathscr{B}_{N,k}$. Furthermore, any element of $\mathscr{B}_{N,k}$ that contains $e>1$ γ_{N+p-1}'s (or one γ_{N+p-1} and one γ with subscript $>N+p-1$) is obtained uniquely by the above transformation from an element of $\mathscr{E}_0$, and any element of $\mathscr{B}_N$ that contains at most one γ_{N+p-1} and has all other subscripts

$<N+p-1$ is in $\mathscr{C}$. Thus $\mathscr{B}_{N,k} \subseteq \mathscr{C} \cup \mathscr{E}$. It is obvious that $\mathscr{C} \cap \mathscr{E} = \varnothing$, which proves (2.1.2).

Proof of (2.1.3). Let

$$\mathscr{B}' = \left\{ \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_d} \mid N-1 \leqslant i_1 \leqslant \cdots \leqslant i_{d-1} \leqslant i_d,\ i_{d-1} < N+p \text{ if } d>1,\ \sum_{j=1}^{d} i_j = k \right\},$$

and let $\mathscr{E}'_0$ be the set of those elements of $\mathscr{B}'$ which contain a certain number of γ_{N-1}'s, say $e \geqslant 1$ of them, and which have the largest subscript i_d satisfying $i_d - ep \geqslant N+p-1$. Then $\mathscr{B}_{N-1,k} \cup \mathscr{B}_{N,k} \subseteq \mathscr{B}'$ and $\mathscr{E}_0 \subseteq \mathscr{E}'_0$. Let $\rho: \mathscr{E}'_0 \amalg \mathscr{B}_{N,k} \to \mathscr{E}'_0 \amalg \mathscr{B}_{N,k}$ be the map which is identity on $\mathscr{B}_{N,k}$ and is defined on $\mathscr{E}'_0$ as follows: if $\gamma \in \mathscr{E}'_0$ then write $\gamma = \gamma_{N-1}\delta\gamma_c$ with $c \geqslant N+2p-1$ and δ a monomial in $\gamma_{N-1}, \gamma_N, \ldots, \gamma_{N+p-1}$, and define $\rho(\gamma) = \delta\gamma_{N+p-1}\gamma_{c-p}$. Further, for such a $\gamma = \gamma_{N-1}\delta\gamma_c \in \mathscr{E}'_0$ define

$$S(\gamma) = \{\delta\gamma_0\gamma_{c+N-1}, \delta\gamma_N\gamma_{c-1}, \delta\gamma_{N+1}\gamma_{c-2}, \ldots, \delta\gamma_{N+p-2}\gamma_{c-p+1}\}.$$

Put $\mathscr{D}_0 = \mathscr{E}_0 \amalg \mathscr{C}$, and for $i \geqslant 1$ let $\mathscr{D}_i = \{\rho(\gamma) \mid \gamma \in \mathscr{D}_{i-1}\}$. Then each $\mathscr{D}_i$ is a subset of $\mathscr{E}'_0 \amalg \mathscr{B}_{N,k}$, ρ is a bijection from $\mathscr{D}_i$ onto $\mathscr{D}_{i-1}$, and $\mathscr{E} \amalg \mathscr{C} = \mathscr{D}_i$ for $i \gg 0$. Let

$$\mathscr{D}_{ij} = \{\gamma \in \mathscr{D}_i \mid \gamma_{N-1} \text{ appears exactly to power } j \text{ in } \gamma\}.$$

Then $\mathscr{D}_i = \amalg_{j \geqslant 0} \mathscr{D}_{ij}$, and for $i, j \geqslant 1$ we have $\rho(\mathscr{D}_{i-1,j}) \subseteq \mathscr{D}_{i,j-1}$ with equality if $j \geqslant 2$. Let $\gamma \in \mathscr{D}_{ij}$ with $j \geqslant 1$. We claim that $S(\gamma) \subseteq \mathscr{D}_{i,j-1}$. This is clear for $i=0$. If $i \geqslant 1$ then $\gamma = \rho(\beta)$ with $\beta \in \mathscr{D}_{i-1,j+1}$, and clearly $S(\gamma) = \{\rho(\alpha) \mid \alpha \in S(\beta)\}$. So the claim follows by induction on i. Now, the set $\{\gamma, \rho(\gamma)\} \cup S(\gamma)$ has $p+2$ elements, and by (1.1) and (1.3) (c) any $p+1$ of these elements form a basis for the vector space spanned by this set. So, as $S(\gamma) \subseteq \mathscr{D}_{i,j-1}$, the sets $\{\gamma\} \cup \mathscr{D}_{i,j-1}$ and $\{\rho(\gamma)\} \cup \mathscr{D}_{i,j-1}$ span the same vector space. Therefore, since $\mathscr{D}_i$ can be obtained from $\mathscr{D}_{i-1}$ in stages by changing

$$(\amalg_{j \geqslant h+1} \rho(\mathscr{D}_{i-1,j})) \cup (\amalg_{j=0}^{h} \mathscr{D}_{i-1,j}) \text{ to } (\amalg_{j \geqslant h} \rho(\mathscr{D}_{i-1,j})) \cup (\amalg_{j=0}^{h-1} \mathscr{D}_{i-1,j}),$$

starting with the highest h, it follows that each $\mathscr{D}_i$ spans the same space. In particular, $\mathscr{E}_0 \amalg \mathscr{C}$ and $\mathscr{E} \amalg \mathscr{C} = \mathscr{B}_{N,k}$ span the same space. The former being a part of a basis for $\mathscr{B}_{N-1,k}$, (2.1.3) is proved. ■

COROLLARY 2.2.

The ideal of all relations among the γ's is generated by the quadratic relations (1.2).

Proof. Only the relations (1.2) were used to reduce the set of all monomials of weight k in the γ's to the basis $\mathscr{B}_{N,k}$. ■

COROLLARY 2.3.

Let $V_{k,d}$ be the subspace of $R_k^{(N)}$ spanned by monomials of weight k and degree $\leqslant d$ in the γ_i ($\deg \gamma_i = 1$ for all $i \geqslant 1$) as in [4, § 2]. *Then $V_{k,d}$ has $\mathbb{Q}$-basis of those monomials in $\mathscr{B}_{N,k}$ of degree $\leqslant d$.*

Proof. The indicated elements are linearly independent since they are part of the basis $\mathscr{B}_{N,k}$. Therefore it suffices to prove that they span $V_{k,d}$. To do this we may assume that $p \geqslant 1$. If $\gamma_i\gamma_j$ is a factor of a monomial in the γ's of weight k, and degree $\leqslant d$ with i and j both $\geqslant N+p$, then as in the proof of (2.1) the quadratic relations (1.2)

can be used to replace $\gamma_i\gamma_j$ by a linear combination of $\gamma_{i+j}, \gamma_N\gamma_{i+j-N}$, $\gamma_{N+1}\gamma_{i+j-N-1}, \cdots, \gamma_{N+p-1}\gamma_{i+j-N-p+1}$ (note that $i+j-N-p+1 \geqslant N+p-1 \geqslant N$ and that the quadratic replacement does not increase degree), from which it follows that the claimed elements span $V_{k,d}$. ■

COROLLARY 2.4. (cf. [4, (2.1)])

We have $\dim V_{k,d} = \binom{p+d-1}{p}$ *for* $k \gg 0$. *More precisely,* $\dim V_{k,d} = \binom{p+d-1}{p}$ *if and only if* $k \geqslant m$, *where* m *is defined as follows*: (1) *if* $p \geqslant 1$ *and* $d \geqslant 2$ *then* $m = (N+p-1)d$; (2) *if* $d = 0$ *then* $m = 1$; (3) *in all other cases* $m = N$ *or* $m = 0$ *accordingly as* $N > 1$ *or* $N = 1$.

Proof. Case (2) is trivial. For, if $d = 0$ then $\binom{p+d-1}{p} = 0$, and the only product of degree zero is the empty product which is 1. So assume that $d \geqslant 1$. Then if the γ_i of highest weight is removed from each element of the basis of $V_{k,d}$ described in (2.3), this basis is put in one-to-one correspondence with a subset of the monomials of degree less than or equal to $d-1$ in the p variables $\gamma_N, \gamma_{N+1}, \cdots, \gamma_{N+p-1}$. If k is large enough we obtain in this manner all monomials of degree less than or equal to $d-1$ in $\gamma_N, \gamma_{N+1}, \cdots, \gamma_{N+p-1}$. Since there are $\binom{p+d-1}{p}$ such monomials, the first part is proved. Assume now that we are in case (1), i.e. $p \geqslant 1$ and $d \geqslant 2$. Then a monomial M of degree $\leqslant d-1$ in $\gamma_N, \gamma_{N+1}, \cdots, \gamma_{N+p-1}$ corresponds to an element of our basis if and only if $k - wt(M)$ is bigger than or equal to any subscript occurring in M. The most critical case is γ_{N+p-1}^{d-1} which requires $k-(d-1)(N+p-1) \geqslant N+p-1$, or $k \geqslant (N+p-1)d = m$, proving case (1). The proof of case (3) is an easy and straightforward verification. ■

Example 2.5. Here is an example to illustrate the algorithm in the proof of (2.1). Let $N = p = 2$. Then $\dim_{\mathbb{Q}} R_{11}^{(1)} = 31$, $\dim_{\mathbb{Q}} R_{11}^{(2)} = 10$. Monomials in the γ's will be represented by listing the subscripts, thus $(1,1,2,7)$ represents $\gamma_1^2\gamma_2\gamma_7$. We have $\mathscr{E}_0 = \{(1,10), (1,1,9), (1,2,8), (1,1,2,7), (1,2,2,6)\}$ and $\mathscr{C} = \{(11), (2,9), (2,2,7), (2,2,2,5), (2,2,2,2,3)\}$. To understand the example it is not necessary to list the elements of $\mathscr{B}_{1,11} - \mathscr{E}_0$ explicitly. We have $\mathscr{D}_0 = \mathscr{E}_0 \amalg \mathscr{C} = \mathscr{D}_{00} \amalg \mathscr{D}_{01} \amalg \mathscr{D}_{02}$ with

$$\mathscr{D}_{00} = \{(11), (2,9), (2,2,7), (2,2,2,5), (2,2,2,2,3)\},$$
$$\mathscr{D}_{01} = \{(1,10), (1,2,8), (1,2,2,6)\} \text{ and } \mathscr{D}_{02} = \{(1,1,9), (1,1,2,7)\}.$$

The following table shows how the transformation proceeds using the linear relation among $\gamma, \rho(\gamma)$ and $S(\gamma)$:

$\gamma =$	Replaced by $\rho(\gamma) =$	Using $S(\gamma) =$
$(1,1,9)$	$(1,3,7)$	$(1,10), (1,2,8)$
$(1,1,2,7)$	$(1,2,3,5)$	$(1,2,8), (1,2,2,6)$
$(1,10)$	$(3,8)$	$(11), (2,9)$
$(1,2,8)$	$(2,3,6)$	$(2,9), (2,2,7)$

(1, 2, 2, 6)	(2, 2, 3, 4)	(2, 2, 7), (2, 2, 2, 5)
(1, 3, 7)	(3, 3, 5)	(3, 8), (2, 3, 6)
(1, 2, 3, 5)	(2, 3, 3, 3)	(2, 3, 6), (2, 2, 3, 4)

The first two rows show how $\mathscr{D}_{02}$ is transformed into $\rho(\mathscr{D}_{02})$ and the next three rows show how $\mathscr{D}_{01}$ is transformed into $\rho(\mathscr{D}_{01})$. This gives $\mathscr{D}_1 = \mathscr{D}_{10} \amalg \mathscr{D}_{11}$ with $\mathscr{D}_{11} = \rho(\mathscr{D}_{02}) = \{(1,3,7),(1,2,3,5)\}$ and $\mathscr{D}_{10} = \rho(\mathscr{D}_{01}) \amalg \mathscr{D}_{00} = \{(3,8),(2,3,6),(2,2,3,4),(11),(2,9),(2,2,7),(2,2,2,5),(2,2,2,2,3)\}$. Finally, the last two rows show how $\mathscr{D}_{11}$ is transformed into $\rho(\mathscr{D}_{11})$, giving $\mathscr{D}_2 = \mathscr{D}_{20} = \rho(\mathscr{D}_{11}) \amalg \mathscr{D}_{10} = \{(3,3,5),(2,3,3,3),(3,8),(2,3,6),(2,2,3,4),(11),(2,9),(2,2,7),(2,2,2,5),(2,2,2,2,3)\} = \mathscr{E} \amalg \mathscr{C} = \mathscr{B}_{2,11}$. Note that for fixed i, j the order in which elements of $\mathscr{D}_{ij}$ are transformed into those of $\rho(\mathscr{D}_{ij})$ is immaterial.

The basis of $V_{3,11}$ given by (2.3) is $\{(11), (2,9), (3,8), (2,2,7), (2,3,6), (3,3,5)\}$. ■

The calculation of the Poincaré series is now just a matter of counting $\mathscr{B}_{N,k}$. The number of partitions of k as sums of integers each $\geqslant N$ and $\leqslant N+p-1$ is the coefficient of t^k in

$$\frac{1}{(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p-1})}. \tag{2.6}$$

Allowing one integer $> N+p-1$ is the same as finding the partitions of the integers from 0 to $k-N-p$ as sums of integers each $\geqslant N$ and $\leqslant N+p-1$ (adding one more integer, which will be greater than $N+p-1$, to each partition to bring the sum up to k), and the number of such partitions is the coefficient of t^k in

$$\frac{t^{N+p}}{(1-t)(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p-1})}. \tag{2.7}$$

Adding (2.6) and (2.7) yields

Theorem 2.8. *Let $P(t)$ be the Poincaré series for the ring $R^{(N)}$*, i.e. *$P(t) = \sum_{k=0}^{\infty} H(k)t^k$, where $H(k) = \dim_{\mathbb{Q}} R_k^{(N)}$. Then*

$$P(t) = \frac{1-t+t^{N+p}}{(1-t)(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p-1})}. \quad ■$$

By a similar argument, using x to keep track of the number of terms added, we obtain that $\dim V_{k,d}$ is the coefficient of $x^d t^k$ in

$$\frac{1-t+xt^{N+p}}{(1-t)(1-x)(1-xt^N)(1-xt^{N+1})\cdots(1-xt^{N+p-1})}. \tag{2.9}$$

3. Relations ideal and the structure of $R^{(N)}$

In this section we determine the structure of $R^{(N)}$ by finding a minimal presentation for it as a graded $\mathbb{Q}$-algebra. We show that $R^{(N)}$ has Krull dimension $p+1$ and

embedding dimension $N+2p$, and that in a presentation of $R^{(N)}$ as a graded quotient of the polynomial algebra in $N+2p$ variables the ideal of relations is generated minimally by $\binom{N+p}{2}$ elements. As corollaries, we show that $R^{(N)}$ is always Cohen–Macaulay; that $R^{(N)}$ is Gorenstein if and only if it is a complete intersection if and only if $N+p\leqslant 2$ (which happens exactly in the three cases $p=0$, $N=1$; $p=0$, $N=2$; $p=1=N$); and that $R^{(N)}$ is regular if and only if $p=0$, $N=1$.

Let $B=\mathbb{Q}[T_N, T_{N+1},\ldots, T_{2N+2p-1}]$ be the polynomial ring in $N+2p$ variables graded by weight $(T_i)=i$, and let $\varphi\colon B\to R^{(N)}$ be the $\mathbb{Q}$-algebra homomorphism given by $\varphi(T_i)=\gamma_i$ for all i. Let $A=\mathbb{Q}[T_N, T_{N+1},\ldots, T_{N+p}]$, let M be the A-submodule of B generated by $1, T_{N+p+1},\ldots, T_{2N+2p-1}$ and let $M'=\varphi(M)$. Then M' is the A'-submodule of $R^{(N)}$ generated by $1, \gamma_{N+p+1},\ldots,\gamma_{2N+2p-1}$, where $A'=\mathbb{Q}[\gamma_N,\gamma_{N+1},\ldots,\gamma_{N+p}]$. (We will see later that $M'=R^{(N)}$.)

Lemma 3.1. *We have $\gamma_i\in M'$ and $\gamma_i\gamma_j\in M'$ for all $i,j\geqslant N$.*

Proof. We prove the first part by induction on i. Clearly we have $\gamma_i\in M'$ for $N\leqslant i\leqslant 2N+2p-1$. Let $i\geqslant 2N+2p$. Then $i-N-p\geqslant N+p$ so by (1.2) γ_i belongs to the $\mathbb{Q}$-span of $\gamma_N\gamma_{i-N}, \gamma_{N+1}\gamma_{i-N-1},\ldots,\gamma_{N+p}\gamma_{i-N-p}$. Now $\gamma_{i-N},\gamma_{i-N-1},\ldots,\gamma_{i-N-p}\in M'$ by induction, since $i>i-N\geqslant i-N-p\geqslant N$. Therefore $\gamma_i\in M'$, and the first part is proved. Now, if at least one of i and j is $\leqslant N+p$ then $\gamma_i\gamma_j\in M'$ by the first part. On the other hand, if both i and j are $>N+p$ then $i+j-N-p+1>N+p-1$ so by (1.2) $\gamma_i\gamma_j$ belongs to the $\mathbb{Q}$-span of $\gamma_{i+j}, \gamma_N\gamma_{i+j-N},\ldots,\gamma_{N+p-1}\gamma_{i+j-N-p+1}$ (just γ_{i+j} if $p=0$) and these $p+1$ monomials belong to M' by the first part. So $\gamma_i\gamma_j\in M'$. ∎

By the Lemma we can write, for $i,j\geqslant N+p+1$, $\gamma_i\gamma_j=\alpha'+\sum_{h=N+p+1}^{2N+2p-1}\beta'_h\gamma_h$ with $\alpha',\beta'_h\in A'$. We may assume that α',β'_h are homogeneous of appropriate weight so that the expression is homogeneous of weight $i+j$. Lift α',β'_h to homogeneous elements α,β_h of A of the same weight and let

$$P_{ij}=T_iT_j-\alpha-\sum_{h=N+p+1}^{2N+2p-1}\beta_hT_h.$$

Then P_{ij} is homogeneous of weight $i+j$.

Theorem 3.2. *The graded $\mathbb{Q}$-algebra $R^{(N)}$ has Krull dimension $p+1$ and embedding dimension $N+2p$, and has a minimal presentation with $N+2p$ generators and $\binom{N+p}{2}$ relations. More precisely, the $\mathbb{Q}$-algebra homomorphism $\varphi\colon B\to R^{(N)}$ is surjective and the ideal $\ker(\varphi)$ of B is generated minimally by the $\binom{N+p}{2}$ elements P_{ij}, $N+p+1\leqslant i\leqslant j\leqslant 2N+2p-1$.*

Proof. By [2,(1.4)], or by (3.1) above, $R^{(N)}$ is generated by $\gamma_N,\gamma_{N+1},\ldots,\gamma_{2N+2p-1}$. This means that φ is surjective, and $R^{(N)}$ is a $\mathbb{Q}$-algebra of finite type. Now, since the quotient field of $R^{(N)}$ is $\mathbb{Q}(x_1,\ldots,x_p,z)$ by [4,(5.2)], we get $\dim(R^{(N)})=p+1$. (That $\dim(R^{(N)})=p+1$ also follows independently from (3.3) below.)

We show next that the set $\{P_{ij}\mid N+p+1\leqslant i\leqslant j\leqslant 2N+2p-1\}$ generates $\ker(\varphi)$ minimally. To do this, let I be the ideal of B generated by this set.

Minimality. Since the P_{ij} are homogeneous, it is enough to show that no P_{ij} belong to the ideal generated by the remaining ones. Suppose for some i,j we hav $P_{ij} = \sum_{(r,s)\neq(i,j)} f_{rs}P_{rs}$ with $f_{rs}\in B$. We may assume that each f_{rs} is homogeneous wit weight $(f_{rs}) = i+j-r-s$ (negative weight means the element is zero). L $Q_{rs} = P_{rs} - T_rT_s$. Then

$$T_iT_j + Q_{ij} = \sum_{(r,s)\neq(i,j)} f_{rs}(T_rT_s + Q_{rs}).$$

Since $N+p+1 \leqslant i, j \leqslant 2N+2p-1$ and Q_{ij} is of degree at most one i $T_{N+p+1},\ldots,T_{2N+2p-1}$, the term T_iT_j is present on the left hand side. Let us look fo this term on the right hand side. First of all, T_iT_j cannot appear in any of the term $f_{rs}T_rT_s$ because $(r,s)\neq(i,j)$ is an unordered pair. It follows that T_iT_j must come fro one of the terms $f_{rs}Q_{rs}$. Since $N+p+1\leqslant i, j\leqslant 2N+2p-1$ and Q_{rs} is of degree at mo one in $T_{N+p+1},\ldots,T_{2N+2p-1}$, in order for T_iT_j to appear in the term $f_{rs}Q_{rs}$ it necessary for f_{rs} to contain a term which is a nonzero rational times T_i or T_j or T_iT Accordingly, we would get $i+j-r-s =$ weight $(f_{rs}) = i$ or j or $i+j$ whence $r+s=$ or i or 0. This is a contradiction, since $r+s\geqslant 2N+2p+2$. This proves the minimalit of the generators.

Generation. By construction, we have $I\subseteq\ker(\varphi)$. So we have the surjective ma $\psi: B/I\to R^{(N)}$ induced by φ. We have to show that ψ is an isomorphism. Note tha M is a free A-module of rank $N+p$, with basis $T_0 := 1, T_{N+p+1},\ldots,T_{2N+2p-1}$. Th module M is graded by weight $(T_i) = i$. Let $\zeta: M\to B/I$ be the restriction of the natur map $B\to B/I$ to M. Given any polynomial in B, we can reduce it modulo I to a element of M. This means that ζ is surjective. Now, let $\sigma = \psi\circ\zeta$. Then $\sigma: M\to R^{(N)}$ an A-linear map which is homogeneous of degree zero and is surjective. Now, denotin by $P_L(t)$ the Poincaré series of a graded A-module L and writing $R = R^{(N)}$, it is enoug to prove that $P_R(t) = P_M(t)$. For, since σ is surjective, this would show that σ is a isomorphism whence also ψ is an isomorphism. Now, by (2.8) we have

$$P_R(t) = \frac{1-t+t^{N+p}}{(1-t)(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p-1})}.$$

On the other hand, since A is the polynomial ring $\mathbb{Q}[T_N, T_{N+1},\ldots,T_{N+p}]$ with weig $(T_i) = i$, we have

$$P_A(t) = \frac{1}{(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p})}.$$

Therefore, since M is A-free with basis $1, T_{N+p+1},\ldots,T_{2N+2p-1}$ and weight $(T_i) =$ we get

$$P_M(t) = P_A(t) + \sum_{i=N+p+1}^{2N+2p-1} t^iP_A(t) = \frac{1+t^{N+p+1}+t^{N+p+2}+\cdots+t^{2N+2p-1}}{(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p})}$$

Now, it is checked readily that $P_R(t) = P_M(t)$. This completes the proof of the equali $I = \ker(\varphi)$.

Finally, we show that the embedding dimension of $R^{(N)}$ is $N+2p$. Recall that f a finitely generated graded ring $C = \oplus_{k\geqslant 0} C_k$ with C_0 a field its embedding dimensi emdim(C) is the minimal number of homogeneous C_0-algebra generators of C, equivalently the minimal number of homogeneous generators of the ide $C_+ = \oplus_{k\geqslant 1} C_k$. In our situation we have $R^{(N)} = B/I$ with I generated by the P

$N+p+1 \leqslant i \leqslant j \leqslant 2N+2p-1$. For such i, j we have $i+j \geqslant 2N+2p+2$. Therefore in the expression $P_{ij} = T_iT_j - \alpha - \sum_{h=N+p+1}^{2N+2p-1} \beta_h T_h$ we have $\alpha \in A_+^2$ and each $\beta_h \in A_+$. This shows that $I \subseteq B_+^2$. Therefore by (graded) Nakayama the minimal number of homogeneous generators of the ideal $R_+^{(N)}$ of $R^{(N)}$ is the same as that of the ideal B_+ of B, which is $N+2p$, since B is the polynomial ring in $N+2p$ variables. This proves that $\operatorname{emdim}(R^{(N)}) = N+2p$. ■

COROLLARY 3.3.

The ring $A' = \mathbb{Q}[\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p}]$ *is the polynomial ring in* $p+1$ *variables over* $\mathbb{Q}$, *and* $R^{(N)}$ *is a finite free* A'*-module with basis* $1, \gamma_{N+p+1}, \ldots, \gamma_{2N+2p-1}$.

Proof. The restriction of the isomorphism $\sigma: M \to R^{(N)}$ to A is a $\mathbb{Q}$-algebra isomorphism of A onto A', sending T_i to γ_i ($N \leqslant i \leqslant N+p$). This implies the first part. The second part follows since $\sigma(T_i) = \gamma_i$ ($i = 0$ or $N+p+1 \leqslant i \leqslant 2N+2p-1$). ■

COROLLARY 3.4.

A $\mathbb{Q}$*-basis for* $R^{(N)}$ *in terms of monomials in* $\gamma_N, \gamma_{N+1}, \ldots, \gamma_{2N+2p-1}$ *is*

$$\{\gamma_N^{q_N}\gamma_{N+1}^{q_{N+1}} \cdots \gamma_{2N+2p-1}^{q_{2N+2p-1}} \mid q_{N+p+1} + q_{N+p+2} + \cdots + q_{2N+2p-1} \leqslant 1\}.$$

Consequently, a $\mathbb{Q}$*-basis for* $R_k^{(N)}$ *is*

$$\left\{\gamma_N^{q_N}\gamma_{N+1}^{q_{N+1}} \cdots \gamma_{2N+2p-1}^{q_{2N+2p-1}} \mid q_{N+p+1} + q_{N+p+2} + \cdots + q_{2N+2p-1} \leqslant 1, \sum_{j=N}^{2N+2p-1} N_j q_j = k\right\}$$

which can also be written, for comparison with (2.1), *as*

$$\left\{\gamma_{i_1}\gamma_{i_2} \cdots \gamma_{i_d} \mid N \leqslant i_1 \leqslant \cdots \leqslant i_{d-1} \leqslant i_d \leqslant 2N+2p-1, \; i_{d-1} \leqslant N+p \text{ if } d > 1, \sum_{j=1}^{d} i_j = k\right\}.$$

Proof. Immediate from (3.3). ■

COROLLARY 3.5.

The sequence $\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p}$ *is* $R^{(N)}$*-regular, and the ring* $R^{(N)}$ *is Cohen–Macaulay.*

Proof. The regularity of the sequence is immediate from (3.3). Therefore the localization of $R^{(N)}$ at the irrelevant maximal ideal $R_+^{(N)}$ of $R^{(N)}$ is Cohen–Macaulay. It is well known that this implies that $R^{(N)}$ is Cohen-Macaulay (e.g. [1, (33.27)]). ■

COROLLARY 3.6.

The following three conditions are equivalent:

(1) $R^{(N)}$ *is Gorenstein;* (2) $R^{(N)}$ *is a complete intersection;* (3) $N+p \leqslant 2$.

Note that, since $N \geqslant 1$, (3) occurs in exactly the following three cases: $p=0$, $N=1$; $p=0$, $N=2$; $p=1=N$.

Proof. (1)$\Leftrightarrow$(3): Since $R^{(N)}$ is graded, it is well known that $R^{(N)}$ is Gorenstein if and only if its localization at the irrelevant maximal ideal $R^{(N)}_+$ is Gorenstein (e.g. [1, (33.27)]). Let C denote this localization and put $D = C/(\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p})$. Then, since C is Cohen–Macaulay and $\gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+p}$ is a regular C-sequence by (3.5), C is Gorenstein if and only if D is Gorenstein. Let $\mathfrak{m}$ be the maximal ideal of D. Then, since $\dim(D)=0$, D is Gorenstein if and only if $\operatorname{ann}(\mathfrak{m})$, the annihilator of $\mathfrak{m}$, is a 1-dimensional space over $D/\mathfrak{m}$. Now, it follows from (3.2) that $\mathfrak{m}$ is generated minimally by $\delta_{N+p+1}, \ldots, \delta_{2N+2p-1}$, where δ_i denotes the natural image of γ_i in D. Consider two cases:

Case 1: $\mathfrak{m}=0$. In this case D is Gorenstein, and this case occurs $\Leftrightarrow \{\delta_{N+p+1}, \ldots, \delta_{2N+2p-1}\} = \varnothing \Leftrightarrow 2N+2p \leqslant N+p+1 \Leftrightarrow N+p \leqslant 1$.

Case 2: $\mathfrak{m} \neq 0$. Then $\operatorname{ann}(\mathfrak{m}) \subseteq \mathfrak{m}$. If $N+p+1 \leqslant i, j \leqslant 2N+2p-1$ then, as noted in the proof of (3.2), we have $P_{ij} = T_iT_j - \alpha - \sum_{h=N+p+1}^{2N+2p-1} \beta_h T_h$ with $\alpha \in A_+^2$ and each $\beta_h \in A_+$. It follows that $\mathfrak{m}^2=0$. Thus $\mathfrak{m} \subseteq \operatorname{ann}(\mathfrak{m})$ whence $\operatorname{ann}(\mathfrak{m}) = \mathfrak{m}$. So D is Gorenstein $\Leftrightarrow \mathfrak{m}$ is generated by one element $\Leftrightarrow 2N+2p-1 = N+p+1 \Leftrightarrow N+p=2$.

(2)$\Leftrightarrow$(3): Since $\dim(R^{(N)}) = p+1$ and $R^{(N)} = B/I$ with I minimally generated by $\binom{N+p}{2}$ homogeneous elements, $R^{(N)}$ is a complete intersection if and only if $N+2p = p+1+\binom{N+p}{2}$. The solutions of this equation with integers $p \geqslant 0$, $N \geqslant 1$ are exactly those given by $N+p \leqslant 2$. ∎

COROLLARY 3.7.

The ring $R^{(N)}$ is regular if and only if $p=0$, $N=1$.

Proof. $R^{(N)}$ is regular $\Leftrightarrow \operatorname{emdim}(R^{(N)}) = \dim(R^{(N)}) \Leftrightarrow N+2p = p+1 \Leftrightarrow N+p=1 \Leftrightarrow p=0$, $N=1$. ∎

Example 3.8. We illustrate the structure theorem (3.2) by computing P_{ij} explicitly in the cases $p=1=N$ and $p=1$, $N=2$.

First, let $p=1=N$. In this case $B = \mathbb{Q}[T_1, T_2, T_3]$, $A = \mathbb{Q}[T_1, T_2]$, $A' = \mathbb{Q}[\gamma_1, \gamma_2]$, M' is the A'-module generated by $1, \gamma_3$, and there is only one relation P_{33}. To find it we have to express γ_3^2 as an A'-linear combination of $1, \gamma_3$. We do this by eliminating γ_4, γ_5 among the relations (1.4.1), (1.4.2), (1.4.3) obtaining

$$\gamma_3^2 = 3\gamma_1^2\gamma_2^2 - 4\gamma_2^3 - 4\gamma_1^3\gamma_3 + 6\gamma_1\gamma_2\gamma_3$$

as the desired linear combination. So $P_{33} = T_3^2 - 3T_1^2T_2^2 + 4T_2^3 + 4T_1^3T_3 - 6T_1T_2T_3$ and $R^{(1)} \cong \mathbb{Q}[T_1, T_2, T_3]/(T_3^2 - 3T_1^2T_2^2 + 4T_2^3 + 4T_1^3T_3 - 6T_1T_2T_3)$.

A similar computation for the case $p=1$, $N=2$ gives

$$R^{(2)} \cong \mathbb{Q}[T_2, T_3, T_4, T_5]/(P_{44}, P_{45}, P_{55})$$

with $P_{44} = T_4^2 - (8/3)T_2T_3^2 + 3T_2^2T_4 - (4/3)T_3T_5$, $P_{45} = T_4T_5 + 12T_3^3 - 16T_2T_3T_4 + 3T_2^2T_5$ and $P_{55} = T_5^2 - 32T_2^2T_3^2 + 36T_2^3T_4 + 9T_3^2T_4 - 14T_2T_3T_5$.

4. The independence of $\mathscr{B}_{N,k}$

In this section we give a new proof of the linear independence of $\mathscr{B}_{N,k}$, which does not depend upon the proof of (2.1). The matrix approach used here gives additional insight into the nature of $R^{(N)}$. In particular, we obtain a sharpening of the independence part of (2.1), in that we prove that a specific minor of a certain matrix is nonzero. Our matrix theoretic techniques are perhaps of interest in their own right.

Before stating our result precisely (Theorem (4.1)) we would like to describe more carefully the relationship between the two bases $\mathscr{B}_{1,k}$ and $\mathscr{G}_k := \{z^k G_t'(k) | t \in \mathscr{T}_k\}$ of $R_k^{(1)}$. In §2 we noted that $\mathscr{T}_k$ is a set of integers indexing (as t ranges over $\mathscr{T}_k$) all sequences of the form $a_t = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k \leqslant p$, $\alpha_{k-1} = \alpha_k$, $\sum \alpha_i \leqslant k$. In [4, §3] we also introduced monomials $b_t = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k}$ (with $x_0 = 1$). If we wish to write an element M of $\mathscr{B}_{1,k}$ (or more generally, any monomial M of weight k in the γ_i) as a linear combination of $\mathscr{G}_k$ we just expand M in terms of monomials b_t. Then the coefficient of $z^k G_t'(k)$ in M is the rational coefficient of b_t (ignoring the power of z). See [4, (3.7) (5)], and for some explicit examples [4, (4.3)]. We shall think of the basis element $z^k G_t'(k)$ as also being indexed by the monomial b_t.

Put $\mathscr{B}'_{N,k} = \{x_1^{a_1} x_2^{a_2} \cdots x_q^{a_q} | 0 \leqslant q \leqslant p,\ a_q \geqslant 2,\ \sum_{i=1}^{q} (i+N-1)a_i \leqslant k\}$. Then $\mathscr{B}'_{N,k}$ has the same cardinality as $\mathscr{B}_{N,k}$. An explicit bijection between $\mathscr{B}'_{N,k}$ and $\mathscr{B}_{N,k}$ is given by $x_1^{a_1} x_2^{a_2} \cdots x_q^{a_q} \leftrightarrow \gamma_N^{a_1} \gamma_{N+1}^{a_2} \cdots \gamma_{N+q-2}^{a_{q-1}} \gamma_{N+q-1}^{a_q} \gamma_\varepsilon$, where $\varepsilon \geqslant N+q-1$ is chosen to yield weight k. Give the set $\{x_1^{a_1} x_2^{a_2} \cdots x_q^{a_q} | 0 \leqslant q \leqslant p, a_q \geqslant 2\}$ the reverse lexicographic order and let $\mathscr{B}'_{N,k}$ have the induced order. Let $\mathscr{B}_{N,k}$ be given the order corresponding to that of $\mathscr{B}'_{N,k}$ under the above-mentioned bijection between $\mathscr{B}_{N,k}$ and $\mathscr{B}'_{N,k}$. This done, let ζ be the matrix over $\mathbb{Q}$ whose ij entry is the coefficient of the jth element of $\mathscr{G}_k$ in the expression of the ith element of $\mathscr{B}_{N,k}$ written as a $\mathbb{Q}$-linear combination of $\mathscr{G}_k$. The linear independence of $\mathscr{B}_{N,k}$ follows immediately from the following theorem.

Theorem 4.1. *Let $p \geqslant 0$ and let ζ be the matrix (with entries in $\mathbb{Q}$) defined above. Let η be the submatrix of ζ consisting of the columns corresponding to $\mathscr{B}'_{N,k}$. Then* $\det(\eta) \neq 0$.

Our first attempt to prove the linear independence of the $\mathscr{B}_{N,k}$ was by proving (4.1), but this turned out to be somewhat elusive. So we ended up proving (2.1) using the basis interchange technique given in §2. However, we were still intrigued by the equality $\operatorname{card}(\mathscr{B}_{N,k}) = \operatorname{card}(\mathscr{B}'_{N,k})$, and we were finally able to prove (4.1), showing that this equality is not a coincidence. This gives an independent, but more difficult, proof of (the linear independence part of) (2.1). In the proof of [4, (4.1)] (the case $N = 1$) the matrix ζ ($= \eta$ in this case) was triangular with nonzero entries down the diagonal so non-singularity was easy to establish. We have not been able to find such a simple argument in the case $N > 1$.

The following example will help explain the meaning of (4.1), as well as illustrate (2.3).

Example 4.2. Let $N = p = 2$, $k = 10$. Then $\mathscr{B}'_{2,10} = \{1, x_1^2, x_1^3, x_1^4, x_1^5, x_2^2, x_1x_2^2, x_1^2x_2^2, x_2^3\}$ and in the corresponding order $\mathscr{B}_{2,10} = \{\gamma_{10}, \gamma_2\gamma_8, \gamma_2^2\gamma_6, \gamma_2^3\gamma_4, \gamma_2^5, \gamma_3\gamma_7, \gamma_2\gamma_3\gamma_5, \gamma_2^2\gamma_3^2, \gamma_2^3\gamma_4\}$. Then $V_{10,1}$ has basis $\{\gamma_{10}\}$, $V_{10,2}/V_{10,1}$ has basis $\{\gamma_2\gamma_8, \gamma_3\gamma_7\}$, $V_{10,3}/V_{10,2}$ has basis $\{\gamma_2^2\gamma_6, \gamma_2\gamma_3\gamma_5, \gamma_3^2\gamma_4\}$, $V_{10,4}/V_{10,3}$ has basis $\{\gamma_2^3\gamma_4, \gamma_2^2\gamma_3^2\}$ and $V_{10,5}/V_{10,4}$ has basis $\{\gamma_2^5\}$. The complete list of monomials corresponding to $\mathscr{G}_k$ is $\{1, x_1^2, x_1^3, x_1^4, x_1^5, x_1^6, x_1^7, x_1^8, x_1^9, x_1^{10}, x_2^2, x_1x_2^2, x_1^2x_2^2, x_1^3x_2^2, x_1^4x_2^2, x_1^5x_2^2, x_1^6x_2^2, x_2^3, x_1x_2^3, x_1^2x_2^3, x_1^3x_2^3, x_1^4x_2^3, x_2^4, x_1x_2^4, x_1^2x_2^4, x_2^5\}$ so the matrix ζ is 9 by 26. Monomials of degree greater than 5 can be omitted since all entries in their columns will be 0. This leaves $\{1, x_1^2, x_1^3, x_1^4, x_1^5, x_2^2, x_1x_2^2, x_1^2x_2^2, x_1^3x_2^2,$

$x_2^3, x_1x_2^3, x_1^2x_2^3, x_2^4, x_1x_2^4, x_2^5\}$ so the non-trivial part of ζ is 9 by 15. We shall not write this matrix down, but the possibly nonzero entries by degree considerations (a row of degree d can have nonzero entries only in a column of degree $\leqslant d$) are indicated by *'s, and only the subscript digits are indicated for the row indices (x being 10). The column indices of $\mathscr{B}'_{2,10}$ (i.e. the columns of η) are underlined.

	$\underline{1}$,	$\underline{x_1^2}$,	$\underline{x_1^3}$,	$\underline{x_1^4}$,	$\underline{x_1^5}$,	$\underline{x_2^2}$,	$\underline{x_1x_2^2}$,	$\underline{x_1^2x_2^2}$,	$x_1^3x_2^2$,	$\underline{x_2^3}$,	$x_1x_2^3$,	$x_1^2x_2^3$,	x_2^4,	$x_1x_2^4$,	x_2^5
x	*														
28	*	*				*									
226	*	*	*			*	*			*					
2224	*	*	*	*		*	*	*		*	*		*		
22222	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
37	*	*				*									
235	*	*	*			*	*			*					
2233	*	*	*	*		*	*	*		*	*		*		
334	*	*	*			*	*			*					

Theorem (4.1) in this case is sharper than (2.1) in that there are several other maximal minors that could be nonzero.

The proof of (4.1) will now occupy the rest of this section. The various constructions involved are illustrated by Example (4.12) below, to which the reader might refer while working through the proof. Suppose $p=0$. Then $\mathscr{G}_k=\{z^k\}$ and $\mathscr{B}'_{1,k}=\{1\}$. Further, $\mathscr{B}_{N,k}=\varnothing$ if $0<k<N$ and $\mathscr{B}_{N,k}=\{\gamma_k\}$ otherwise. So ζ is either the 0×1 empty matrix or the 1×1 identity matrix, and (4.1) holds trivially in either case. Similarly, (4.1) is trivial in case $k=0$. Assume therefore that $p\geqslant 1$ and $k\geqslant 1$. The integers p, k and $N\geqslant 1$ are fixed in what follows. Let $d=\lfloor (k/N)\rfloor$. In the notation of [4, (3.5)] let $\mathscr{A}'=\{(\alpha_1,\ldots,\alpha_d)\in\mathbb{Z}^d \mid 0\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_{d-1}=\alpha_d\leqslant p\}$. For $i\geqslant 1$ let a_i be the number of times i occurs in $(\alpha_1,\ldots,\alpha_d)$. Then the correspondence $(\alpha_1,\ldots,\alpha_d)\leftrightarrow(a_1,\ldots,a_p)$ identifies $\mathscr{A}'$ with the following subset U of $(\mathbb{Z}^+)^p$:

$$U=\{(0,\ldots,0)\}\cup\left\{(a_1,\ldots,a_p)\;\middle|\;\sum_{i=1}^{p}a_i\leqslant d,\ \exists\, j \text{ with } a_j\geqslant 2 \text{ and } a_i=0\ \forall\, i>j\right\}.$$

For $a=(a_1,\ldots,a_p)\in(\mathbb{Z}^+)^p$ define the *weight* of a to be $wt(a)=\sum_{i=1}^{p}(N+i-1)a_i$. Let $V=\{a\in U \mid wt(a)\leqslant k\}$. Define $V_0=W_0=\{(0,\ldots,0)\}$ and for $1\leqslant j\leqslant p$ define $V_j=\{(a_1,\ldots,a_p)\in V \mid a_j\geqslant 2 \text{ and } a_i=0\ \forall\, i>j\}$ and $W_j=\{(a_1,\ldots,a_{j-1},a_j-1,0,\ldots,0) \mid (a_1,\ldots,a_{j-1},a_j,0,\ldots,0)\in V_j\}$. Put $W=\amalg_{j=0}^{p}W_j$.

We use the reverse lexicographic order on U. Namely, $(a_1,\ldots,a_p)<(b_1,\ldots,b_p)$ (or $(a_1,\ldots,a_p)$ "precedes" $(b_1,\ldots,b_p)$) if the last nonzero entry of $(a_1,\ldots,a_p)-(b_1,\ldots,b_p)$ is negative. Let V and W have the induced order. This order is such that the elements of V_{j-1} (resp. W_{j-1}) precede those of V_j (resp. W_j).

Let $S=\mathbb{Q}[x_1,\ldots,x_p,T]$ and let $F(T)=\sum_{i=0}^{p}\binom{T}{i}x_i$ (where $x_0=1$). If $(a_1,\ldots,a_p)$ is the ith element of W define

$$F_i(T)=F(N)^{a_1}F(N+1)^{a_2}\cdots F(N+p-1)^{a_p}$$
$$\times F(T-a_1N-a_2(N+1)-\cdots-a_p(N+p-1)).$$

Note that $F(n)=(\gamma_n)_{z=1}$, and that $F_i(k)$ is the ith element of $\mathscr{B}_{N,k}$ (with z set equal to 1). The reason for decreasing the last index in defining the elements of W is to take

into account the adjustment of the last index to obtain weight k when defining the elements of $\mathscr{B}_{N,k}$. If $(b_1,\dots,b_p)$ is the jth element of V then $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ is the jth element of $\mathscr{B}'_{N,k}$ (where the latter has the same order as before). Let $r=\mathrm{card}(V)=\mathrm{card}(W)$. Let $M(T)$ be the $r\times r$ matrix $(M_{ij}(T))_{1\leqslant i,j\leqslant r}$ with $M_{ij}(T)\in\mathbb{Q}[T]$ the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $F_i(T)$, where $(b_1,\dots,b_p)$ is the jth element of V. (Note that the rows of M are indexed by W and that the columns are indexed by V.) By the discussion preceding Theorem (4.1), $M_{ij}(k)$ is the coefficient of $z^kG'_t(k)$ (t corresponding to the jth element of $\mathscr{B}'_{N,k}$) in the expansion of the ith element of $\mathscr{B}_{N,k}$. Therefore $\eta=M(k)$, so (4.1) is equivalent to $M(k)$ being invertible. If $p=1$ then $M(k)$ is lower triangular with nonzero entries down the diagonal, hence trivially invertible. The argument that follows is needed only for $p\geqslant 2$.

Note that if j corresponds to an element of V_h then

$$\deg_T M_{ij}(T)\leqslant h. \tag{4.3}$$

Therefore

$$\deg_T \det(M(T))\leqslant \delta := \sum_{h=1}^{p} h\cdot\mathrm{card}(V_h)=\sum_{h=1}^{p} h\cdot\mathrm{card}(W_h). \tag{4.4}$$

Our intention is to show that $M(k)$ is invertible by finding δ roots for $\det(M(T))$, each less than k, and then showing that the coefficient of T^δ in $\det(M(T))$ is not identically zero. The δ roots will be found by obtaining coincidences of the rows of the matrix $M(s)$, as s ranges between N and $k-1$.

We begin by proving a few lemmas.

Lemma 4.5. *Let $\tilde{M}(T)$ be an $r\times r$ matrix with entries in $\mathbb{Q}[T]$. Let $\mu\in\mathbb{Q}$. Let $\mathscr{R}$ be the set of rows of $\tilde{M}(T)$ and let $\mathscr{S}$ be the set of all nonempty subsets of $\mathscr{R}$. Suppose there exists a subset $\mathscr{E}$ of $\mathscr{S}$ such that*

(1) *The sets in $\mathscr{E}$ are disjoint.*

(2) *For each $E\in\mathscr{E}$, all the rows in E coincide when T is specialized to μ.*

Let $c=c(\mathscr{E})=\sum_{E\in\mathscr{E}}(\mathrm{card}(E)-1)$. Then $(T-\mu)^c$ divides $\det(\tilde{M}(T))$.

Proof. It is clear that $\mathrm{rank}(\tilde{M}(\mu))\leqslant r-c$. By elementary row and column operations over $\mathbb{Q}[T]$ the matrix $\tilde{M}(T)$ can be reduced to a diagonal matrix $\tilde{D}(T)$ with diagonal entries $\{f_1(T),\dots,f_r(T)\}$. (This is well known, and easily proved using that $\mathbb{Q}[T]$ is an Euclidean domain.) Then (since the same operations can be carried out with T set equal to μ) we have $\mathrm{rank}(\tilde{D}(\mu))=\mathrm{rank}(\tilde{M}(\mu))\leqslant r-c$. Thus $(T-\mu)$ divides at least c of the f_i. Since (up to a nonzero scalar) $\det(\tilde{M}(T))=\det(\tilde{D}(T))=\Pi_{i=1}^r f_i$, the lemma follows. ∎

Before stating the next lemma we introduce some notation. For $a=(a_1,\dots,a_p)\in(\mathbb{Z}^+)^p$ put $\gamma^a=\gamma_N^{a_1}\gamma_{N+1}^{a_2}\cdots\gamma_{N+p-1}^{a_p}$. Then $\mathscr{B}_{N,k}=\{\gamma^a\gamma_{k-wt(a)}\mid a\in W\}$. Since the rows of $M(T)$ correspond to $\mathscr{B}_{N,k}$, those of $M(s)$ correspond to $\mathscr{B}_{N,k}(s):=\{\gamma^a\gamma_{s-wt(a)}\mid a\in W\}$. Here the elements $\gamma^a\gamma_{s-wt(a)}$ are treated as *symbolic monomials* with $s-wt(a)$ allowed to be negative. Given symbolic monomials $\gamma^a\gamma_t$, $\gamma^b\gamma_u$ with $a,b\in(\mathbb{Z}^+)^p$ and $t,u\in\mathbb{Z}$, we say they are *formally equal* if at least one of the following two conditions holds: (1) $(a,t)=(b,u)$; (2) both t and u belong to the set $\{0\}\cup[N,N+p-1]$ and $\gamma^a\gamma_t$ and $\gamma^b\gamma_u$ coincide as formal monomials in $\gamma_N,\dots,\gamma_{N+p-1}$ on replacing γ_0 by 1. We say that a row R of $M(s)$ is *labeled* by a symbolic monomial $\gamma^a\gamma_t$ if the symbolic monomial in

$\mathscr{B}_{N,k}(s)$ corresponding to R formally equals $\gamma^a\gamma_t$. Clearly two rows of $M(s)$ labeled by the same symbolic monomial are equal.

Let $Q_j=\{(b_1,\ldots,b_j,0,\ldots,0)\in(\mathbb{Z}^+)^p \mid b_j\neq 0\}$. For $b\in Q_j$ put $E(b)=W\cap\{b-e_i \mid 0\leqslant i\leqslant j\}$, where $e_0=(0,\ldots,0)$ and for $1\leqslant i\leqslant p$, $e_i=(0,\ldots,1,\ldots,0)$ is the standard basis vector with 1 in the ith place.

Lemma 4.6. *Let $b\in Q_j$. Then the rows of $M(wt(b))$ which are labeled by $\gamma^b(=\gamma^b\gamma_0)$ are precisely those indexed by $E(b)$. Moreover, if $b, c\in Q_j$ with $b\neq c$ and $wt(b)=wt(c)$ then $E(b)\cap E(c)=\varnothing$.*

Proof. It is clear that the rows of $M(wt(b))$ indexed by $E(b)$ are labeled by γ^b. Let R be a row of $M(wt(b))$ which is labeled by γ^b. Let a be the element of W corresponding to R. Then the symbolic monomial of $\mathscr{B}_{N,k}(wt(b))$ corresponding to R is $\gamma^a\gamma_{wt(b)-wt(a)}$. Comparing the subscripts and exponents of this symbolic monomial with those of γ^b we conclude that $a\in E(b)$. This proves the first part. Now, let $b, c\in Q_j$ with $wt(b)=wt(c)=s$, say. Suppose $E(b)$ and $E(c)$ have a common element, say a. Let R be the row of $M(s)$ indexed by a. Then R is labeled by γ^b as well as by γ^c whence we get $b=c$. ∎

Lemma 4.7. *For an element b of Q_j the following three conditions are equivalent:*
(1) *$card(E(b))\geqslant 2$;* (2) *$b-e_i\in W$ for some i, $0\leqslant i<j$;* (3) *$b-e_j\in W$ and $b-e_i\in W$ for some i, $0\leqslant i<j$.*
Moreover, if any of these conditions holds then $wt(b)<k$.

Proof. Assume (2). Then $b-e_j\in Q_h$ for some h, $i\leqslant h\leqslant j$. Since $b-e_i\in W_j$ we have $wt(b-e_j)<wt(b-e_i)\leqslant k-(N+j-1)\leqslant k-(N+h-1)$ whence $b-e_j\in W_h$. This proves (2)⇒(3). Also, the inequality $wt(b-e_j)<k-(N+j-1)$ gives $wt(b)<k$. The implications (1)⇒(2) and (3)⇒(1) are trivial. ∎

Put $Q=\{b\in\amalg_{j=1}^{p}Q_j \mid \text{card}(E(b))\geqslant 2\}$.

Lemma 4.8. *The product $\amalg_{b\in Q}(T-wt(b))^{\text{card}(E(b))-1}$ divides* $\det(M(T))$.

Proof. Writing $Q(s)=\{b\in Q \mid wt(b)=s\}$, it is enough to prove that $\amalg_{b\in Q(s)}(T-wt(b))^{\text{card}(E(b))-1}$ divides $\det(M(T))$ for every s. But this is immediate from (4.6) and (4.5), since rows labeled by the same symbolic monomial are equal. ∎

Lemma 4.9. $\sum_{b\in Q}(\text{card}(E(b))-1)=\delta$.

Proof. For $b\in Q\cap Q_j$ put $E'(b)=\{(b, b-e_i) \mid 0\leqslant i<j, b-e_i\in W\}$. It follows from (4.7) that $\text{card}(E'(b))=\text{card}(E(b))-1$. Let $\mathscr{E}=\amalg_{b\in Q}E'(b)$. The second projection induces a map $\eta\colon\mathscr{E}\to W$. Let $a\in W_j$ and let i be an integer with $0\leqslant i<j$. Then $a+e_i\in Q$ by (4.7). It follows that $\eta^{-1}(a)=\{(a+e_i, a) \mid 0\leqslant i<j\}$. Thus there are exactly j elements in the fibre of η over each element of W_j. Therefore we get $\sum_{b\in Q}(\text{card}(E(b))-1)=\sum_{b\in Q}\text{card}(E'(b))=\text{card}(\mathscr{E})=\sum_{j=1}^{p}j\cdot\text{card}(W_j)=\delta$. ∎

Now, since $\deg_T\det(M(T))\leqslant\delta$ by (4.4), and since (4.7)–(4.9) taken together exhibit δ roots of $\det(M(T))$ each less than k, it remains only to show that $\det(M(T))$ is not

identically zero. We do this by showing that the coefficient of T^{δ} is not zero. Let σ_{ij} be the coefficient in $M_{ij}(T)$ of T^h if j corresponds to an index in V_h (by (4.3) h is highest power of T with a potentially nonzero coefficient in $M_{ij}(T)$). It then suffices to show that $\det((\sigma_{ij}))\neq 0$. For $1\leqslant i,\ j\leqslant r$ (where as before M is $r\times r$) let $H_i(T)=F(T)^{a_1}F(T+1)^{a_2}\cdots F(T+p-1)^{a_p}$, where $(a_1,\ldots,a_p)$ is the ith element of W, and let τ_{ij} be the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $H_i(N)$, where $(b_1,\ldots,b_p)$ is the jth element of W. Let h be the index for which $(b_1,\ldots,b_p)\in W_h$. Then, since $(b_1,\ldots,1+b_h,0,\ldots,0)$ is the corresponding element of V_h and since $F_i(T)=H_i(N)F(T-c)$ for some integer c, we get $\sigma_{ij}=(1/h!)\tau_{ij}$. So it suffices to prove that $\det(\tau)\neq 0$ (where $\tau=(\tau_{ij})$). Rearrange the rows and columns of τ by reordering W by degree (where degree $(a_1,\ldots,a_p)=\sum a_i$). Then τ is lower-block triangular with degree blocks down the diagonal. It suffices to show that each of these blocks has a nonzero determinant. Therefore for u $(0\leqslant u\leqslant d-1)$ let S_u be the submatrix of τ with rows and columns indexed by elements of W and V of degree u. It suffices to show that $\det(S_u)\neq 0$.

The matrix S_u is obtained as follows: Let $W(u)$ be the elements of W of degree u, and let $r(u)=\operatorname{card}(W(u))$. For $1\leqslant i,\ j\leqslant r(u)$ let $(a_1,\ldots,a_p)$ be the ith element of $W(u)$ and let $(b_1,\ldots,b_p)$ be the jth element of $W(u)$. Define an $r_u\times r_u$ matrix $L_u(T)$ by setting the (i,j) entry to be the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $H_i(T)$. Then $S_u=L_u(N)$. Thus it suffices to show that N is not a root of $\det(L_u(T))$. Since we are now dealing with the homogeneous case we can replace $F(T)$ by $\tilde{F}(T)=\sum_{i=1}^{p}\binom{T}{i}x_i$ and $H_i(T)$ by $\tilde{H}_i(T)=\tilde{F}(T)^{a_1}\tilde{F}(T+1)^{a_2}\cdots\tilde{F}(T+p-1)^{a_p}$ without changing $L_u(T)$. We now note that $\tilde{H}_i(T)$ is divisible by $T^{a_1}(T+1)^{a_2}\cdots(T+p-1)^{a_p}$, or equivalently, the ith row of L_u is divisible by $T^{a_1}(T+1)^{a_2}\cdots(T+p-1)^{a_p}$. Factoring out these entries from the rows of $L_u(T)$ we obtain a matrix $K_u(T)$ which can be defined directly as follows: let $G(T)=\sum_{i=1}^{p}(1/i)\binom{T-1}{i-1}x_i$ (so that $TG(T)=\tilde{F}(T)$) and define $\tilde{L}_i(T)=G(T)^{a_1}G(T+1)^{a_2}\cdots G(T+p-1)^{a_p}$. Then the (i,j) entry of $K_u(T)$ is the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $\tilde{L}_i(T)$. Noting that the roots of the factors $(T+i)^{a_i}$ are all $\leqslant 0$, it suffices to prove that N is not a root of $\det(K_u(T))$. In fact, $\det(K_u(T))$ is a nonzero constant, as we show next.

For $a=(a_1,\ldots,a_j,0,\ldots,0)\in Q_j$ define $aw(a)$, the *augmented weight* of a, to be $N+j-1+\sum_{i=1}^{j}(N+i-1)a_i$. Also define $aw(0)=0$. If $a\in W$ then $aw(a)$ is the weight of the corresponding element of V, and $aw(a)\leqslant k$ for all $a\in W$. Now, order the elements of $W(u)$ by augmented weight with small weights coming first, and order elements of the same weight by reverse lexicographic order as was done previously. This ordering is such that

(4.10) if for $j<i$ we decrease a_i by one and increase a_j by one then we get an earlier element in the ordering.

Furthermore $W(u)$ is a leading segment in the set $\tilde{W}(u)$ of all elements of degree u in $(\mathbb{Z}^+)^p$ (where $\tilde{W}(u)$ is ordered in the same manner). The matrix $K_u(T)$ can be constructed with $W(u)$ ordered in this way without changing the value of $\det(K_u(T))$.

Now we shall work with $\tilde{W}(u)$. Let $\tilde{r}(u)=\operatorname{card}(\tilde{W}(u))$ and let $\tilde{K}_u(T)$ be the $\tilde{r}_u\times\tilde{r}_u$ matrix whose (i,j) entry is the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $\tilde{L}_i(T):=G(T)^{a_1}G(T+1)^{a_2}\cdots G(T+p-1)^{a_p}$ where $(a_1,\ldots,a_p)$ and $(b_1,\ldots,b_p)$ are respectively the ith and the jth

elements of $\tilde{W}(u)$ (for convenience of notation we are changing the meaning of $\tilde{L}_i$ rather than introducing a new symbol). Let $u=1$. If we take out the factors $1/i$ from the columns then $\tilde{K}_1(T)$ is reduced to the matrix $J=\left[\binom{T+i-2}{j-1}\right]_{1\leqslant i,j\leqslant p}$. If we subtract each row of J from the next (performing the operations in the order replace pth row by pth $-(p-1)$st, replace $(p-1)$st by $(p-1)$st $-(p-2)$nd etc.) and use the binomial identities $\binom{T+i-1}{j-1}-\binom{T+i-2}{j-1}=\binom{T+i-2}{j-2}$ then J row reduces to $\begin{pmatrix}1 & * \\ 0 & J'\end{pmatrix}$ where $J'=\left[\binom{T+i-2}{j-1}\right]_{1\leqslant i,j\leqslant p-1}$. (Performing row operations in this manner was suggested to us by Sue Geller.) Continued row reduction of this type (subtracting from a row $\mathbb{Q}$-linear combinations of previous rows) will reduce $\tilde{K}_1(T)$ to an upper triangular matrix with ones down the diagonal. We conclude that $\det(J)=1$ whence $\det(\tilde{K}_1(T))=1/p!$, a nonzero constant. Now, let $E=(E_{ij})$ by any $p\times p$ matrix with entries in $\mathbb{Q}[T]$. If R_i is the ith row of E let us identify R_i with the element $E_{i1}x_1+E_{i2}x_2+\cdots+E_{ip}x_p$ of $\mathbb{Q}[T,x_1,\ldots,x_p]$. Let $f_u(E)$ be the $\tilde{r}_u\times\tilde{r}_u$ matrix whose (i,j) entry is the coefficient of $x_1^{b_1}x_2^{b_2}\cdots x_p^{b_p}$ in $R_1^{a_1}R_2^{a_2}\cdots R_p^{a_p}$, where as before $(a_1,\ldots,a_p)$ and $(b_1,\ldots,b_p)$ are respectively the ith and the jth elements of $\tilde{W}(u)$. This construction is such that $f_u(\tilde{K}_1(T))=\tilde{K}_u(T)$. Furthermore if we change E into a matrix E' by row operations of the above type (i.e. subtracting from a row $\mathbb{Q}$-linear combinations of previous rows) then because of (4.10) $f_u(E)$ is changed into $f_u(E')$ by row operations of the same type. We have that f_u of an upper triangular matrix is upper triangular, so $\tilde{K}_u(T)$ can be converted to an upper triangular matrix with nonzero constant entries down the diagonal by a succession of row operations in which from a given row we subtract a $\mathbb{Q}$-linear combination of previous rows. These row operations leave invariant the subspaces spanned by the first i rows $(1\leqslant i\leqslant\tilde{r}_u)$. Since $W(u)$ is an initial segment of $\tilde{W}(u)$ we conclude that $\det K_u(T)$ is a nonzero constant, completing the proof of (4.1). ■

Example 4.11. If $N=3$, $p=4$, then in reverse lexicographic order we have $(1,0,2,0)<(0,1,2,0)<(0,0,3,0)<(2,0,0,1)$ with augmented weights respectively 18, 19, 20, 18. Therefore if we order reverse lexicographically instead of by augmented weights the argument above will fail for $k=18$ since then $W(3)$ will not be an initial segment of $\tilde{W}(3)$.

Example 4.12. Let us return to (4.2), where $N=p=2$, $k=10$. Here we have $V_0=\{(0,0)\}$, $V_1=\{(2,0),(3,0),(4,0),(5,0)\}$, $V_2=\{(0,2),(1,2),(2,2),(0,3)\}$, $W_0=\{(0,0)\}$, $W_1=\{(1,0),(2,0),(3,0),(4,0)\}$ and $W_2=\{(0,1),(1,1),(2,1),(0,2)\}$. The rows of $M(10)$ are indexed by the monomials $\mathscr{B}_{2,10}=\{\gamma_{10},\gamma_2\gamma_8,\gamma_2^2\gamma_6,\gamma_2^3\gamma_4,\gamma_2^5,\gamma_3\gamma_7,\gamma_2\gamma_3\gamma_5,\gamma_2^2\gamma_3^2,\gamma_3^2\gamma_4\}$ as noted in (4.2). Thus the rows of $M(s)$ are indexed by $\mathscr{B}_{2,10}(s)=\{\gamma_s,\gamma_2\gamma_{s-2},\gamma_2^2\gamma_{s-4},\gamma_2^3\gamma_{s-6},\gamma_2^4\gamma_{s-8},\gamma_3\gamma_{s-3},\gamma_2\gamma_3\gamma_{s-5},\gamma_2^2\gamma_3\gamma_{s-7},\gamma_3^2\gamma_{s-6}\}$. The polynomial $\det(M(T))$ is of degree $\operatorname{card}(V_1)+2\operatorname{card}(V_2)=4+2\cdot4=12$, and we have $Q=\{(1,0),(2,0),(3,0),(4,0),(0,1),(1,1),(2,1),(3,1),(0,2),(1,2)\}$. Taking $b=(1,0)$ we get $E(b)=\{(0,0),(1,0)\}$. This corresponds to the pair $\gamma_s,\gamma_2\gamma_{s-2}$, indexing the first two rows, which become equal when we set $s=2$. The complete set of row coincidences is obtained similarly and is given by the following table:

b	$E(b)$	elts of $\mathscr{B}_{2,10}(s)$	rows	roots of $\det(M(T))$
$(1,0)$	$(0,0), (1,0)$	$\gamma_s, \gamma_2\gamma_{s-2}$	1,2	2
$(2,0)$	$(1,0), (2,0)$	$\gamma_2\gamma_{s-2}, \gamma_2^2\gamma_{s-4}$	2,3	4
$(3,0)$	$(2,0), (3,0)$	$\gamma_2^2\gamma_{s-4}, \gamma_2^3\gamma_{s-6}$	3,4	6
$(4,0)$	$(3,0), (4,0)$	$\gamma_2^3\gamma_{s-6}, \gamma_2^4\gamma_{s-8}$	4,5	8
$(0,1)$	$(0,0), (0,1)$	$\gamma_s, \gamma_3\gamma_{s-3}$	1,6	3
$(1,1)$	$(1,0), (0,1), (1,1)$	$\gamma_2\gamma_{s-2}, \gamma_3\gamma_{s-3}, \gamma_2\gamma_3\gamma_{s-5}$	2,6,7	5,5
$(2,1)$	$(2,0), (1,1), (2,1)$	$\gamma_2^2\gamma_{s-4}, \gamma_2\gamma_3\gamma_{s-5}, \gamma_2^2\gamma_3\gamma_{s-7}$	3,7,8	7,7
$(3,1)$	$(3,0), (2,1)$	$\gamma_2^3\gamma_{s-6}, \gamma_2^2\gamma_3\gamma_{s-7}$	4,8	9
$(0,2)$	$(0,1), (0,2)$	$\gamma_3\gamma_{s-3}, \gamma_3^2\gamma_{s-6}$	6,9	6
$(1,2)$	$(1,1), (0,2)$	$\gamma_2\gamma_3\gamma_{s-5}, \gamma_3^2\gamma_{s-6}$	7,9	8

By direct computation $\det(M(T))$ turns out to be

$$2^4 3^5 (T-9)(T-8)^2(T-7)^2(T-6)^2(T-5)^2(T-4)(T-3)(T-2),$$

which is in agreement with the roots (together with multiplicities) obtained from the above table. We have that $\det(M(T))$ does not vanish at $T=10$, as claimed.

Now we shall illustrate some features of the last part of the proof. Here $G(T) = x_1 + ((T-1)/2)x_2$, and $G(T+1) = x_1 + (T/2)x_2$, so $K_1(T) = \tilde{K}_1(T) = \begin{pmatrix} 1 & (T-1)/2 \\ 1 & (T/2) \end{pmatrix}$. We have $W(3) = \{(3,0), (2,1)\}$ and $\tilde{W}(3) = \{(3,0), (2,1), (1,2), (0,3)\}$. The respective augmented weights of the elements of $\tilde{W}(3)$ are 8 $(=4\cdot 2)$, 10 $(=2\cdot 2 + 2\cdot 3)$, 11 $(=1\cdot 2 + 3\cdot 3)$ and 12 $(=4\cdot 3)$. The last two have weights greater than 10 and so are not included in $W(3)$. If $p=2$ the reverse lexicographic ordering is also an ordering by weight, but this need not be the case for larger p, as we saw in (4.11). Set $R_1 = G(T)$ and $R_2 = G(T+1)$. Then the matrix $\tilde{K}_3$ has rows $\{R_1^3, R_1^2R_2, R_1R_2^2, R_2^3\}$, (or more precisely the 4×4 matrix obtained by taking the coefficients of $\{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$ in these polynomials). The rows of $\tilde{K}_3$ will be denoted as $\{r_1, r_2, r_3, r_4\}$. The row operation that reduces $\tilde{K}_1(T)$ to upper triangular form is to replace $\{R_1, R_2\}$ by $\{R_1, R_2 - R_1\}$. Then $f_3(\{R_1, R_2 - R_1\})$ has row corresponding to $\{R_1^3, R_1^2(R_2-R_1), R_1(R_2-R_1)^2, (R_2-R_1)^3\} = \{R_1^3, R_1^2R_2 - R_1^3, R_1R_2^2 - 2R_1^2R_2 + R_1^3, R_2^3 - 3R_1R_2^2 + 3R_1^2R_2 - R_1^3\}$ so the row operation to reduce $\tilde{K}_3$ to upper triangular form (with nonzero diagonal entries) replaces $\{r_1, r_2, r_3, r_4\}$ by $(r_1, r_2 - r_1, r_3 - 2r_2 + r_1, r_4 - 3r_3 + 3r_2 + r_1\}$. The matrix K_3 is the upper left 2×2 submatrix of $\tilde{K}_3$, to which these row operations restrict, so $\det \tilde{K}_3$ is also a nonzero constant. If we had used weight 11 rather than 10, then $\tilde{K}_3$ would have been the upper left 3×3 block of K_3, which also has determinant a nonzero constant, for the same reason.

5. Hilberty polynomials

The graded ring $R^{(N)}$ has Hilbert function H given by $H(n) = \dim_{\mathbb{Q}} R_n^{(N)}$. We consider the problem of expressing $H(n)$ as one or more polynomials in n. The Hilbert function

of a graded ring which is standard (i.e. finitely generated over a field by elements of weight 1) is given for $n \gg 0$ by its Hilbert polynomial. Our ring $R^{(N)}$ is finitely generated but is not standard except in the trivial case $p=0$, $N=1$. For such a ring there exist, by [5, Corollary 2], a positive integer d and polynomials $H_0, H_1, \ldots, H_{d-1}$ such that

$$H(n) = H_i(n) \quad \text{if} \quad n \gg 0 \text{ and } n \equiv i \pmod{d}. \tag{*}$$

In general, it is of interest to quantify precisely the condition "$n \gg 0$". In particular, in the standard case, if the Hilbert function coincides with the Hilbert polynomial for all $n \geqslant 0$ then the ring is called a *Hilbertian* ring. So we may call a general finitely generated graded ring Hilbertian if (*) holds for all $n \geqslant 0$. In our first result (5.1) we show that $R^{(N)}$ is Hilbertian if $p \geqslant 2$, and determine the minimal d satisfying (*).

If $p=0$ then $H(n)=1$ if $n=0$ or $n \geqslant N$, so in this case (*) holds with $d=1$, $H_0=1$, and $R^{(N)}$ is Hilbertian if and only if $N=1$.

Now, in general, to say that $R^{(N)}$ is Hilbertian is the same as saying that its Hilbert function H is a *quasi-polynomial* in the language of [6, (4.4)]. The integer d appearing in (*) is then a *quasi-period* of H.

Theorem 5.1. *Let* $d = \operatorname{lcm}(N, N+1, \ldots, N+p-1)$. *If* $p \geqslant 2$ *then* H *is a quasi-polynomial with minimum quasi-period* d, *and in particular* $R^{(N)}$ *is Hilbertian. If* $p=1$ *then the function* $\tilde{H}$ *given by* $\tilde{H}(n) = H(n)$ *for* $n \geqslant 1$ *and* $\tilde{H}(0) = H(0) - 1 = 0$ *is a quasi-polynomial with minimum quasi-period* d.

Proof. Let $P(t) = \sum_{n=0}^{\infty} H(n)t^n$ and $\tilde{P}(t) = \sum_{n=0}^{\infty} \tilde{H}(n)t^n$, where we put $\tilde{H} = H$ if $p \geqslant 2$. Then by (2.8) we have

$$\tilde{P}(t) = P(t) = \frac{1 - t + t^{N+p}}{(1-t)(1-t^N)(1-t^{N+1})\cdots(1-t^{N+p-1})} \quad \text{if } p \geqslant 2,$$

and $\tilde{P}(t) = P(t) - 1 = t^N/((1-t)(1-t^N))$ if $p=1$. In either case write $\tilde{P}(t) = f(t)/g(t)$ with $f(t)$, $g(t)$ polynomials without a common factor. Then $\deg f(t) < \deg g(t)$ and the zeros of $g(t)$ are the dth roots of 1. So by [6, (4.4.1)] $\tilde{H}$ is a quasi-polynomial with quasi-period d.

To prove the minimality of d, we claim first that d is the lcm of the orders of the roots of unity which occur as zeros of $g(t)$. This is clear if $p=1$. Hence assume that $p \geqslant 2$. If λ is a root of unity as well as a zero of $1-t+t^{N+p}$ then 1, $-\lambda$ and λ^{N+p} are three roots of unity whose sum is zero. This is the case if and only if $\{1, -\lambda, \lambda^{N+p}\}$ are the three cube roots of unity. Thus $-\lambda$ is a primitive cube root of unity, so λ is a primitive sixth root of unity and $\lambda^{N+p} = (-\lambda)^2$ is the other primitive cube root of unity, whence $N+p \equiv 2 \pmod 6$. Obviously $1-t+t^{N+p}$ has no repeated factors, so if $N+p \equiv 2 \pmod 6$ then we can cancel the cyclotomic polynomial $1-t+t^2$ of primitive sixth roots of unity once, otherwise there is no cancellation. The cancellation still leaves us with roots of unity of order 2 and 3 as zeros of $g(t)$, proving our claim.

Now let D be the minimum quasi-period. Then we can write $\tilde{P}(t)$ in the form $\tilde{P}(t) = \sum_{j=0}^{D-1} \sum_{i \geqslant 0} H_j(j+Di)t^{j+Di}$ for some polynomials H_j. Multiplying by $1-t^D$ amounts to differencing the coefficients (except in low degrees) so $(1-t^D)^e \tilde{P}(t)$ is a polynomial in t for some positive integer e. Therefore the roots of unity that occur as zeros of $g(t)$ must have orders which divide D. Thus d divides D, proving the minimality of d. ∎

Theorem 5.2. *The polynomials H_i in* (5.1) *are all of degree p.*

Proof. This is seen by examining the partial fraction expansion of $\tilde{P}(t)$. We have that 1 is a root of the denominator of $\tilde{P}(t)$ of multiplicity $p+1$, and that all other roots are of smaller multiplicity. Setting $X=\lambda T$ in the well-known expansion $\frac{1}{(1-X)^r}=\sum_{n\geqslant 0}\binom{n+r-1}{r-1}X^n$ (in which the coefficient of X^n is a polynomial in n of degree $r-1$), we see that a root λ of multiplicity m of the denominator contributes a polynomial of degree $m-1$ to each of the H_j. Thus 1 contributes degree p to each H_j and the other roots contribute a lower degree, so the highest degree terms cannot cancel leaving all the H_j of degree p. ∎

Now, we give an example to show that the various H_j need not be distinct. Consider the case $N=2$, $p=3$, where our Poincaré series

$$\frac{1-t+t^5}{(1-t)(1-t^2)(1-t^3)(1-t^4)}$$

has partial fraction expansion

$$\frac{a}{(1-t)^4}+\frac{b}{(1+t)^2}+\frac{1/8}{1+t^2}+\frac{2/9}{1+t+t^2}$$

with a of degree 3 and b of degree 1 which need not be stated explicitly. The power series expansions of $1/(1+t^2)$ and $1/(1+t+t^2)$ are

$$1-t^2+t^4-t^6+t^8-\cdots$$
$$1-t+t^3-t^4+t^6-\cdots$$

of periods 4 and 3 respectively, with coefficients in each period being $1,0,-1,0$ and $1,-1,0$ respectively. The "non-polynomial" contribution to the various $H(i)$ are given by the following table (with rows corresponding to t^i for $i=0,1,2,\ldots$ and columns corresponding respectively to the roots of order 1, 2, 4, 3):

i				
0	1	1	1	1
1	1	−1	0	−1
2	1	1	−1	0
3	1	−1	0	1
4	1	1	1	−1
5	1	−1	0	0
6	1	1	−1	1
7	1	−1	0	−1
8	1	1	1	0
9	1	−1	0	1
10	1	1	−1	−1
11	1	−1	0	0
12	1	1	1	1

The polynomials coincide if and only if the rows are the same. By inspection of the table we see that the period is indeed 12, as given by (5.1), and that $H_1=H_7$, $H_3=H_9$,

and $H_5 = H_{11}$, with the polynomials H_j $(0 \leqslant j \leqslant 11)$ being otherwise distinct. Th equality of the H_j's here comes from the 0's in the power series expansion of th cyclotomic polynomial of primitive fourth roots of unity. Note that the possibiliti are determined only by the columns corresponding to roots of order 4 and 3. Obvious the first column plays no role in deciding on the cases, and the second does not eith since whenever entries in columns three and four are equal, so are the entries column two.

By explicit computation we obtain $H_0(t) = 1 + \frac{t}{6} + \frac{t^2}{48} + \frac{t^3}{144}$, $H_1 = H_7 = -\frac{19}{144}$ $\frac{5t}{48} + \frac{t^2}{48} + \frac{t^3}{144}$, etc. with the polynomials all of degree 3 as claimed by our theorem, ar with polynomials equal and distinct as claimed above. The coefficients of t^2 and are the same in all polynomials, which can be explained by the fact that only th root 1 has multiplicity greater than two, and the coefficient of t is periodic with perio 2 since only the root -1 has multiplicity 2.

In another example that we have worked out, equality of the various H_j arose a seemingly accidental way from primitive roots of unity of order other than powe of two. The general situation seems to be quite complicated.

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Flat connections, geometric invariants and energy of harmonic functions on compact Riemann surfaces

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Abstract. A geometric invariant is associated to the space of flat connections on a G-bundle over a compact Riemann surface and is related to the energy of harmonic functions.

Keywords. Principal G-bundle; flat connections; Chern-Simons forms; energy of maps; harmonic maps.

Introduction

This work grew out of an attempt to generalize the construction of Chern–Simons invariants. In this paper, we associate a geometric invariant to the space of flat connection on a $SU(2)$-bundle on a compact Riemann surface and relate it to the energy of harmonic functions on the surface.

Our set up is as follows. Let $G = SU(2)$ and M be a compact Riemann surface and $E \to M$ be the trivial G-bundle. (Any $SU(2)$-bundle over M is topologically trivial). Let $\mathscr{C}$ be the space of all connections and $\mathscr{F}$ the subspace of all flat connections on this G-bundle. We endow on $\mathscr{C}$ the Frechet topology and the subspace topology on $\mathscr{F}$.

Given a loop $\sigma: S^1 \to \mathscr{F}$, we can extend σ to the closed unit disc $\tilde{\sigma}: D^2 \to \mathscr{C}$ since $\mathscr{C}$ is contractible. On the trivial G-bundle $E \times D^2 \to M \times D^2$ we define a *tautological* connection form ϑ^σ as follows

$$\vartheta^\sigma|_{(e,t)} = \tilde{\sigma}(t) \forall (e,t) \in E \times D^2.$$

Clearly restriction of ϑ^σ to the bundle $E \times \{t\} \to M \times \{t\}$ is $\tilde{\sigma}(t) \forall t \in D^2$. Let $K(\vartheta^\sigma)$ be the curvature form of ϑ^σ. Evaluation of the second Chern polynomial on this curvature form $K(\vartheta^\sigma)$ gives a closed 4-form on $M \times D^2$, which when integrated along D^2 yields a 2-form on M. This 2-form is closed since $\dim M = 2$ and thus defines an element in $H^2(M, R) \approx R$. It is seen that this class is independent of the extension of σ. We thus have a map

$$\chi: \Omega(\mathscr{F}) \to H^2(M, R) \approx R$$

where $\Omega(\mathscr{F})$ is the loop-space of $\mathscr{F}$.

We assume that the genus of $M \geqslant 2$. The energy $E(f)$ of any smooth function $f: M \to G$ is defined using the Poincare metric on M and the bi-invariant metric on $G = SU(2)$ given by the Killing form.

Any smooth function $f:M\to G$ defines a flat connection $\omega_f = f^*(\mu)$ on the trivial bundle $M\times G\to M$, where μ is the Maurer–Cartan form on G. By a result of Hitchin ([H]), the loop in $\mathscr{C}$ is given by

$$\sigma_f(t) = \frac{1}{2}(\omega_f + (\cos t)\omega_f + (\sin t)(*\omega_f)) \text{ for } t\in[0,2\pi],$$

where $*:\Lambda^1(M,\mathscr{G})\to\Lambda^1(M,\mathscr{G})$ is the Hodge star operator, is actually a loop in $\mathscr{F}$ if and only if f is harmonic. ($\mathscr{G}$ is the Lie-Algebra of G).

The main result of this paper is

Theorem *If $f:M\to G$ is a harmonic map, then*

$$\chi(\sigma_f) = -\frac{1}{4\pi}E(f).$$

1. Construction of the basic geometric invariant

In this paper we suppose M is a compact Riemann surface of genus $g\geqslant 2$, $G = SU(2)$ with Lie algebra $\mathscr{G} = su(2)$ and $\pi:E\to M$ is the trivial G-bundle on M. $\mathscr{C}$ is the space of connections and $\mathscr{F}$ is the subspace of all flat connections on $E\to M$. D^2 is the closed unit disc in R^2 and $\partial D^2 = S^1$ is the unit circle. $\Omega(\mathscr{F}) = \mathrm{Map}(S^1,\mathscr{F})$ is the loop-space of $\mathscr{F}$. Given a loop $\sigma:S^1\to\mathscr{F}$ we extend σ to $\tilde{\sigma}:D^2\to\mathscr{C}$ ($\mathscr{C}$ is contractible). On the trivial bundle $E\times D^2\to M\times D^2$, let ϑ^σ be the tautological connection defined in the introduction. Let $K(\vartheta^\sigma)$ be the curvature 2-form of the connection ϑ^σ. Let C_2 be the second Chern polynomial on $\mathscr{G}$. For the Lie algebra $\mathscr{G} = su(2)$, C_2 is essentially the determinant. More particularly $C_2(A) = -(1/4\pi^2)\det(A)$ for $A\in su(2)$ (cf. [KN], Chap. XII). Now an easy computation shows that

$$C_2(A) = \frac{1}{8\pi^2}\mathrm{trace}(A^2) \text{ for } A\in\mathscr{G}.$$

Evaluation of C_2 on $K(\vartheta^\sigma)$ gives a closed 4-form $\overline{C_2(K(\vartheta^\sigma))}$ on $E\times D^2$ which projects to a closed 4-form $C_2(K(\vartheta^\sigma))$ on $M\times D^2$. Integrating $C_2(K(\vartheta^\sigma))$ along D^2 yields a closed 2-form on M (dim $M = 2$) and thus defines a cohomology class in $H^2(M,R)$ i.e.

$$\left\{\int_{D^2} C_2(K(\vartheta^\sigma))\right\}\in H^2(M,R)\approx \mathrm{R}.$$

We outline the proof of the following lemma (cf. [G], § 1 and [GS], § 2, 3).

Lemma 1.1. *$\int_{D^2}C_2(K(\vartheta^\sigma))$ is independent of the extension of $\sigma:S^1\to\mathscr{C}$ to $\tilde{\sigma}:D^2:\to\mathscr{C}$.*

Proof. Let $\tilde{\sigma}_1,\tilde{\sigma}_2$ be two extensions of σ with corresponding connection forms ϑ_1^σ, ϑ_2^σ and curvature forms $K(\vartheta_1^\sigma), K(\vartheta_2^\sigma)$ on the bundle $E\times D^2\to M\times D^2$. On $E\times D^2$ we have

$$\mathrm{d}TC_2(\vartheta_1^\sigma) = \overline{C_2(K(\vartheta_1^\sigma))}$$

$$\mathrm{d}TC_2(\vartheta_1^\sigma) = \overline{C_2(K(\vartheta_2^\sigma))}$$

where $TC_2(\vartheta_1^\sigma)$, $TC_2(\vartheta_2^\sigma)$ are the Chern–Simons secondary forms with respect to $\vartheta_1^\sigma, \vartheta_2^\sigma$ respectively (cf. [CS, 3]). We can easily check that $\overline{C_2(K(\vartheta_1^\sigma))} - \overline{C_2(K(\vartheta_2^\sigma))}$ is an exact form on E (cf. [G, 1]). Since $\pi^*: H^2(M,R) \to H^2(E,R)$ is an isomorphism it follows that $\{C_2(K(\vartheta_1^\sigma))\} = \{C_2(K(\vartheta_1^\sigma))\} \in H^2(M,R)$ and this proves the lemma.

We thus have a map

$$\Omega(\mathscr{F}) \xrightarrow{\chi} H^2(M,R) \approx R$$

$$\sigma \mapsto \chi(\sigma) = \left\{ \int_{D^2} C_2(K(\vartheta^\sigma)) \right\}$$

where $\Omega(\mathscr{F})$ is the loop-space of $\mathscr{F}$. It is easy to check that $\chi(\sigma \circ \sigma') = \chi(\sigma) + \chi(\sigma')$ where $\sigma \circ \sigma'$ is the composite of two loops in $\mathscr{F}$. We call this map χ the geometric invariant.

2. Energy of functions and a class of special loops

We recall the definition of energy of a function. Let X and Y be Riemannian manifolds. Given a smooth map $f: X \to Y$, the energy density of f is a function $e(f): X \to R$ defined by

$$e(f)(x) = \| \mathrm{d}f(x) \|^2$$

where $\| \mathrm{d}f(x) \|$ denotes the Hilbert–Schmidt norm of the differential $\mathrm{d}f(x) \in T_x^*(x) \otimes T_{f(x)}(Y)$. If X is compact and oriented, the energy of f, denoted by $E(f)$ is given by

$$E(f) = \left(\int_M e(f)(x) \mathrm{d}x \right)^{1/2}$$

where $\mathrm{d}x$ is the volume form of X with respect to its Riemannian metric. f is *harmonic* if it is a critical point of the energy functional.

Using the Poincare metric on the compact Riemann surface of genus $\geqslant 2$ and the bi-invariant metric on $G = SU(2)$ given by the Killing form, we can define the energy $E(f)$ of a smooth function $f: M \to G$ by the above formula.

Any smooth function $f: M \to G$ defines a flat connection $\omega_f = f^*(\mu)$ on the trivial bundle $E \to M$ where

$$\mu = \begin{pmatrix} i\mu_1 & \mu_2 + i\mu_3 \\ -\mu_2 + i\mu_3 & -i\mu_1 \end{pmatrix}$$

is the Maurer–Cartan form on G. In the case of the trivial bundle $E \to M$, clearly the space of all connections $\mathscr{C}$ can be identified with the space $\Lambda^1(M, \mathscr{G})$ of all $\mathscr{G}$-valued 1-forms on M. For any smooth function $f: M \to G$, consider the loop in $\mathscr{C}$ given by $\sigma_f(t) = \frac{1}{2}(\omega_f + (\cos t)\omega_f + (\sin t)(*\omega_f))$ for $t \in [0, 2\pi]$, where $*: \Lambda^1(M, \mathscr{G}) \to \Lambda^1(M, \mathscr{G})$ is the Hodge star operator. By a result of Hitchin ([H]), we know that $\sigma_f([0, 2\pi]) \subset \mathscr{F}$ iff f is harmonic, i.e. σ_f is a loop in $\mathscr{F}$ iff f is harmonic.

3. Relation between the geometric invariant and the energy of harmonic maps

We prove the following result

Theorem 3.1. *If $f:M\to G$ is a harmonic map, then* $\chi(\sigma_f)=-\dfrac{1}{4\pi}E(f)$.

Proof. At the outset we show that the closed 2-form which represents $\chi(\sigma_f)\in H^2(M,R)$ is $\dfrac{1}{2\pi}(*\omega_1\Lambda\omega_1+*\omega_2\Lambda\omega_2+*\omega_3\Lambda\omega_3)$ where

$$\omega_f=f^*\mu=\begin{pmatrix} i\omega_1 & \omega_2+i\omega_3 \\ -\omega_2+i\omega_3 & -i\omega_1 \end{pmatrix}.$$

We extend the loop σ_f in $\mathscr{F}$ to a map $\tilde{\sigma}_f:D^2\to\mathscr{C}$ in an obvious way. We drop the suffix f and simply use σ and $\tilde{\sigma}$ in the computations that follow.

Let (s,t) be the polar coordinates on $D^2=\{(s,t),0\leqslant s\leqslant 1,0\leqslant t\leqslant 2\pi\}$.
Set $\tilde{\sigma}(s,t)=s\sigma(t)$. We now compute the curvature $K(\vartheta^\sigma)$ of the connection form ϑ^σ on the bundle $E\times D^2\to M\times D^2$.

$$\begin{aligned} K(\vartheta^\sigma)&=d\vartheta^\sigma+\tfrac{1}{2}[\vartheta^\sigma,\vartheta^\sigma],\\ &=d\vartheta^\sigma+\vartheta^\sigma\wedge\vartheta^\sigma,\\ &=d_E\vartheta^\sigma+d_{D^2}\vartheta^\sigma+\vartheta^\sigma\wedge\vartheta^\sigma,\\ &=d_{D^2}^{\vartheta^\sigma}+K(\tilde{\sigma}(s,t)), \end{aligned}$$

where $K(\tilde{\sigma}(s,t)$ is the curvature of $\tilde{\sigma}(s,t))$ and d_E and d_{D^2} are respectively the exterior differentials on E and D^2.

If we set

$$\sigma(t)=\begin{pmatrix} i\alpha(t) & \beta(t)+i\gamma(t) \\ -\beta(t)+i\gamma(t) & -i\alpha(t) \end{pmatrix}$$

as a form on M for each $t\in S^1$, then after a straightforward calculation (see [G], Lemma 4.1), it follows that $\int_{D^2}C_2(K(\vartheta^\sigma))$ is cohomologous to the form

$$\frac{1}{4\pi^2}\int_{S^1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)\wedge\alpha(t)+\frac{\mathrm{d}}{\mathrm{d}t}\beta(t)\wedge\beta(t)+\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\wedge\gamma(t)\right)\mathrm{d}t$$

Now

$$\omega=f^*\mu=\begin{pmatrix} i\omega_1 & \omega_2+i\omega_3 \\ -\omega_2+i\omega_3 & -i\omega_1 \end{pmatrix}$$

so that

$$\sigma(t)=\begin{pmatrix} i(\omega_1+\cos t\omega_1+\sin t*\omega_1) & (\omega_2+\cos t\omega_2+\sin t*\omega_2)+ \\ & i(\omega_3+\cos t\omega_3+\sin t*\omega_3) \\ -(\omega_2+\cos t\omega_2+\sin t*\omega_2)+ & -i(\omega_1+\cos t\omega_1+\sin t*\omega_2) \\ i(\omega_3+\cos t\omega_3+\sin t*\omega_3) & \end{pmatrix}$$

i.e.

$$\alpha(t) = (\omega_1 + \cos t\omega_1 + \sin t * \omega_1)$$

$$\beta(t) = (\omega_2 + \cos t\omega_2 + \sin t * \omega_2)$$

$$\gamma(t) = (\omega_3 + \cos t\omega_3 + \sin t * \omega_3)$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)\Lambda\alpha(t) = ((-\sin t)\omega_1 + \cos t * \omega_1)\Lambda(\omega_1 + \cos t * \omega_1 + \sin t * \omega_1)$$

$$= -\sin^2 t\omega_1\Lambda * \omega_1 + \cos^2 t * \omega_1\Lambda\omega_1$$

$$= * \omega_1\Lambda\omega_1.$$

Similarly

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) \wedge \beta(t) = * \omega_2 \wedge \omega_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \wedge \gamma(t) = * \omega_3 \wedge \omega_3.$$

It follows that $\int_{D^2} C_2(K(\vartheta^\sigma))$ is cohomologous to the form

$$\frac{1}{4\pi^2}\int_{S^1} (* \omega_1 \wedge \omega_1 + * \omega_2 \wedge \omega_2 + * \omega_3 \wedge \omega_3)\mathrm{d}t$$

$$= \frac{1}{2\pi}(* \omega_1 \wedge \omega_1 + * \omega_2 \wedge \omega_2 + * \omega_3 \wedge \omega_3).$$

Thus the closed 2-form on M representing $\chi(\sigma_f) \in H^2(M, R)$ is $\frac{1}{2\pi}(* \omega_1 \wedge \omega_1 + * \omega_2 \wedge \omega_2 + * \omega_3 \wedge \omega_3)$.

To prove that $\chi(\sigma_f) = -\frac{1}{4\pi}E(f)$, we check using local coordinates that the forms

$$\frac{1}{2\pi}\left(\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) \wedge \alpha(t) + \frac{\mathrm{d}}{\mathrm{d}t}\beta(t) \wedge \beta(t) + \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \wedge \gamma(t)\right)$$

and $-\frac{1}{4\pi}e(f)(m)\mathrm{d}m$ ($\mathrm{d}m$ is the volume form on M) are equal at any arbitrary point.

Since any left translation in G is an isometry, for any $m \in M$, $\|\mathrm{d}f(m)\| = \|\mathrm{d}(L_{f(m)^{-1}} \circ f)(m)\|$ where $L_{f(m)^{-1}}: G \to G$ is left translation by $f(m)^{-1}$. We can therefore assume that f maps some point $m \in M$ to the identity element in G, i.e. $f(m) = 1$.

Since we intend to use local coordinates to prove the equality of forms, we can go to the universal cover D^2 of M with Poincare metric and assume $f: D^2 \to G$ and $f(m) = 1$ for some fixed $m \in D^2$. Since there exist an isometry of D^2 which maps the origin to m, we can assume $f(0) = 1$ and check equality of forms at the origin.

At the origin we have

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle = 1 = \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle$$

and

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = 0$$

where $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ are the usual coordinate vector fields. Let dx and dy be the dual 1-forms. Clearly at the origin $*dx = dy$ and $*dy = -dx$. Since $dm = dx \wedge dy$ we have

$$e(f)(m)dm\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = e(f)(m).$$

We prove that

$$\frac{1}{2\pi}(*\omega_1 \wedge \omega_1 + *\omega_2 \wedge \omega_2 + *\omega_3 \wedge \omega_3)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = -\frac{1}{4\pi}e(f)(m).$$

If $\omega_j = a_j dx + b_j dy$ $(1 \leqslant j \leqslant 3$, a_j, b_j are functions on $D^2)$ then $*\omega_j = a_j dy - b_j dx$ for $1 \leqslant j \leqslant 3$ so that $*\omega_j \wedge \omega_j = -(a_j^2 + b_j^2)dx \wedge dy$ for $1 \leqslant j \leqslant 3$

$$\Rightarrow \frac{1}{2\pi}(*\omega_1 \wedge \omega_1 + *\omega_2 \wedge \omega_2 + *\omega_3 \wedge \omega_3) = -\frac{1}{2\pi}(a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2)dx \wedge dy$$

For $f: D^2 \to SU(2)$ with $f(0) = 1$

$$\| df(0) \|^2 = \left\| df(0)\left(\frac{\partial}{\partial x}\right) \right\|^2 + \left\| df(0)\left(\frac{\partial}{\partial y}\right) \right\|^2$$

$$= \left\| \frac{\partial f(0)}{\partial x} \right\|^2 + \left\| \frac{\partial f(0)}{\partial y} \right\|^2$$

By definition of Maurer–Cartan form

$$\frac{\partial f(0)}{\partial x} = \mu\left(\frac{\partial f(0)}{\partial x}\right) =$$

$$\begin{pmatrix} i\mu_1\left(\frac{\partial f(0)}{\partial x}\right) & \mu_2\left(\frac{\partial f(0)}{\partial x}\right) + i\mu_3\left(\frac{\partial f(0)}{\partial x}\right) \\ -\mu_2\left(\frac{\partial f(0)}{\partial x}\right) + i\mu_3\left(\frac{\partial f(0)}{\partial x}\right) & -i\mu_1\left(\frac{\partial f(0)}{\partial x}\right) \end{pmatrix}$$

The pairing $(A, B) \mapsto \text{trace}(AB)$ for $A, B \in su(2)$ gives the Killing form on $su(2)$ so that

$$\left\| \frac{\partial f(0)}{\partial x} \right\|^2 = \text{trace}\left(\frac{\partial f(0)}{\partial x}\frac{\partial f(0)}{\partial x}\right)$$

$$= 2\left\{\left(\mu_1\left(\frac{\partial f(0)}{\partial x}\right)\right)^2 + \left(\mu_2\left(\frac{\partial f(0)}{\partial x}\right)\right)^2 + \left(\mu_3\left(\frac{\partial f(0)}{\partial x}\right)\right)^2\right\}.$$

Similarly

$$\left\| \frac{\partial f(0)}{\partial x} \right\|^2 = 2\left\{\left(\mu_1\left(\frac{\partial f(0)}{\partial y}\right)\right)^2 + \left(\mu_2\left(\frac{\partial f(0)}{\partial y}\right)\right)^2 + \left(\mu_3\left(\frac{\partial f(0)}{\partial y}\right)\right)^2\right\}.$$

$$\Rightarrow \| df(0) \|^2 = 2\left\{\sum_{j=1}^{3}\left(\mu_j\left(\frac{\partial f(0)}{\partial x}\right)\right)^2 + \left(\mu_j\left(\frac{\partial f(0)}{\partial y}\right)\right)^2\right\}.$$

Noting that $f^*\mu_j = \omega_j (1 \leqslant j \leqslant 3)$ we have

$$\omega_j\left(\frac{\partial}{\partial x}\right)(0) = (f^*\mu_j)\left(\frac{\partial}{\partial x}\right)(0) = \mu_j\left(\frac{\partial f}{\partial x}(0)\right).$$

Now

$$\omega_j\left(\frac{\partial}{\partial x}\right) = (a_j \mathrm{d}x + b_j \mathrm{d}y)\left(\frac{\partial}{\partial x}\right) = a_j.$$

Therefore

$$\omega_j\left(\frac{\partial f(0)}{\partial x}\right) = a_j \quad (1 \leqslant j \leqslant 3).$$

Similarly

$$\omega_j\left(\frac{\partial f(0)}{\partial y}\right) = b_j \quad (1 \leqslant j \leqslant 3).$$

Thus

$$\|\mathrm{d}f(0)\|^2 = 2\{a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2\} = e(f)(m).$$

Therefore we have

$$(-(*\omega_1 \wedge \omega_1 + *\omega_2 \wedge \omega_2 + *\omega_3 \wedge \omega_3)) = \frac{1}{2} e(f)(m) \mathrm{d}x \wedge \mathrm{d}y.$$

In other words

$$\left(\frac{1}{2\pi}(*\omega_1 \wedge \omega_1 + *\omega_2 \wedge \omega_2 + *\omega_3 \wedge \omega_3)\right) = -\frac{1}{4\pi} e(f)(m)\mathrm{d}m$$

Consequently $\chi(\sigma_f) = -\frac{1}{4\pi} E(f)$ and the theorem follows.

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Fibred Frobenius theorem

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Abstract. We give a version of Frobenius Theorem for fibred manifolds whose proof is shorter than the "short proofs" of the classical Frobenius Theorem. In fact, what shortens the proof is the fibred form of the statement, since it permits an inductive process which is not possible from the standard statement.

Keyword. Frobenius theorem

Theorem. *Let $\pi: M \to N$ be a submersion, $\dim N = n$, $\dim M = m, + n$, and let $E \subset V(\pi)$ be an involutive sub-bundle of rank r of the vertical bundle $V(\pi)$ of π. Given a point $y_0 \in M$ and any coordinate system $(x_1, \ldots, x_n)$ on a neighbourhood of $x_0 = \pi(y_0)$, there exist functions $(y_1, \ldots, y_m)$ on M such that:*

(i) *$(x_1 \circ \pi, \ldots, x_n \circ \pi; y_1, \ldots, y_m)$ is a coordinate system on an open neighbourhood U of y_0,*

(ii) $\Gamma(U, E) = \langle \partial/\partial y_1, \ldots, \partial/\partial y_r \rangle$.

Proof. By induction on r. For $r = 1$ there exists an open neighbourhood U_0 of y_0 and a non-singular vector field Y such that $\Gamma(U_0, E) = \langle Y \rangle$. Since π is a submersion, given $(x_1, \ldots, x_n)$ there exist functions $(y'_1, \ldots, y'_m)$ on M satisfying (i) and, since Y is vertical, we have $Y = \sum_{i=1}^{m} f_i(\partial/\partial y'_i)$. As Y is non-singular, we can apply the theorem of reduction of vector fields to normal form (see [1, Lemma 2]) by considering $(x_1, \ldots, x_n)$ as parameters, thus obtaining a system $(x_1 \circ \pi, \ldots, x_n \circ \pi; y_1, \ldots, y_m)$ such that $Y = \partial/\partial y_1$.

Assume $r > 1$. There is an open neighbourhood U_0 of y_0 on which E admits a basis: $\Gamma(U_0, E) = \langle Y_1, \ldots, Y_r \rangle$. Applying the above case to Y_1, we obtain a system $(x_1 \circ \pi, \ldots, x_n \circ \pi; y'_1, \ldots, y'_m)$ such that $Y_1 = \partial/\partial y'_1$. Let

$$Y'_i = Y_i - Y_i(y'_1) Y_1, \quad 2 \leqslant i \leqslant r,$$

so that $Y'_2, \ldots, Y'_r$ span an involutive sub-bundle $E' \subset E$ of rank $r - 1$. In fact, as E is involutive, we have for $2 \leqslant i, j \leqslant r$:

$$[Y'_i, Y'_j] = \sum_{k=1}^{r} f_{ij}^k Y_k = \sum_{k=2}^{r} f_{ij}^k Y'_k + f_{ij} Y_1, \quad f_{ij} = f_{ij}^1 + \sum_{k=2}^{r} f_{ij}^k Y_k(y'_1).$$

As $Y'_i(y'_1) = 0$, one has $[Y'_i, Y'_j]\,(y'_1) = 0$. Hence $f_{ij} = 0$.

Let $\pi': M \to N \times \mathbb{R}$ be the submersion $\pi' = (\pi, y'_1)$. Since $Y'_i(y'_1) = 0$, we have $E' \subset V(\pi')$. Let $x_{n+1}: N \times \mathbb{R} \to \mathbb{R}$ be the projection onto the second factor, which makes $(x_1, \ldots, x_n, x_{n+1})$ a coordinate system on $N \times \mathbb{R}$. By the induction hypothesis, there

exist functions $(y''_2,\dots,y''_m)$ satisfying conditions (i) and (ii) with respect to E'. Consequently, there exists an open neighbourhood U of y_0 such that $\Gamma(U,E)=\langle Y_1,\partial/\partial y''_2,\dots,\partial/\partial y''_r\rangle$, and from $x_{n+1}\circ\pi'=y'_1$, we deduce $Y_1=\partial/\partial y'_1+\sum_{i=2}^{m}f_i(\partial/\partial y''_i)$. Substituting $Y'_1=\partial/\partial y'_1+\sum_{i=r+1}^{m}f_i(\partial/\partial y''_i)$ for Y_1, we also have $\Gamma(U,E)=\langle Y'_1,\partial/\partial y''_2,\dots,\partial/\partial y''_r\rangle$, and since E is involutive for every $2\leqslant j\leqslant r$, we have:

$$\left[\frac{\partial}{\partial y''_j},Y'_1\right]=\sum_{i=1}^{m-r}\frac{\partial f_{i+r}}{\partial y''_j}\frac{\partial}{\partial y''_{i+r}}=g_1Y'_1+\sum_{i=2}^{r}g_i\frac{\partial}{\partial y''_i}.$$

Applying both sides to y'_1 we conclude $g_1=0$. Hence $\partial f_{i+r}/\partial y''_j=0$; that is, $(f_{r+1},\dots,f_m)$ depend only on $(x_1\circ\pi,\dots,x_n\circ\pi;\ y'_1,y''_{r+1},\dots,y''_m)$. Consequently, there exists a change of coordinates

$$\begin{aligned}
y_1&=\phi_1(x_1\circ\pi,\dots,x_n\circ\pi;y'_1,y''_{r+1},\dots,y''_m),\\
y_{r+1}&=\phi_{r+1}(x_1\circ\pi,\dots,x_n\circ\pi;y'_1,y''_{r+1},\dots,y''_m),\\
&\dots\\
y_m&=\phi_m(x_1\circ\pi,\dots,x_n\circ\pi;y'_1,y''_{r+1},\dots,y''_m),
\end{aligned}$$

which reduces Y'_1 to $Y'_1=\partial/\partial y_1$. Now, writing $y_j=y''_j$, $2\leqslant j\leqslant r$, we have $\Gamma(U,E)=\langle\partial/\partial y_1,\partial/\partial y_2,\dots,\partial/\partial y_r\rangle$, thus finishing the proof.

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On infinitesimal *h*-conformal motions of Finsler metric

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Abstract. The conformal theory of Finsler spaces was initiated by Knebelman in 1929 and lately Kikuchi [7] gave the conditions for a Finsler space to be conformal to a Minkowski space. However under the *h*-condition, the third author [4] obtained the conditions for a Finsler space to be *h*-conformal to a Minkowski space.

The purpose of the paper is to investigate the infinitesimal *h*-conformal motions of Finsler metric and its application to an *H*-recurrent Finsler space. We obtain the following results.

A. Theorem 2.1. If an HR-F_n space is a Landsberg space, then the tensor F^i_{hjk} is recurrent.

B. Proposition 3.3. An infinitesimal *h*-conformal motion satisfies

$$L_X G^i_{jk} = \rho_j \delta^i_k + \rho_k \delta^i_j - \rho^i g_{jk} - \phi_1 l^i l_j l_k,$$
$$L_X G^i_j = \rho \delta^i_j + \phi_j y^i - y_j \rho^i.$$

C. Proposition 3.6. An infinitesimal *h*-conformal motion satisfies $L_X P^i_{jk} = \rho C^i_{jk}$.

D. Theorem 3.7. In order that an infinitesimal *h*-conformal motion preserves Landsberg spaces, it is necessary and sufficient that the transformation be an infinitesimal homothetic motion.

E. Theorem 3.8. An infinitesimal *h*-conformal motion preserves *P-Finsler spaces.

F. Theorem 3.10. An infinitesimal *h*-conformal motion preserves *h*-conformally flat Finsler spaces.

G. Theorem 4.1. An infinitesimal homothetic motion preserves *H*-recurrent Finsler spaces.

H. Theorem 4.2. If an *H*-recurrent Finsler space admits an infinitesimal homothetic motion, then Lie derivatives of the tensor F^i_{hjk} and all its successive covariant derivatives by x^i or y^i vanish.

Keywords. Infinitesimal h-conformal motion; h-conformal tensor; infinitesimal homothetic motion.

1. Preliminaries

1.1 Berwald connection

Let F_n be an n-dimensional Finsler space with the Finsler metric $F(x, y)$. The metric and angular metric tensors are given by $g_{ij} := \dot\partial_i \dot\partial_j F^2/2$ and $h_{ij} := g_{ij} - l_i l_j$, where $\dot\partial_i := \dfrac{\partial}{\partial y_i}$, $l_i := \dot\partial_i F$ and $l^i := \dfrac{y^i}{F}$.

We use the following:

$$\text{(a)}\quad \gamma^i_{jk} := \tfrac{1}{2}g^{ih}(\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk}),$$

$$\text{(b)}\quad G^i_{jk} := \dot\partial_k G^i_j, \quad G^i_j := \dot\partial_j G^i, \quad G^i := \tfrac{1}{2}\gamma^i_{jk}y^j y^k. \tag{1.1}$$

Two types of covariant derivatives for a vector X^i are given by

$$\text{(a)}\quad X^i_{:k} := d_k X^i + G^i_{hk}X^h, \quad d_k := \partial_k - G^m_k\dot\partial_m,$$

$$\text{(b)}\quad X^i_{|k} := \dot\partial_k X^i, \qquad\qquad \partial_k := \frac{\partial}{\partial x^k},$$

and the Cartan tensor is defined by $C^i_{hk} := \frac{1}{2}g^{im}\dot\partial_k g_{mh}$. This connection is known as th Berwald connection, which is not metrial, that is,

$$g_{ij:k} = -2P_{ijk} = -2C_{ijk:m}y^m, \quad g^{ij}_{:k} = 2P^{ij}_k. \quad \text{(cf. [2])} \tag{1.2}$$

When a Finsler space satisfies the condition $P^i_{jk} = 0$, the space is called a *Landsber* space.

The curvature tensor H^i_{hjk} is defined by

$$H^i_{hjk} := d_k G^i_{hj} + G^m_{hj}G^i_{mk} - j|k, \tag{1.3}$$

where $j|k$ means the interchange of indices j and k in the foregoing terms. We see

$$\begin{aligned} &H^i_{jk} := H^i_{0jk}, \qquad H^i_k := H^i_{0k}, \qquad H_{hj} := H^i_{hji}, \\ &H^i_{hjk} = \dot\partial_h H^i_{jk}, \quad 3H^i_{jk} = \dot\partial_j H^i_k - j|k, \end{aligned} \tag{1.4}$$

where the index 0 means the transvection by y.

The Ricci identities are denoted by

$$\text{(a)}\quad T^i_{h:j:k} - T^i_{h:k:j} = H^i_{mjk}T^m_h - T^i_m H^m_{hjk} - H^m_{jk}T^i_{h|m},$$

$$\text{(b)}\quad T^i_{h:j|k} - T^i_{h|k:j} = G^i_{mjk}T^m_h - T^i_m G^m_{hjk}, \quad G^i_{hjk} := \dot\partial_k G^i_{hj}.$$

In the theory of conformal transformation, the h-conformal tensor F^i_{hjk} is define by ([4], (4.15))

$$\begin{aligned} F^i_{hjk} := H^i_{hjk} &- \frac{1}{n-2}(H_{hj}\delta^i_k + g_{hj}H_{mk}g^{im} - j|k) + \bar H(g_{hj}\delta^i_k - g_{hk}\delta^i_j) \\ &- \frac{1}{(n-1)(n-2)}\{l_j(h^i_k M_h - M^i h_{hk} - (n-2)h^i_h M_k) - j|k\}, \end{aligned} \tag{1.5}$$

$$\bar H := \frac{g^{ik}H_{ik}}{(n-1)(n-2)}, \quad M_h := (H_{mh} - H_{hm})l^m.$$

1.2 Lie derivative

We consider an infinitesimal extended point transformation in a Finsler spac generated by the vector $X = v^i(x)\partial_i$, i.e.

$$\bar x^i = x^i + v^i\mathrm{d}t, \quad \bar y^i = y^i + (\partial_j v^i)y^j\,\mathrm{d}t. \tag{1.6}$$

The well-known commutation formulae ([1], [5], [6], [8–10], etc.) involving L

and covariant derivatives are given by

$$\begin{aligned} &\text{(a)}\ L_X T^i_{j:k} - (L_X T^i_j)_{:k} = T^m_j A^i_{mk} - T^i_m A^m_{jk} - A^m_k \dot{\partial}_m T^i_j, \\ &\text{(b)}\ L_X H^i_{hjk} = A^i_{hj:k} + A^m_j G^i_{hkm} - J|k, \end{aligned} \tag{1.7}$$

where

$$A^m_j := L_X G^m_j, \quad A^i_{jk} := L_X G^i_{jk} = v^i_{:j:k} + H^i_{jkh} v^h + G^i_{jkm} v^m_{:0}.$$

In the usual way we raise or lower indices by means of the metric tensors g^{ij} or g_{ij}.

2. An HR-F_n space

A Finsler space F_n is said to be an *H-recurrent* Finsler space (denoted by an HR-F_n space), if the Berwald curvature tensor H^i_{hjk} satisfies the relation

$$H^i_{hjk:m} = K_m H^i_{hjk}, \quad H^i_{hjk} \neq 0, \tag{2.1}$$

where K_m is a nonzero vector. As $y^i_{:k} = 0$, we have

$$H^i_{j:k} = K_k H^i_j, \quad H_{ij:k} = K_k H_{ij}$$

and from (1.5) we obtain

$$\begin{aligned} F^i_{hjk:m} = K_m F^i_{hjk} &+ \frac{2}{n-2}(P_{hjm} H_{tk} g^{it} - g_{hj} H_{tk} P^{it}_m - j|k) \\ &+ \frac{2}{(n-1)(n-2)}\{P^{ts}_m H_{ts} g_{hj} \delta^i_k - g^{ts} H_{ts} P_{hjm} \delta^i_k \\ &+ l_j (P^{it}_m M_t g_{hk} - M^i P_{hkm}) - j|k\}. \end{aligned} \tag{2.2}$$

Thus we have

Theorem 2.1. *If an HR-F_n space is a Landsberg space, then the tensor F^i_{hjk} is recurrent.*

3. An infinitesimal *h*-conformal motion

3.1 An i.c.m.

The condition for an infinitesimal transformation (1.6) to be an infinitesimal conformal motion (denoted by an i.c.m.) is that there exists a function ϕ of x such that

$$L_X g_{jk} = 2\phi(x) g_{jk}, \quad L_X g^{jk} = -2\phi(x) g^{jk}. \quad \text{(cf. [1], [5], etc.)} \tag{3.1}$$

If the function ϕ is a constant, the i.c.m. (3.1) is called an *infinitesimal homothetic motion* (denoted by an i.h.m.) and when $\phi = 0$, the (3.1) called an *infinitesimal isometric motion* (denoted by an i.i.m.).

It is well known that an i.c.m. (3.1) satisfies $L_X C^i_{jk} = 0$ and $L_X y^i = 0$. We can easily see

$$\begin{aligned} &\text{(a)}\ L_X C_j = 0,\ C_j := C^i_{ji},\ L_X C^i = -2\phi C^i, \quad \text{(b)}\ L_X F = \phi F, \\ &\text{(c)}\ L_X l^i = -\phi l^i,\ L_x l_j = \phi l_j, \quad \text{(d)}\ L_X h^i_j = 0,\ L_X(g^{ih} g_{jk}) = 0. \end{aligned} \tag{3.2}$$

Since the Lie derivative is commutative with ∂_k or $\dot{\partial}_k$, we see from (1.1) (a) and (3.1)

$$L_X\gamma^i_{jk} = \phi_j\delta^l_k + \phi_k\delta^i_j - \phi^i g_{jk}, \quad \phi_j := \partial_j\phi.$$

Transvecting the above equation by $y^j y^k$ we have

$$L_X G^i = B^{ih}\phi_h, \quad B^{ih} := y^i y^h - \tfrac{1}{2}F^2 g^{ih}. \tag{3.3}$$

Differentiating (3.3) by y^j and y^k, we get

$$\text{(a)}\ L_X G^i_j = B^{ih}_j\phi_h, \quad B^{ih}_j := \dot{\partial}_j B^{ih} = \delta^i_j y^h + \delta^h_j y^i - g^{ih}y_j + F^2 C^{ih}_j,$$

$$\text{(b)}\ L_X G^i_{jk} = B^{ih}_{jk}\phi_h, \quad B^{ih}_{jk} := \dot{\partial}_k B^{ih}_j,$$

$$B^{ih}_{jk} = \delta^i_j\delta^h_k + \delta^i_k\delta^h_j - g^{ih}g_{jk} + 2y_j C^{ih}_k + 2y_k C^{ih}_j + F^2\dot{\partial}_k C^{ih}_j. \tag{3.4}$$

Using (3.2) we have

PROPOSITION 3.1.
An infinitesimal conformal motion satisfies the following:

$$L_X B^{ih} = 0 \leftrightarrow L_X B^{ih}_j = 0 \leftrightarrow L_X B^{ih}_{jk} = 0.$$

3.2 An i.a.m.

If an infinitesimal transformation (1.6) satisfies $L_X G^i_{jk} = 0$, then the transformation is called an *infinitesimal affine motion* (denoted by an i.a.m.).

First, we shall show

Theorem 3.2 ([1], (VII), Theorem 5.1). *In order for an infinitesimal transformation be homothetic, it is necessary and sufficient that the transformation be conformal and affine motion at the same time.*

Proof. We see from (3.4)

$$0 = y^j y^k L_X G^i_{jk} = B^{ih}_{00}\phi_h = 2B^{ih}\phi_h = F^2(2l^i l^h - g^{ih})\phi_h.$$

Transvecting the above equation with $2l_i l_k - g_{ik}$, we have $F^2\phi_k = 0$. Q.E.D.

Remark.. This theorem was first proved by Takano (Japanese, 1952).

3.3 An i.h-c.m.

If we impose the h-condition on the vector ϕ_j, i.e.

$$FC^h_{ij}\phi_h = \phi_1 h_{ij}, \quad \phi_1 := \frac{FC^h\phi_h}{n-1}. \quad \text{(cf. [4], §3)} \tag{3.5}$$

the transformation is called an *infinitesimal h-conformal motion* (denoted by an i.h-c.m.).

Because the function $\phi_1(x)$ is proved to be a function of x only (see [4], Lemma 3.2), we get

(a) $F\dot{\partial}_k h^i_j = -h_{jk}l^i - h^i_k l_j$,

(b) $F^2(\dot{\partial}_k C^{ih}_j)\phi_h = F\dot{\partial}_k(FC^{ih}_j\phi_h) - FC^{ih}_j\phi_h\dot{\partial}_k F$

$= F\dot{\partial}_k(\phi_1 h^i_j) - \phi_1 h^i_j l_k = -\phi_1(h_{jk}l^i + h^i_k l_j + h^i_j l_k)$. (3.6)

Using the above calculations, we obtain

$$A^i_{jk} = L_X G^i_{jk} = \rho_j\delta^i_k + \rho_k\delta^i_j - g_{jk}\rho^i - \phi_1 l_j l_k l^i,$$
$$A^i_j = L_X G^i_j = \rho\delta^i_j + \phi_j y^i - y_j\rho^i, \tag{3.7}$$

$$\rho_j(x,y) := \phi_j + \phi_1 l_j = \left(\delta^h_j + \frac{FC^h l_j}{n-1}\right)\phi_h, \quad \rho := \rho_0 = \phi_0 + F\phi_1.$$

Hence we have

PROPOSITION 3.3.
An infinitesimal h-conformal motion satisfies

$$L_X G^i_{jk} = \rho_j\delta^i_k + \rho_k\delta^i_j - \rho^i g_{jk} - \phi_1 l^i l_j l_k,$$
$$L_X G^i_j = \rho\delta^i_j + \phi_j y^i - y_j\rho^i.$$

The vector ρ_j is called an *associated vector* with a vector ϕ_j and satisfies the conditions:

(a) $FC^h_{jk}\rho_h = \phi_1 h_{jk}$, (*h*-condition)

(b) $\rho_j|_k := \dot{\partial}_k\rho_j - C^h_{jk}\rho_h = 0$.
(Cartan's covariant derivative by y^k) (3.8)

A vector which satisfies (3.8) (a) (b) is called an *h*-vector.

PROPOSITION 3.4 ([4], Proposition 3.4).
Let $v_i(x,y)$ be a vector in a Finsler space. If v_i satisfies the conditions $v_i|_k = 0$ and $FC^h_{jk}v_h = v_1 h_{jk}$, then the function v_1 and the vector $^*v_i := v_i - v_1 l_i$ are independent of y.
Here we shall show

Lemma 3.5. We have

$$F^2\rho^m\dot{\partial}_m C^i_{jk} = \phi_1(FC^i_{jk} - h^i_j l_k - h^i_k l_j - h_{jk}l^i).$$

Proof. We see

$$\dot{\partial}_k g^{ih} = -2C^{ih}_k, \quad \dot{\partial}_m C^i_{jk} = \dot{\partial}_m(g^{ih}C_{hjk}) = -2C^{ih}_m C_{hjk} + g^{ih}\dot{\partial}_m C_{hjk},$$

and using $\dot{\partial}_m C_{hjk} = \dot{\partial}_k C_{hjm}$, (3.6) (a) and (3.8) (b), we get

$$\begin{aligned} F^2\rho^m\dot{\partial}_m C^i_{jk} &= -2F\phi_1 C^i_{jk} + Fg^{ih}\{\dot{\partial}_k(FC_{hjm}\rho^m) - C_{hjm}\dot{\partial}_k(F\rho^m)\} \\ &= -2F\phi_1 C^i_{jk} + g^{ih}\phi_1 F\dot{\partial}_k h_{hj} - FC^i_{jm}(l_k\rho^m - FC^m_{tk}\rho^t) \\ &= \phi_1(FC^i_{jk} - h^i_j l_k - h^i_k l_j - h_{jk}l^i). \end{aligned}$$

From (1.7)*(a) we see

$$L_X C^i_{jk:l} - (L_X C^i_{jk})_{:l} = A^i_{ml}C^m_{jk} - A^m_{jl}C^i_{mk} - A^m_{kl}C^i_{jm} - A^m_l\dot{\partial}_m C^i_{jk}.$$

In consideration of $L_X C^i_{jk} = 0$ and transvecting the above equation by y^l, we have

$$L_X P^i_{jk} = L_X C^i_{jk:0} = A^i_m C^m_{jk} - A^m_j C^i_{mk} - A^m_k C^i_{jm} - A^m_0 \dot{\partial}_m C^i_{jk}.$$

Substituting (3.7) into the above equation, we have

$$L_X P^i_{jk} = \phi_1(h^i_j l_k + h^i_k l_j + h_{jk} l^i) + \phi_0 C^i_{jk} + F^2 \rho^m \dot{\partial}_m C^i_{jk}.$$

Using Lemma 3.5, we obtain

$$L_X P^i_{jk} = (\phi_0 + F\phi_1) C^i_{jk} = \rho C^i_{jk}. \tag{3.9}$$

Thus we have

PROPOSITION 3.6
An infinitesimal h-conformal motion satisfies

$$L_X P^i_{jk} = \rho C^i_{jk}.$$

Remark. If we denote the *deformed* tensor (cf. [1]) of P^i_{jk} with respect to an i.*h*-c.m. (1.6) by $\tilde{P}^i_{jk}$, we see

$$\tilde{P}^i_{jk} = P^i_{jk} + (\rho C^i_{jk})\,\mathrm{d}t.$$

This means that the deformed space of a Landsberg space ($P^i_{jk} = 0$) is not necessarily a Landsberg space.

However we can state the following.

Theorem 3.7. *In order that an infinitesimal h-conformal motion preserves Landsberg spaces, it is necessary and sufficient that the transformation be an infinitesimal homothetic motion.*

Proof. It is sufficient to show $\phi_i = 0$. In fact, we have

$$0 = \dot{\partial}_j \rho = \rho_j = \left(\delta^i_j + \frac{FC^i l_j}{n-1}\right)\phi_i, \quad \phi_j = \left(\delta^i_j - \frac{FC^i l_j}{n-1}\right)\rho_i = 0.$$

It is evident that the theorem holds. Q.E.D.

*3.4 *P-Finsler space*

If the tensor ${}^*P^i_{jk} := P^i_{jk} - \lambda C^i_{jk}$ vanishes, the space is called a **P-Finsler* space (cf. [3]). The *P-condition: $P^i_{jk} = \lambda C^i_{jk}$ is invariant under any *h*-conformal change of Finsler metric.

From (3.9) we have

$$L_X P_j = \rho C_j, \quad P_j := P^i_{ji}. \tag{3.10}$$

Using (3.10) we see

$$0 = L_X(P_j - \lambda C_j) = (\rho - L_X\lambda) C_j, \quad L_X\lambda = \rho. \tag{3.11}$$

This means $L_X {}^*P^i_{jk} = 0$. Hence we have

Theorem 3.8. *An infinitesimal h-conformal motion preserves *P-Finsler spaces.*

3.5 An h-conformally flat Finsler space

If a Finsler space is h-conformal to a Minkowski space, the space is called an *h-conformally flat Finsler space.*

An h-conformally flat Finsler space is proved to be one of $*P$-Finsler space (cf. [4], (5.2)). Here we shall show

*Lemma 3.9. In a *P-Finsler space an infinitesimal h-conformal motion satisfies*

$$L_X\lambda_h = \rho_h, \quad L_X{}^*\lambda_j = \phi_j, \quad {}^*\lambda_j := \left(\delta^i_j - \frac{C^i y_j}{n-1}\right)\lambda_i, \quad \lambda_i := \dot\partial_i\lambda. \tag{3.12}$$

Proof. Differentiating (3.11) w.r.t. y^h we have $L_X\lambda_h = \rho_h$. Next from (3.2) we see

$$L_X\left(\delta^i_j - \frac{C^i y_j}{n-1}\right) = 0, \quad C^i y_j = g^{ih}g_{jk}C_h y^k.$$

Hence we have $L_X{}^*\lambda_j = \left(\delta^i_j - \frac{C^i y_j}{n-1}\right)\rho_i = \phi_j$. Q.E.D.

On the other hand, we know the theorem ([4], Theorem 6.6):
The necessary and sufficient conditions for a Finsler space to be h-conformally flat are that $\dot\partial_l \Pi^i_{jk} = 0$ and $\Pi^i_{hkl} = 0$ and λ_i is an h-vector, where

$$\Pi^i_{jk} := G^i_{jk} - B^{ih}{}^*\lambda_h,$$
$$\Pi^i_{hkl} := {}^*d_l\Pi^i_{hk} + \Pi^m_{hk}\Pi^i_{ml} - k|l, \quad {}^*d_k = \partial_k - \Pi^m_{0k}\dot\partial_m. \tag{3.13}$$

The parameter Π^i_{jk} and the tensor Π^i_{hkl} are invariant under an h-conformal transformation and these are independent of y.

We shall show

Theorem 3.10. *An infinitesimal h-conformal motion preserves h-conformally flat Finsler spaces.*

Proof. It is sufficient to prove $L_X\Pi^i_{jk} = 0$. We see $L_X B^{ih}_{jk} = 0$ from Proposition 3.1. Moreover, we have from (3.4) (b) and (3.12)

$$L_X\Pi^i_{jk} = L_X(G^i_{jk} - B^{ih}{}^*\lambda_h) = B^{ih}\phi_h - B^{ih}\phi_h = 0.$$

It is easy to prove $\Pi^i_{hkl} = 0$. Q.E.D.

4. An infinitesimal homothetic motion in HR-F_n spaces

In this section we shall consider an i.h.m. only, that is,

$$L_X g_{ij} = 2cg_{ij}, \quad L_X g^{ij} = -2cg^{ij}, \quad c = \text{constant}. \tag{4.1}$$

From Theorem 3.2 and (1.7) (b), we have $L_X H^i_{hjk} = 0$.

From (1.7) (a) and (2.1) we see

$$L_X H^i_{hjk:m} = (L_X H^i_{hjk})_{:m} = 0.$$
$$L_X H^i_{hjk:m} = L_X(K_m H^i_{hjk}) = (L_X K_m)H^i_{hjk} = 0,$$

which means

$$L_X K_m = 0 \quad \text{and} \quad L_X(H^i_{hjk:m} - K_m H^i_{hjk}) = 0.$$

Thus we have

Theorem 4.1. *An infinitesimal homothetic motion preserves H-recurrent Finsler spaces and satisfies* $L_X K_m = 0$.

An i.h.m. (4.1) satisfies

$$L_X l_j = cl_j, \quad L_X l^i = -cl^i, \quad L_X h^i_j = 0. \tag{4.2}$$

From Proposition 3.6 we see

$$L_X P^i_{jk} = 0, \quad L_X P_{jik} = 2cP_{jik}, \quad L_X P^{ij}_k = -2cP^{ij}_k,$$
$$L_X M_h = -cM_h, \quad L_X M^i = -3cM^i, \quad L_X \bar{H} = -2c\bar{H}. \tag{4.3}$$

After some calculations we obtain $L_X F^i_{hjk} = 0$.

Moreover we see from (2.2), (4.2) and (4.3)

$$L_X F^i_{hjk:m} = (L_X K_m) F^i_{hjk} + K_m L_X F^i_{hjk} = 0.$$

Hence we have

Theorem 4.2. *If an H-recurrent Finsler space admits an infinitesimal homothetic motion, then Lie derivatives of the tensor* F^i_{hjk} *and all its successive covariant derivatives w.r.t.* x^i *or* y^i *vanish.*

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A bibasic hypergeometric transformation associated with combinatorial identities of the Rogers–Ramanujan type

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Abstract. During the last five decades, a number of combinatorial generalizations and interpretations have occurred for the identities of the Rogers–Ramanujan type. The object of this paper is to give a most general known analytic auxiliary functional generalization which can be used to give combinatorial interpretations of generalized q-identities of the Rogers–Ramanujan type. The derivation realise the theory of basic hypergeometric series with two unconnected bases.

Keywords. Auxiliary functions; unibasic hypergeometric series; bibasic hypergeometric series; q-hypergeometric identities.

1. Introduction

The two celebrated Rogers–Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+\alpha n}}{(q;q)_n} = \prod_{n=1}^{\infty} (1-q^n)^{-1}, \qquad |q|<1, \tag{1}$$

$$n \equiv \pm(\alpha+1) (\mathrm{mod}\ 5)$$

where $\alpha = 0$ or 1, were first given by Rogers [12] in 1894 and then rediscovered (without proof) by Ramanujan in 1911.

In 1916, MacMahon ([11]; § 7, Chap. III) gave the following combinatorial interpretation of these two identities:

"The number of partitions of n into parts that differ by at least 2 with each part $> \alpha$ is equal to the number of partitions of n into parts $\equiv \pm(\alpha+1) (\mathrm{mod}\ 5)$, where α may be either 0 or 1".

In 1917, while scanning some old volumes of the *Proceedings of the London Mathematical Society*, Ramanujan came across the remarkable papers of Rogers [12–14] which not only contained analytical proofs of these identities but also contained other similar identities for the moduli 7, 10, 14, 15, 20 and 21. In 1919, in a joint paper, Rogers and Ramanujan [15] gave several proofs of these identities which are based on the general transformation formula:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1-aq^{2n})(a;q)_n}{(1-a)(q;q)_n} a^{2n} q^{(1/2)n(5n-1)} = (aq;q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n}, \tag{2}$$

proved by them.

Later, in 1929, Watson [22] gave an elegant and straightforward proof of these identities with the help of the following transformation formula connecting a terminating well-poised ${}_8\Phi_7$ and a terminating Saalschützian ${}_4\Phi_3$ series:

$$
{}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, c, d, e, f, q^{-n}; q; a^2q^{2+n}/cdef \\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, aq^{n+1} \end{matrix}\right]
$$

$$
= \frac{(aq, aq/ef; q)_n}{(aq/e, aq/f; q)_n} {}_4\Phi_3\left[\begin{matrix} aq/cd, e, f, q^{-n}; q; q \\ aq/c, aq/d, efq^{-n}/a \end{matrix}\right], \qquad n \geqslant 0. \tag{3}
$$

In 1936, with the help of certain difference-equations, Selberg [17] obtained, besides a number of other identities, the Rogers–Ramanujan identities (1) by means of his auxiliary function

$$
C_{k,r}(x; q) = (xq; q)_\infty Q_{k,r}(x; q), \tag{4}
$$

in which

$$
Q_{k,r}(x; q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(1/2)(2k+1)n(n+1)-rn} x^{kn}(1 - x^r q^{(2n+1)r})}{(q; q)_n (xq^{n+1}; q)_\infty}, \tag{5}
$$

where k is real and $> -\frac{1}{2}$.

In 1947, Bailey [6, 7] outlined a technique of obtaining a large variety of transformations of basic hypergeometric series from which he deduced known as well as new identities of the Rogers–Ramanujan type on different moduli by specializing the parameters suitably. Shortly afterwards, Slater [19, 20] made a systematic use of Bailey's technique to give a list of 130 identities of the Rogers–Ramanujan type involving prime factors 2, 3, 5 and 7 in the moduli.

A generalization of Rogers–Ramanujan type of identities in a different direction was given by Alder [3] in 1954. He used Selberg's auxiliary function (4) to prove the following generalizations of the Rogers–Ramanujan identities (1):

$$
\sum_{n=0}^{\infty} \frac{G_{k,n}(q)}{(q; q)_n} = \prod_{n=1}^{\infty} (1 - q^n)^{-1}, \tag{6}
$$

$$
n \not\equiv 0, \pm k \pmod{2k+1}
$$

$$
\sum_{n=0}^{\infty} \frac{G_{k,n}(q) q^n}{(q; q)_n} = \prod_{n=1}^{\infty} (1 - q^n)^{-1}, \tag{7}
$$

$$
n \not\equiv 0, \pm 1 \pmod{2k+1}
$$

where $G_{k,n}(q)$, $k \geqslant 2$, are certain polynomials which reduce to q^{n^2} for $k = 2$, the Rogers–Ramanujan case. Singh [18] extended these results of Alder by giving r-generalizations of the above two identities with the help of a transformation theorem for basic hypergeometric series given by Sears ([16]; §4).

In 1974, Andrews [4] obtained another analytic generalization of the Rogers–Ramanujan identities (1) with the help of Selberg's auxiliary function (4) by using the q-difference equations

$$
Q_{k,r}(x; q) - Q_{k,r-1}(x; q) = x^{r-1} q^{r-1} Q_{k,k-r+1}(xq; q), \quad 0 < r \leqslant k,
$$

iteratively. Later, he [5] considered the auxiliary function

$$\sum_{n\geqslant 0}\frac{(a,b,c;q)_n(1-aq^{2n})(-a^kq^k/bc)^nq^{(2k-1)n(n-1)/2}}{(1-a)(q,aq/b,aq/c;q)_n}, \tag{8}$$

and showed that it is equal to

$$\frac{(aq,aq/bc;q)_\infty}{(aq/b,aq/c;q)_\infty}\sum_{m_1,\ldots,m_{k-1}\geqslant 0}\frac{(b,c;q)_{m_1+\ldots+m_{k-1}}}{(q;q)_{m_1}\cdots(q;q)_{m_{k-1}}}\left(\frac{q}{bc}\right)^{m_1+\ldots+m_{k-1}}$$
$$\times a^{(k-1)m_1+\ldots+2m_{k-2}+m_{k-1}}q^{m_1^2+(m_1+m_2)^2+\ldots+(m_1+\ldots+m_{k-2})^2}. \tag{9}$$

In 1980, Bressoud [8] also obtained an analytic generalization of the Rogers–Ramanujan identities (1) by considering the following auxiliary function for $0<r\leqslant k$ with $M_i=m_i+m_{i+1}+\ldots+m_{k-1}$:

$$C_{k,r}(b^{-1},c^{-1};a;q)=\frac{1}{(a;q)_\infty}\sum_{n\geqslant 0}\frac{(a,b,c;q)_n(-1)^n}{(q,aq/b,aq/c;q)_n}(1-a^rq^{2rn})a^{kn}$$
$$\times(bc)^{-n}q^{(1/2)\{n^2(2k-1)+n(3-2r)\}}, \tag{10}$$

$$=\sum_{m_1,\ldots,m_{k-1}\geqslant 0}\frac{(aq/bc;q)_{m_{k-1}}a^{M_1+\ldots+M_{k-1}}q^{-(M_1+\ldots+M_{r-1})+M_1^2+\ldots+M_{k-1}^2}}{(q;q)_{m_1}\cdots(q;q)_{m_{k-1}}(aq/b,aq/c;q)_{m_{k-1}}}, \tag{11}$$

and also gave a combinatorial interpretation of these identities in the following form:

Let δ,r,k be integers satisfying $\delta=0$ or $1, 0<r<(2k+\delta)/2$. Let $B_{k,r,\delta}(n)$ denote the number of partitions of n such that, if f_i denotes the number of times i appears as a part in the partition, then $f_i\leqslant r-1, f_i+f_{i+1}\leqslant k-1$ for all i and $f_i+f_{i+1}=k-1$ implies that $if_i+(i+1)f_{i+1}\equiv r-1 \pmod{2-\delta}$. Also let $A_{k,r,\delta}(n)$ denote the number of partitions of n in which no part is $\equiv 0, \pm r \pmod{2k+\delta}$. Then, for each positive integer n,

$$A_{k,r,\delta}(n)=B_{k,r,\delta}(n). \tag{12}$$

In another paper, Bressoud [9] gave a further analytic generalization of the Rogers–Ramanujan identities by using the following auxiliary function:

$$H_{\lambda,k,r}((b_\lambda);a);q)=\frac{((aq/b_\lambda);q)_\infty}{(a;q)_\infty}\sum_{n\geqslant 0}\frac{(-1)^{n(\lambda+1)}a^{kn}(b_1\ldots b_\lambda)^{-n}}{(q;q)_n}$$
$$\times\frac{q^{(1/2)(2k-\lambda+1)n^2+(1/2)(\lambda+1-2r)n}(a,(b_\lambda);q)_n(1-a^rq^{2nr})}{((aq/b_\lambda);q)_n}. \tag{13}$$

He also proved that

$$H_{2k-1,k,1}(b_1,\ldots,b_{2k-1};a;q)$$
$$=\sum_{m_1,\ldots,m_{k-1}\geqslant 0}\frac{a^{M_1+\ldots+M_{k-1}}q^{M_1^2+\ldots+M_{k-1}^2}(q^{1-M_1}/b_1;q)_{M_1}}{(q;q)_{m_1}\cdots(q;q)_{m_{k-1}}}$$

$$\times \prod_{s=2}^{k} \{(q^{1-M_s}/b_s, q^{1-M_s}/b_{2k+1-s}; q)_{M_s}(aq/b_s b_{2k+1-s}; q)_{M_{s-1}}$$

$$(aq^{1+M_{s-1}}/b_s, aq^{1+M_{s-1}}/b_{2k+1-s}; q)_\infty\}, \qquad (14)$$

and gave a very general combinatorial interpretation of the identities obtained by him.

A close examination of the auxiliary functions (from Rogers–Ramanujan to Bressoud) stated above raises some very natural questions of the type:

(i) Is it necessary to take a and a^r, simultaneously, in the auxiliary function as has been done by Bressoud?
(ii) Is it necessary to take two related bases q and q^r instead of two general unconnected bases q and q_1?

Since the general transformation theory for basic hypergeometric series with two unconnected bases has already been developed in 1967 by Agarwal and Verma [1, 2], the object of the present paper is to establish a general bibasic transformation formula similar to (14) with a parameter λ in place of a^r and then discuss a few interesting particular and limiting cases of this transformation.

2. Notation

For $|q| < 1$, let

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)\ldots(1-aq^{n-1}), \quad n \geqslant 1;$$

$$(a_1, \ldots, a_r; q)_n = (a_1; q)_n \ldots (a_r; q)_n; (a;q)_\infty = \prod_{n=1}^{\infty} (1-aq^{n-1}),$$

$$(a_1, \ldots, a_r; q)_\infty = (a_1; q)_\infty \ldots (a_r; q)_\infty.$$

A generalized multibasic hypergeometric series, whenever it converges, is defined as

$${}_R\Phi_S^{(m+1)}\left[\begin{matrix} a:c_1:\ldots:c_m \\ b:d_1:\ldots:d_m \end{matrix}; q, q_1, \ldots, q_m : z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n Z^n}{(q,b;q)_n}\{(-1)^n q^{n(n-1)/2}\}^{1+s-r}$$

$$\times \prod_{t=1}^{m} \frac{(c_t;q_t)_n}{(d_t;q_t)_n}\{(-1)^n q_t^{n(n-1)/2}\}^{s_t - r_t},$$

where $R = r + r_1 + \ldots + r_m$, $S = s + s_1 + \ldots + s_m$, $a = (a_1, \ldots, a_r)$, $b = (b_1, \ldots, b_s)$, $c_t = (c_{t,1}, \ldots, c_{t,r_t})$, $d_t = (d_{t,1}, \ldots, d_{t,s_t})$.

The superscript $(m+1)$ in the Φ-symbol denotes the number of bases in the series.

3.

We shall first prove the following general bibasic transformation formula:

Theorem. *For* $|q|<1, |q_1|<1$,

$$
{}_{2k+2m+4}\Phi^{(2)}_{2k+2m+3}\left[\begin{matrix} a, b, c, a_1, \ldots, a_{2k-1} & : & q_1\sqrt{\lambda}, -q_1\sqrt{\lambda}, \\ aq/b, aq/c, aq/a_1, \ldots, aq/a_{2k-1} & : & \sqrt{\lambda}, -\sqrt{\lambda}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} b_1, \ldots, b_{2m} \\ \lambda q_1/b_1, \ldots, \lambda q_1/b_{2m} \end{matrix}; q, q_1; \frac{\lambda^{m+k/r} q^{1+k} q_1^{m-1}}{bca_1 \ldots a_{2k-1} b_1 \ldots b_{2m}}\right]
$$

$$
= \sum_{m_1, \ldots, m_{k-1} \geqslant 0} \frac{(aq/bc; q)_{m_{k-1}} (aq/a_1 a_2; q)_{m_{k-2}} \ldots (aq/a_{2k-5} a_{2k-4}; q)_{m_1}}{(q; q)_{m_1} \ldots (q; q)_{m_{k-1}}}
$$

$$
\times \frac{(a_1, a_2; q)_{M_{k-1}} \ldots (a_{2k-5}, a_{2k-4}; q)_{M_2} (a_{2k-3}, a_{2k-2}, a_{2k-1}; q)_{M_1}}{(aq/b, aq/c; q)_{M_{k-1}} (aq/a_1, aq/a_2; q)_{M_{k-2}} \ldots (aq/a_{2k-5}, \ldots, aq/a_{2k-1}; q)_{M_1}}
$$

$$
\times \frac{(a; q)_{2M_1} (q_1\sqrt{\lambda}, -q_1\sqrt{\lambda}, b_1, \ldots, b_{2m}; q_1)_{M_1} (-1)^{M_1} \lambda^{(m+k/r)M_1}}{(\sqrt{\lambda}, -\sqrt{\lambda}, \lambda q_1/b_1, \ldots, \lambda q_1/b_{2m}; q_1)_{M_1} a^{(k-1)M_1 - M_2 - \ldots - M_{k-1}}}
$$

$$
\times \frac{q^{-(1/2)M_1^2 + (5/2)M_1 + M_2 + \ldots + M_{k-1}} q_1^{(m-1)M_1}}{(a_1 a_2)^{M_{k-1}} \ldots (a_{2k-5} a_{2k-4})^{M_2} (a_{2k-3} a_{2k-2} a_{2k-1})^{M_1} (b_1 \ldots b_{2m})^{M_1}}
$$

$$
{}_{2m+6}\Phi^{(2)}_{2m+5}\left[\begin{matrix} aq^{2M_1}, a_{2k-3} q^{M_1}, a_{2k-2} q^{M_1}, a_{2k-1} q^{M_1} & : & q_1^{1+M_1}\sqrt{\lambda}, \\ aq^{1+M_1}/a_{2k-3}, aq^{1+M_1}/a_{2k-2}, aq^{1+M_1}/a_{2k-1} & : & q_1^{M_1}\sqrt{\lambda}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} -q_1^{1+M_1}\sqrt{\lambda}, b_1 q_1^{M_1}, \ldots, b_{2m} q_1^{M_1} \\ -q_1^{M_1}\sqrt{\lambda}, \lambda q_1^{(1+M_1)}/b_1, \ldots, \lambda q_1^{(1+M_1)}/b_{2m} \end{matrix}; q, q_1; \right.
$$

$$
\left.\frac{\lambda^{m+k/r} q^{2-M_1} q_1^{m-1}}{a^{k-1} a_{2k-3} a_{2k-2} a_{2k-1} b_1 \ldots b_{2m}}\right], \qquad (15)
$$

where $M_i = m_i + m_{i+1} + \ldots + m_{k-1}$, m, r *and* k *are positive integers.*

Proof. By the q-analogue of Saalshütz's theorem ([10]; eqn. (1.7.2)), we have

$$
\frac{(b, c; q)_n}{(aq/b, aq/c; q)_n} \left(\frac{aq}{bc}\right)^n = \sum_{s=0}^{n} \frac{(aq/bc, aq^n, q^{-n}; q)_s}{(q, aq/b, aq/c; q)_s} q^s. \qquad (16)
$$

Using (16) in the left hand side of (15), one can easily write it in the form

$$
\sum_{m_{k-1}=0}^{\infty} \frac{(aq/bc; q)_{m_{k-1}} (-1)^{m_{k-1}}}{(q, aq/b, aq/c; q)_{m_{k-1}}} q^{m_{k-1}(m_{k-1}+1)/2}
$$

$$
\times \sum_{n=m_{k-1}}^{\infty} \frac{(a; q)_{n+m_{k-1}} (a_1, \ldots, a_{2k-1}; q)_n}{(q; q)_{n-m_{k-1}} (aq/a_1, \ldots, aq/a_{2k-1}; q)_n}
$$

$$
\times \frac{(q_1\sqrt{\lambda}, -q_1\sqrt{\lambda}, b_1, \ldots, b_{2m}; q_1)_n \lambda^{n(m+k/r)} q^{n(k-m_{k-1})} q_1^{n(m-1)}}{(\sqrt{\lambda}, -\sqrt{\lambda}, \lambda q_1/b_1, \ldots, \lambda q_1/b_{2m}; q_1)_n (aa_1 \ldots a_{2k-1} b_1 \ldots b_{2m})^n}.
$$

Putting $n = m_{k-1} + t$, the last expression is equal to

$$
\sum_{m_{k-1}=0}^{\infty} \frac{(aq/bc, a_1, \ldots, a_{2k-1}; q)_{m_{k-1}} (a; q)_{2m_{k-1}}}{(q, aq/b, aq/c, aq/a_1, \ldots, aq/a_{2k-1}; q)_{m_{k-1}}}
$$

$$\times \frac{(q_1\sqrt{\lambda}, -q_1\sqrt{\lambda}, b_1, \ldots, b_{2m}; q_1)_{m_{k-1}}}{(\sqrt{\lambda}, -\sqrt{\lambda}, \lambda q_1/b_1, \ldots, \lambda q_1/b_{2m}; q_1)_{m_{k-1}}}$$

$$\times q^{m_{k-1}(m_{k-1}+1)/2}\left\{-\frac{\lambda^{m+k/r}q^{k-m_{k-1}}q_1^{m-1}}{a\,a_1\ldots a_{2k-1}b_1\ldots b_{2m}}\right\}^{m_{k-1}}$$

$$\times {}_{2k+2m+2}\Phi^{(2)}_{2k+2m+1}\left[\begin{matrix} aq^{2m_{k-1}}, a_1q^{m_{k-1}}, \ldots, a_{2k-1}q^{m_{k-1}} : q_1^{1+m_{k-1}}\sqrt{\lambda}, -q_1^{1+m_{k-1}}\sqrt{\lambda}, \\ aq^{1+m_{k-1}}/a_1, \ldots, aq^{1+m_{k-1}}/a_{2k-1} : q_1^{m_{k-1}}\sqrt{\lambda}, -q_1^{m_{k-1}}\sqrt{\lambda}, \end{matrix}\right.$$

$$\left.\begin{matrix} b_1q_1^{m_{k-1}}, \ldots, b_{2m}q_1^{m_{k-1}} \\ \lambda q_1^{1+m_{k-1}}/b_1, \ldots, \lambda q_1^{1+m_{k-1}}/b_{2m} \end{matrix}; q, q_1; \frac{\lambda^{m+k/r}q^{k-m_{k-1}}q_1^{m-1}}{a\,a_1\ldots a_{2k-1}b_1\ldots b_{2m}}\right]. \tag{17}$$

We now iterate the procedure used in transforming the left hand side of (15) to the form (17) (such that the parameters b, c are "shifted out" from the Φ series). Then, after $(k-2)$ iterations, we find that the left hand side of (15) can be written in the following form:

$$\sum_{m_1,\ldots,m_{k-1}=0}^{\infty} \frac{(aq/bc;q)_{m_{k-1}}(aq/a_1a_2;q)_{m_{k-2}}\ldots(aq/a_{2k-5}a_{2k-4};q)_{m_1}}{(q;q)_{m_1}\ldots(q;q)_{m_{k-1}}}$$

$$\times \frac{(a_1,a_2;q)_{m_{k-1}}(a_3,a_4;q)_{m_{k-2}+m_{k-1}}\ldots(a_{2k-5},a_{2k-4};q)_{m_{k-1}+\cdots+m_2}}{(aq/b,aq/c;q)_{m_{k-1}}(aq/a_1,aq/a_2;q)_{m_{k-1}+m_{k-2}}\cdots(aq/a_{2k-5},\ldots,aq/a_{2k-1};q)_{m_{k-1}+\cdots+m_1}}$$

$$\times \frac{(a_{2k-3},a_{2k-2},a_{2k-1};q)_{m_{k-1}+\ldots+m_1}(a;q)_{2m_{k-1}+\ldots+2m_1}(-q_1^{m-1})^{m_1+\ldots+m_{k-1}}}{(\sqrt{\lambda},-\sqrt{\lambda},\lambda q_1/b_1,\ldots,\lambda q_1/b_{2m};q_1)_{m_{k-1}+\ldots+m_1}}$$

$$\times \frac{(q_1\sqrt{\lambda},-q_1\sqrt{\lambda},b_1,\ldots,b_{2m};q_1)_{m_{k-1}+\ldots+m_1}q^{(m_1/2)(m_1+1)+\ldots+(m_{k-1}/2)(m_{k-1}+1)}}{a^{m_{k-1}+2m_{k-2}+\ldots+(k-1)m_1}(a_1a_2)^{m_{k-1}}(a_3a_4)^{m_{k-1}+m_{k-2}}\ldots(a_{2k-5}a_{2k-4})^{m_{k-1}+\ldots+m_2}}$$

$$\times \frac{\lambda^{(m+k/r)(m_{k-1}+\ldots+m_1)}q^{(k-m_{k-1})m_{k-1}+(k-1-m_{k-1}-m_{k-2})m_{k-2}+\ldots+(2-m_{k-1}-\ldots-m_1)m_1}}{(a_{2k-3}a_{2k-2}a_{2k-1})^{m_{k-1}+\ldots+m_1}}$$

$$\times \frac{q_1^{(m-1)(m_{k-1}+\ldots+m_1)}}{(b_1\ldots b_{2m})^{m_{k-1}+\ldots+m_1}}$$

$$\times {}_{2m+6}\Phi^{(2)}_{2m+5}\left[\begin{matrix} aq^{2m_{k-1}+\ldots+2m_1}, a_{2k-3}q^{m_{k-1}+\ldots+m_1}, a_{2k-2}q^{m_{k-1}+\cdots+m_1}, \\ aq^{1+m_{k-1}+\ldots+m_1}/a_{2k-3}, aq^{1+m_{k-1}+\ldots+m_1}/a_{2k-2}\lambda, \\ a_{2k-1}q^{m_{k-1}+\ldots+m_1} : q_1^{(1+m_{k-1}+\ldots+m_1)}\sqrt{\lambda}, -q_1^{1+m_{k-1}+\ldots+m_1}\sqrt{\lambda}, \\ aq^{1+m_{k-1}+\ldots+m_1}/a_{2k-1} : q_1^{(m_{k-1}+\ldots+m_1)}\sqrt{\lambda}, -q_1^{(m_{k-1}+\ldots+m_1)}\sqrt{\lambda}, \\ b_1q_1^{m_{k-1}+\ldots+m_1}, \ldots, b_{2m}q_1^{m_{k-1}+\ldots+m_1} \\ \lambda q_1^{1+m_{k-1}+\ldots+m_1}/b_1, \ldots, \lambda q_1^{1+m_{k-1}+\ldots+m_1}/b_{2m} \end{matrix}; \right.$$

$$\left. q, q_1; \frac{\lambda^{m+k/r}q^{k-m_{k-1}-\ldots-m_1}q_1^{m-1}}{a^{k-1}q^{k-2}a_{2k-3}a_{2k-2}a_{2k-1}b_1\ldots b_{2m}}\right].$$

Introducing the summatory symbols M_i, it is easy to see that the last expression is equivalent to the right hand side of (15).

4. Particular cases

We shall now discuss a few interesting particular and limiting cases of the above transformation (15).

Case I. Let us take $k=2$, $m=0$, $r=1$, $\lambda=a$ and $q_1=q$ in (15) and make $b, c, a_1, a_2, a_3 \to \infty$. Then, with the help of a well-poised ${}_6\Phi_5$ summation formula ([10]; eq. (2.7.1)) and some simplification, we get the transformation (2) proved earlier by Rogers and Ramanujan [15].

Case II. We now consider the following auxiliary function which is a multiple of the left hand side of (15):

$$C^*_{k,r}(b^{-1}, c^{-1}, (a_{2k-1}), (b_{2m}); a, \lambda; q, q_1)$$
$$= \frac{(1-\lambda)}{(a;q)_\infty} {}_{2k+2m+4}\Phi^{(2)}_{2k+2m+3}\left[\begin{matrix} a, b, c, (a_{2k-1}): \\ aq/b, aq/c, (aq/a_{2k-1}): \end{matrix}\right.$$
$$\left.\begin{matrix} q_1\sqrt{\lambda}, -q_1\sqrt{\lambda}, (b_{2m}) \\ \sqrt{\lambda}, -\sqrt{\lambda}, (\lambda q_1/b_{2m}) \end{matrix}; q, q_1; \frac{\lambda^{m+k/r} q^{1+r} q_1^{m-1}}{bc a_1 \ldots a_{2k-1} b_1 \ldots b_{2m}}\right]. \tag{18}$$

If we first take $m=0$, $\lambda=a^r$, $q_1=q^r$ in (18) and then make $b, c, a_1, \ldots, a_{2k-1} \to \infty$, we get

$$C^*_{k,r}(a;q) = \frac{1}{(a;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (1-a^r q^{2rn})(-a^k)^n q^{(1/2)(2k+1)n^2+(n/2)-rn}, \tag{19}$$

which is equivalent to Selberg's auxiliary function (5).

Case III. If we take $m=0$, $\lambda=a$, $q_1=q, r=1$ and $a_{2k-1}=q^{-N}$ in (15), then the inner series on the right hand side of it can be summed up by a well-poised ${}_6\Phi_5$ summation formula ([10]; eq. (2.7.1)). We easily get the following identity:

$${}_{2k+4}\Phi_{2k+3}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, a_1, \ldots, a_{2k-1}, q^{-N}; \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/a_1, \ldots, aq/a_{2k-1}, aq^{N+1} \end{matrix}; q; \frac{a^k q^{k+N}}{bc a_1 \ldots a_{2k-1}}\right]$$
$$= \sum_{m_1, \ldots, m_{k-1} \geqslant 0} \frac{(aq/bc;q)_{m_{k-1}} (aq/a_1 a_2; q)_{m_{k-2}} \ldots (aq/a_{2k-5} a_{2k-4}; q)_{m_1}}{(q;q)_{m_1} \ldots (q;q)_{m_{k-1}}}$$
$$\times \frac{(a_1, a_2; q)_{m_{k-1}} (a_3, a_4; q)_{m_{k-1}+m_{k-2}} \ldots (a_{2k-3}, a_{2k-2}; q)_{m_{k-1}+\ldots+m_1}}{(aq/b, aq/c; q)_{m_{k-1}} (aq/a_1, aq/a_2; q)_{m_{k-2}+m_{k-1}} \ldots (aq/a_{2k-5}, aq/a_{2k-4}; q)_{m_{k-1}+\ldots+m_1}}$$
$$\times \frac{(q^{-N}; q)_{m_{k-1}+\ldots+m_1} (aq)^{(k-2)m_{k-1}+\ldots+2m_3+m_2} q^{m_{k-1}+\ldots+m_1}}{(a_{2k-3} a_{2k-2} q^{-N}/a; q)_{m_{k-1}+\ldots+m_1} (a_1 a_2)^{m_{k-1}} (a_3 a_4)^{m_{k-1}+m_{k-2}} \ldots (a_{2k-5} a_{2k-4})^{m_{k-1}+\ldots+m_1}}, \tag{20}$$

which is seen to be equivalent to the identity ([15]; Theorem 4) due to Andrews.

Case IV. Again, let us take $\lambda = a^r, q_1 = q^r (0 < r \leqslant k)$ in (15) and make

$$a_{2k-2}, a_{2k-1} \to \infty,$$
$$b_1, \ldots, b_m \to \infty,$$
$$b_{m+1}, \ldots, b_{2m} \to 0.$$

The inner series on the right hand side of (15) is then summable by a well-poised ${}_6\Phi_5$ summation formula ([10]; eq. (2.7.1)) and we get the following transformation:

$$\begin{aligned} & H_{2k-1,k,1}(b, c, a_1, \ldots, a_{2k-3}; a; q) \\ &= \sum_{m_1, \ldots, m_{k-1} \geqslant 0} \frac{a^{M_1 + \ldots + M_{k-1}} q^{M_1^2 + \ldots + M_{k-1}^2} (q^{1-M_1}/a_{2k-3}; q)_{M_1}}{(q;q)_{m_1} \ldots (q;q)_{m_{k-1}}} \\ &\quad \times (q^{1-M_{k-1}}/a_1, q^{1-M_{k-1}}/a_2; q)_{M_{k-1}} (q^{1-M_{k-2}}/a_3, q^{1-M_{k-2}}/a_4; q)_{M_{k-2}} \\ &\qquad \ldots (q^{1-M_2}/a_{2k-5}, q^{1-M_2}/a_{2k-4}; q)_{M_2} \\ &\quad \times (aq^{1+M_{k-1}}/b, aq^{1+M_{k-1}}/c; q)_\infty (aq^{1+M_{k-2}}/a_1, aq^{1+M_{k-2}}/a_2; q)_\infty \\ &\qquad \ldots (aq^{1+M_1}/a_{2k-5}, aq^{1+M_1}/a_{2k-4}; q)_\infty \\ &\quad \times (aq/bc; q)_{m_{k-1}} (aq/a_1 a_2; q)_{m_{k-2}} \ldots (aq/a_{2k-5} a_{2k-4}; q)_{m_1}. \end{aligned} \tag{21}$$

The transformation (21) is easily seen to be equivalent to Bressoud's transformation (14).

If, in addition to these changes, we make

$$a_1, \ldots, a_{2k-3} \to \infty$$

in (18), we get the auxiliary function (10) due to Bressoud. However, if we make all these changes in (15), then, on making use of a well-poised ${}_6\Phi_5$ summation formula ([10]; eq. (2.7.1)), we get the transformation (11) due to Bressoud.

Case V(a). Let us take

$$k = p - 2,\ m = 2,\ r = 1,\ \lambda = a,\ q_1 = q,$$
$$a_{2p-7} = e,\ a_{2p-6} = x,\ a_{2p-5} = -x,$$
$$b_1 = y,\ b_2 = -y,\ b_3 = -q^{-N},\ b_4 = q^{-N}$$

in (15), and transform the inner series on the right hand side of the resulting transformation by another transformation ([21]; eq. (1.3)). Then, we easily get a general transformation which is seen to be equivalent to the result ([21]; eq. (4.1)) due to Verma and Jain.

In (15), let us first replace q by q^2 and then take

$$k = p - 2,\ \lambda = a,\ q_1 = q^2,\ m = 2,\ r = 1,$$
$$a_{2p-7} = e,\ a_{2p-6} = x,\ a_{2p-5} = xq,$$
$$b_1 = y,\ b_2 = yq,\ b_3 = q^{-N+1},\ b_4 = q^{-N}.$$

We can now transform the resulting inner series in (15) by the transformation formula

([21]; eq. (1.4)). We thus get a general transformation which is easily seen to be equivalent to the result ([21]; eq. (4.3)) due to Verma and Jain.

Case V(*b*). Let us now take

$$k = p-3,\ m = 3,\ r = 1,\ \lambda = a,\ q_1 = q,$$
$$a_{2p-9} = x,\ a_{2p-8} = \omega x,\ a_{2p-7} = \omega^2 x,$$
$$b_1 = y,\ b_2 = \omega y,\ b_3 = \omega^2 y,$$
$$b_4 = q^{-N},\ b_5 = \omega q^{-N},\ b_6 = \omega^2 q^{-N}$$

in (15), and then transform the inner series on the right hand side of the resulting transformation by another transformation ([21]; eq. (1.5)), we thus get a general transformation equivalent to the result ([21]; eq. (4.4)) due to Verma and Jain.

Next, we first replace q by q^3 in (15) and set

$$k = p-3,\ m = 3,\ r = 1,\ \lambda = a,\ q_1 = q^3,$$
$$a_{2p-9} = x,\ a_{2p-8} = xq,\ a_{2p-7} = xq^2,$$
$$b_1 = y,\ b_2 = yq,\ b_3 = yq^2,$$
$$b_4 = q^{-N},\ b_5 = q^{-N+1},\ b_6 = q^{-N+2}.$$

Then, by using the transformation formula ([21]; eq. (1.6)), we get a general transformation which is also seen to be equivalent to the result ([21]; eq. (4.5)) due to Verma and Jain.

Case VI. In (15), let us replace q and q_1 by q^2 and q^{2N+1}, respectively, where N is a positive integer and make

$$b, c, a_1, \ldots, a_{2k-1} \to \infty, c_1, \ldots, c_{2m} \to \infty. \tag{22}$$

Then, on setting $a = q^2$, we easily get the following interesting identity which is believed to be new:

$$\sum_{n, m_1, \ldots, m_{k-1} \geq 0} \frac{(q^{2+2n}; q^2)_{2M_1}(1 - \lambda q^{(2N+1)(2M_1+2n)}}{(q^2; q^2)_{m_1} \ldots (q^2; q^2)_{m_{k-1}}} \{-\lambda^{m+k/r}\}^{M_1+n}$$
$$\times q^{2\{M_2^2 + \ldots + M_{k-1}^2 + M_2 + \ldots + M_{k-1}\} + (2Nm+m+2)M_1^2 - (2N+2k-3)M_1}$$
$$\times q^{(3+2Nm+m)n^2 + 2\{2M_1 + (2N+1)M_1 m - N - k + 1\}n}$$
$$= \lim_{a,b \to \infty} {}_3\Phi_2 \left[\begin{matrix} q^*, a, b; q^*; \dfrac{-\lambda^{m+k/r} q^* q^{-2N}}{ab} \\ 0, 0 \end{matrix} \right]$$
$$- \lambda \lim_{a,b \to \infty} {}_3\Phi_2 \left[\begin{matrix} q^*, a, b; q^*; \dfrac{-\lambda^{m+k/r} q^* q^{2N+2}}{ab} \\ 0, 0 \end{matrix} \right], \tag{23}$$

where $q^* = q^{2k+2mN+m+1}$.

Again, if we replace q and q_1 by q^3 and q^4, respectively, in (15), take the limits indicated in (22) and then set $a = q^3$, we obtain the following interesting identity which is also believed to be new:

$$\sum_{n,m_1,\ldots,m_{k-1}\geq 0} \frac{(q^{3\div 3n};q^3)_{2M_1}(1-\lambda q^{8M_1+8n})}{(q^3;q^3)m_1\ldots(q^3;q^3)m_{k-1}} \{-\lambda^{m+k/r}\}^{M_1+n}$$
$$\times q^{3\{M_2^2+\ldots+M_{k-1}^2+M_2+\ldots+M_{k-1}\}+(3+4m)M_1^2-(3k-2)M_1(4m+9/2)n^2}$$
$$\times q^{(8mM_1+6M_1-3k+1/2)n}$$
$$= \lim_{a,b\to\infty} {}_3\Phi_2\left[\begin{matrix} q',a,b;q';\dfrac{-\lambda^{m+k/r}q'q^{(5/2)}}{ab} \\ 0,0 \end{matrix}\right]$$
$$-\lambda \lim_{a,b\to\infty} {}_3\Phi_2\left[\begin{matrix} q',a,b;q';\dfrac{-\lambda^{m+k/r}q'q^{11/2}}{ab} \\ 0,0 \end{matrix}\right], \qquad (24)$$

where $q' = q^{3k+4m+3/2}$.

5. Conclusion

We have not tried to list all the special cases of our general result but have only drawn attention to the fact that multidimensional transformations of bibasic hypergeometric series perhaps provide the best way of unifying the enormous number of partition – theoretic analytical identities. We hope to exploit this viewpoint in a future communication.

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Some theorems on the general summability methods

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Abstract. In this paper a new theorem which covers many methods of summability is proved. Several results are also deduced.

Keywords. Summability methods.

1. Introduction

Let Σa_n be an infinite series with partial sums s_n. Let σ_n^δ and η_n^δ denote the nth Cesàro mean of order $\delta(\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$, respectively. The series Σa_n is said to be summable (C, δ) with index k, or simply summable $|C, \delta|_k$, $k \geqslant 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty,$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1} |\eta_n^\delta|^k < \infty.$$

Let $\{p_n\}$ be a sequence of real or complex constants with

$$P_n = p_0 + p_1 + p_2 \cdots + p_n, \quad p_{-r} = P_{-r} = 0, \quad r = 1, 2, \ldots.$$

The series Σa_n is said to be summable $|N, p_n|$, if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty, \tag{1}$$

where

$$t_n = P_n^{-1} \sum_{v=0}^{n} p_{n-v} s_v \quad (t_{-1} = 0).$$

We write $p = \{p_n\}$ and

$$M = \left\{ p : p_n > 0 \quad \text{and} \quad \frac{p_{n+1}}{p_n} \leqslant \frac{p_{n+2}}{p_{n+1}} \leqslant 1, \quad n = 0, 1, \ldots \right\}$$

It is known that for $p \in M$, (1) holds if and only if (Das [4])

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right| < \infty.$$

DEFINITION 1 (Sulaiman [5])

For $p \in M$, we say that Σa_n is summable $|N, p_n|_k$, $k \geqslant 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right|^k < \infty.$$

In the special case in which $p_n = A_n^{r-1}$, $r > -1$, where A_n^r is the coefficient of x^n in the power series expansion of $(1-x)^{-r-1}$ for $|x| < 1$, $|N, p_n|_k$ summability reduces to $|C, r|_k$ summability.

The series Σa_n is said to be summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_n - T_{n-1}|^k < \infty \quad \text{(Bor [1])}$$

where

$$T_n = P_n^{-1} \sum_{v=0}^{n} p_v s_v.$$

If we take $p_n = 1$, then $|\bar{N}, p_n|_k$ summability is equivalent to $|C, 1|_k$ summability. In general, these two summabilities are not comparable.

We set

$$\Delta f_n = f_n - f_{n+1}$$

$$Q_n = q_0 + q_1 + \cdots + q_n, \quad q_{-1} = Q_{-1} = 0$$

$$U_n = u_0 + u_1 + \cdots + u_n, \quad u_{-1} = U_{-1} = 0$$

$$V_n = v_0 + v_1 + \cdots + v_n, \quad v_{-1} = V_{-1} = 0$$

$$R_n = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0$$

$$W_n = u_0 v_n + u_1 v_{n-1} + \cdots + u_n v_0$$

and assume that P_n, U_n, R_n and W_n all tend to ∞.

DEFINITION 2 (Sulaiman [6])

Let $\{p_n\}$, $\{q_n\}$ be sequences of positive real constants such that $q \in M$. We say that Σa_n is summable $|N, p_n, q_n|_k$, $k \geqslant 1$, if

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n R_{n-1}^k} \left| \sum_{v=1}^{n} P_{v-1} q_{n-v} a_v \right|^k < \infty.$$

Clearly $|N, p_n, 1|_k$ and $|N, 1, q_n|_k$ are equivalent to $|\bar{N}, p_n|_k$ and $|N, q_n|_k$ respectively. We prove the following:

Theorem 1. *Let* $\{p_n\}$, $\{q_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *be sequences of positive real constants such that* $q, v \in M$, $q_n = O(v_n)$, $\{p_n/P_n R_{n-1}^k v_n^k\}$ *nonincreasing and that* $a_n \geqslant 0$ *if* $v_n \neq c$. *Suppose* $\{\varepsilon_n\}$ *is a sequence of constants and write* $W_{n-1} G_n = \Sigma_{r=1}^n U_{r-1} v_{n-r} a_r$. *If*

$$\sum_{n=r+1}^{\infty} \frac{p_r}{P_r R_{r-1}} \frac{q_{n-r-1}}{v_{n-r-1}^k} = O(1/P_r v_r^k), \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} \left(\frac{W_{n-1}}{v_n U_{n-1}} \right)^k |\varepsilon_n|^k |G_n|^k < \infty, \tag{3}$$

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left(\frac{u_n}{U_n} \right)^k \left(\frac{W_{n-1}}{v_n U_{n-1}} \right)^k |\varepsilon_n|^k |G_n|^k < \infty, \tag{4}$$

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left(\frac{W_{n-1}}{v_n U_{n-1}} \right)^k |\Delta \varepsilon_n|^k |G_n|^k < \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{p_r}{P_r} \left(\frac{P_{r-1}}{R_{r-1}} \right)^k \left(\frac{W_{r-1}}{v_r U_{r-1}} \right)^k |\varepsilon_r|^k |G_r|^k < \infty, \tag{6}$$

then the series $\Sigma a_n \varepsilon_n$ *is summable* $|N, p_n, q_n|_k$, $k \geqslant 1$.

2. Lemmas

Lemma 1 (Sulaiman [6]). *Let* $q \in M$, *then for* $0 < v \leqslant 1$,

$$\sum_{n=r}^{\infty} \frac{q_{n-r}}{n^v Q_r} = O(r^{-v}).$$

Lemma 2. $\{p_n/P_n R_{n-1}^k v_n^k\}$ *nonincreasing implies*

$$\sum_{n=r+1}^{m} \frac{p_n}{P_n R_{n-1}^k} \frac{|\Delta_r q_{n-r}|}{v_{n-r}^k} = O\left\{ \frac{p_r}{P_r R_{r-1}^k v_r^k} \sum_{n=1}^{m} |\Delta q_n| \right\}$$

Proof. Since

$$\frac{p_n}{P_n R_{n-1}^k} = \frac{p_n v_n^k}{P_n R_{n-1}^k v_n^k} \leqslant \frac{p_{n-1} v_n^k}{P_{n-1} R_{n-2}^k v_{n-1}^k} \leqslant \frac{p_n}{P_{n-1} R_{n-2}^k}$$

therefore $\{p_n/P_n R_{n-1}^k\}$ is nonincreasing. We have

$$\sum_{n=r+1}^{m} \frac{p_n}{P_n R_{n-1}^k} \frac{|\Delta_r q_{n-r}|}{v_{n-r}^k} = \left\{ \sum_{n=r+1}^{2r} + \sum_{n=2r+1}^{m} \right\} = J_1 + J_2, \quad \text{say}$$

$$J_1 = O\left\{ \frac{p_r}{P_r R_{r-1}^k} \right\} O\left(\frac{1}{v_r^k} \right) \sum_{n=r+1}^{2r} |\Delta_r q_{n-r}| = O\left\{ \frac{p_r}{P_r R_{r-1}^k v_r^k} \sum_{n=1}^{m} |\Delta q_n| \right\}.$$

$$J_2 = \sum_{u=r+1}^{m-r} \frac{p_{r+u}}{P_{r+u} R_{r+u-1}^k} \frac{|\Delta q_u|}{v_u^k} = O\left\{ \sum_{u=r+1}^{m} \frac{p_u}{P_u R_{u-1}^k} \frac{|\Delta q_u|}{v_u^k} \right\} =$$

$$O\left\{ \frac{p_r}{P_r R_{r-1}^k v_r^k} \sum_{u=1}^{m} |\Delta q_u| \right\}.$$

3. Proof of theorem 1

Write

$$F_n = \sum_{r=1}^{n} P_{r-1} q_{n-r} a_r \varepsilon_r,$$

then, by Abel's transformation

$$\begin{aligned}
F_n &= \sum_{r=1}^{n} U_{r-1} v_{n-r} a_r \left(\frac{P_{r-1}}{U_{r-1}} \frac{q_{n-r}}{v_{n-r}} \varepsilon_r \right) \\
&= \sum_{r=1}^{n-1} \left(\sum_{s=1}^{r} U_{s-1} v_{n-s} a_s \right) \Delta_r \left(\frac{P_{r-1}}{U_{r-1}} \frac{q_{n-r}}{v_{n-r}} \varepsilon_r \right) + W_{n-1} G_n \frac{P_{n-1}}{U_{n-1}} \frac{q_0}{v_0} \varepsilon_n \\
&\leqslant \sum_{r=1}^{n-1} W_{r-1} |G_r| \left\{ \frac{|\Delta_r q_{n-r}|}{v_{n-r}} \frac{P_{r-1}}{U_{r-1}} |\varepsilon_r| + q_{n-r-1} \left| \Delta_r \left(\frac{1}{v_{n-r}} \right) \right| \frac{P_{r-1}}{U_{r-1}} |\varepsilon_r| \right. \\
&\quad \left. + \frac{q_{n-r-1}}{v_{n-r-1}} \frac{p_r}{U_{r-1}} |\varepsilon_r| + \frac{q_{n-r-1}}{v_{n-r-1}} \frac{u_r P_r}{U_r U_{r-1}} |\varepsilon_r| + \frac{q_{n-r-1}}{v_{n-r-1}} \frac{P_r}{U_r} |\Delta \varepsilon_r| \right\} \\
&\quad + W_{n-1} |G_n| \frac{P_{n-1}}{U_{n-1}} \frac{q_0}{v_0} |\varepsilon_n| \\
&= F_{n,1} + F_{n,2} + F_{n,3} + F_{n,4} + F_{n,5} + F_{n,6}, \quad \text{say.}
\end{aligned}$$

In order to prove the theorem, by Minkowski's inequality, it is therefore suff
to show that

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n R_{n-1}^k} F_{n,r}^k < \infty, \quad r = 1, 2, 3, 4, 5, 6,$$

where $k > 1$. Applying Hölder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} F_{n,1}^k &= \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \left\{ \sum_{r=1}^{n-1} \frac{|\Delta_r q_{n-r}|}{v_{n-r}} \frac{P_{r-1}}{U_{r-1}} W_{r-1} |\varepsilon_r| |G_r| \right\}^k \\
&\leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \sum_{r=1}^{n-1} \frac{|\Delta_r q_{n-r}|}{v_{n-r}^k} \frac{P_{r-1}^k}{U_{r-1}^k} W_{r-1}^k |\varepsilon_r|^k |G_r|^k \\
&\quad \times \left\{ \sum_{r=1}^{n-1} |\Delta_r q_{n-r}| \right\}^{k-1} \\
&= O(1) \sum_{r=1}^{m} \frac{P_{r-1}^k}{U_{r-1}^k} W_{r-1}^k |\varepsilon_r|^k |G_r|^k \sum_{n=r+1}^{m+1} \frac{p_n}{P_n^k R_{n-1}^k} \frac{|\Delta_r q}{v_n^k} \\
&= O(1) \sum_{r=1}^{m} \frac{p_r}{P_r} \left(\frac{P_{r-1}}{R_{r-1}} \right)^k \left(\frac{W_{r-1}}{v_r U_{r-1}} \right)^k |\varepsilon_r|^k |G_r|^k = O(1
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} F_{n,2}^k &\leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n R_{n-1}^k} \sum_{r=1}^{n-1} \frac{q_{n-r-1}^k}{v_{n-r-1}^k} \frac{|\Delta_r v_{n-r}|}{v_{n-r}^k} \frac{P_{r-1}^k}{U_{r-1}^k} W_{r-1}^k |\varepsilon_r \\
&\quad \times \left\{ \sum_{r=1}^{n-1} |\Delta_r v_{n-r}| \right\}^{k-1}
\end{aligned}$$

$$= O(1)\sum_{r=1}^{m}\left(\frac{P_{r-1}}{U_{r-1}}\right)^k W_{r-1}^k|\varepsilon_r|^k|G_r|^k \sum_{n=r+1}^{m+1}\frac{p_n}{P_n R_{n-1}^k}\frac{|\Delta_r v_{n-r}|}{v_{n-r}^k}$$

$$= O(1)\sum_{r=1}^{m}\frac{p_r}{P_r}\left(\frac{P_{r-1}}{R_{r-1}}\right)^k\left(\frac{W_{r-1}}{v_r U_{r-1}}\right)^k|\varepsilon_r|^k|G_r|^k = O(1).$$

$$\sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}^k}F_{n,3}^k \leqslant \sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}}\sum_{r=1}^{n-1}\frac{q_{n-r-1}}{v_{n-r-1}^k}\frac{p_r}{U_{r-1}^k}W_{r-1}^k|\varepsilon_r|^k|G_r|^k$$
$$\times\left\{\sum_{r=1}^{n-1}\frac{p_r q_{n-r-1}}{R_{n-1}}\right\}^{k-1}$$

$$= O(1)\sum_{r=1}^{m}\frac{p_r}{U_{r-1}^k}W_{r-1}^k|\varepsilon_r|^k|G_r|^k\sum_{n=r+1}^{m+1}\frac{p_n}{P_n R_{n-1}}\frac{q_{n-r-1}}{v_{n-r-1}^k}$$

$$= O(1)\sum_{r=1}^{m}\frac{p_r}{P_r}\left(\frac{W_{r-1}}{v_r U_{r-1}}\right)^k|\varepsilon_r|^k|G_r|^k = O(1).$$

$$\sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}^k}F_{n,4}^k \leqslant \sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}^k}\sum_{r=1}^{n-1}p_r\frac{q_{n-r-1}}{v_{n-r-1}^k}\frac{P_r^k}{p_r^k}\frac{u_r^k}{U_r^k U_{r-1}^k}W_{r-1}^k|\varepsilon_r|^k|G_r|^k$$
$$\times\left\{\sum_{r=1}^{n-1}\frac{p_r q_{n-r-1}}{R_{n-1}}\right\}^{k-1}$$

$$= O(1)\sum_{r=1}^{m}p_r\frac{P_r^k}{p_r^k}\frac{u_r^k}{U_r^k U_{r-1}^k}W_{r-1}^k|\varepsilon_r|^k|G_r|^k$$
$$\times\sum_{n=r+1}^{m+1}\frac{p_n}{P_n R_{n-1}}\frac{q_{n-r-1}}{v_{n-r-1}^k}$$

$$= O(1)\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{k-1}\left(\frac{u_r}{U_r}\right)^k\left(\frac{W_{r-1}}{v_r U_{r-1}}\right)^k|\varepsilon_r|^k|G_r|^k = O(1).$$

$$\sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}^k}F_{n,5}^k \leqslant \sum_{n=2}^{m+1}\frac{p_n}{P_n R_{n-1}}\sum_{r=1}^{n-1}p_r\frac{q_{n-r-1}}{v_{n-r-1}^k}\frac{P_r^k}{p_r^k}\frac{W_{r-1}^k}{U_r^k}|\Delta\varepsilon_r|^k|G_r|^k$$
$$\times\left\{\sum_{r=1}^{n-1}\frac{p_r q_{n-r-1}}{R_{n-1}}\right\}^{k-1}$$

$$= O(1)\sum_{r=1}^{m}p_r\left(\frac{P_r}{p_r}\right)^k\left(\frac{W_{r-1}}{U_r}\right)^k|\Delta\varepsilon_r|^k|G_r|^k\sum_{n=r+1}^{m+1}\frac{p_n}{P_n R_{n-1}}\frac{q_{n-r-1}}{v_{n-r-1}^k}$$

$$= O(1)\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{k-1}\left(\frac{W_{r-1}}{v_r U_{r-1}}\right)^k|\Delta\varepsilon_r|^k|G_r|^k = O(1).$$

$$\sum_{n=1}^{m}\frac{p_n}{P_n R_{n-1}^k}F_{n,6}^k = \sum_{n=1}^{m}\frac{p_n}{P_n R_{n-1}^k}\left(\frac{q_0}{v_0}\right)^k P_{n-1}^k\left(\frac{W_{n-1}}{U_{n-1}}\right)^k|\varepsilon_n|^k|G_n|^k$$

$$= O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}\left(\frac{P_{n-1}}{R_{n-1}}\right)^k\left(\frac{W_{n-1}}{v_n U_{n-1}}\right)^k|\varepsilon_n|^k|G_n|^k = O(1).$$

is completes the proof of the theorem.

4. Applications

Theorem 2. (Bor [1] and [2]). *If* $nu_n = O(U_n)$, $U_n = O(nu_n)$, *then the series* Σa_n *is summable* $|C,1|_k$ *if and only if it is summable* $|N,u_n|_k$, $k \geqslant 1$.

Proof.

($\Rightarrow$) follows from theorem 1 by putting $p_n = 1$, $q_n = 1$, $v_n = 1$, and $\varepsilon_n = 1$.
($\Leftarrow$) follows from theorem 1 by putting $q_n = 1$, $u_n = 1$, $v_n = 1$, and $\varepsilon_n = 1$.

Theorem 3. (Bor and Thorpe [3]). *Let* $\{p_n\}$, $\{u_n\}$ *be sequences of positive real constants. If* $p_n U_n = O(P_n u_n)$ and $P_n u_n = O(p_n U_n)$, *then the series* Σa_n *is summable* $|\bar{N}, p_n|_k$ *whenever it is summable* $|\bar{N}, u_n|_k$, $k \geqslant 1$.

Proof. Follows from theorem 1 by putting $q_n = 1$, $v_n = 1$ and $\varepsilon_n = 1$.

Theorem 4. *If the sequences* $\{p_n\}, \{q_n\}, \{u_n\}, \{v_n\}$, *satisfy the conditions of theorem* 1 *except* (3)–(5) *and if* $p_n U_n = O(P_n u_n)$, $P_n u_n = O(p_n U_n)$ *and* $W_{n-1} = O(v_n U_{n-1})$, *then the series* Σa_n *is summable* $|N, p_n, q_n|_k$ *whenever it is summable* $|N, u_n, v_n|_k$, $k \geqslant 1$.

Proof. Follows from theorem 1 by putting $\varepsilon_n = 1$.

COROLLARY 5

Let $\{q_n\}$, $\{u_n\}$ *be sequences of positive real constants such that* $q \in M$, $U_n = O(nu_n)$ *and* $nu_n = O(U_n)$. *Then the series* Σa_n *is summable* $|N, q_n|_k$ *whenever it is summable* $|\bar{N}, u_n|_k$, $k \geqslant 1$.

Proof. Follows from theorem 4, by putting $p_n = 1$, $v_n = 1$, and making use of lemma 1.

COROLLARY 6

If the sequences $\{p_n\}$, $\{q_n\}$, $\{u_n\}$, $\{v_n\}$ *satisfy the conditions of theorem* 1 *except* (3)–(5), *and if* $p_n U_n = O(P_n u_n)$ *and* $P_n u_n = O(p_n U_n)$, *then sufficient conditions that* $\Sigma a_n \varepsilon_n$ *is summable* $|N, p_n, q_n|_k$ *whenever it is summable* $|N, u_n, v_n|_k$, $k \geqslant 1$ *are*

$$\text{(i)}\ |\Delta\varepsilon_n| = O\left\{\frac{p_n}{P_n}\frac{v_n U_{n-1}}{W_{n-1}}\right\}, \quad \text{(ii)}\ |\varepsilon_n| = O\left\{\frac{v_n U_{n-1}}{W_{n-1}}\right\}.$$

Proof. Follows from theorem 1.

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Symmetrizing a Hessenberg matrix: Designs for VLSI parallel processor arrays

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Abstract. A symmetrizer of a nonsymmetric matrix A is the symmetric matrix X that satisfies the equation $XA = A^tX$, where t indicates the transpose. A symmetrizer is useful in converting a nonsymmetric eigenvalue problem into a symmetric one which is relatively easy to solve and finds applications in stability problems in control theory and in the study of general matrices. Three designs based on VLSI parallel processor arrays are presented to compute a symmetrizer of a lower Hessenberg matrix. Their scope is discussed. The first one is the Leiserson systolic design while the remaining two, viz., the double pipe design and the fitted diagonal design are the derived versions of the first design with improved performance.

Keywords. Complexity; equivalent symmetric matrix; Hessenberg matrix; symmetrizer; systolic array; VLSI processor array.

1. Introduction

A symmetrizer [3, 7, 14, 16, 19, 20] of an $n \times n$ nonsymmetric matrix A is the solution X satisfying the equations $XA = A^tX$ and $X = X^t$. A symmetrizer is used in transforming a nonsymmetric matrix into an equivalent symmetric matrix [14, 20] whose eigenvalues are the same as those of the nonsymmetric matrix and is useful in many engineering problems, specifically stability problems in control theory and in the study of general matrices [14].

Let

$$B = \begin{bmatrix} b_{11} & b_{12} & 0 & \cdots & 0 \\ b_{21} & b_{22} & b_{23} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \cdots & b_{n-1,n} \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix} \tag{1}$$

be a lower Hessenberg matrix with $b_{i,i+1} \neq 0$ for $i = 1(1)n-1$, where $i = 1(1)n-1$ denotes $i = 1, 2, \ldots, n-1$. Also, let $\mathbf{x}_i$ be the i-th row of the symmetrizer X for $i = n(-1)1$. Then, from $XB = B^tX$, we write the serial algorithm [3] as follows

STEP 1: Choose $\mathbf{x}_n \neq 0$ arbitrarily.

STEP 2: Compute $\mathbf{x}_{n-1}, \mathbf{x}_{n-2}, \ldots, \mathbf{x}_1$ recursively from

$$\mathbf{x}_i = \frac{1}{b_{i,i+1}} \left(\mathbf{x}_{i+1} * B - \sum_{p=i+1}^{n} b_{p,i+1} \mathbf{x}_p \right) \quad i = n-1(-1)1$$

As an illustration, consider

$$B = \begin{bmatrix} 3 & -4 & 0 & 0 \\ -1 & 2 & -4 & 0 \\ 2 & -1 & 6 & -2 \\ 5 & 3 & -2 & 4 \end{bmatrix}$$

Choose $\mathbf{x}_4 = [1 \quad -2 \quad 0 \quad -1]$. $\mathbf{x}_3, \mathbf{x}_2$, and then $\mathbf{x}_1$ are computed following the foregoing algorithm. Hence the symmetrizer is

$$X = \begin{bmatrix} 1{\cdot}8438 & 3{\cdot}8750 & 2{\cdot}0000 & 1{\cdot}0000 \\ 3{\cdot}8750 & 3{\cdot}2500 & 1{\cdot}5000 & -2{\cdot}0000 \\ 2{\cdot}0000 & 1{\cdot}5000 & -5{\cdot}0000 & 0{\cdot}0000 \\ 1{\cdot}0000 & -2{\cdot}0000 & 0{\cdot}0000 & -1{\cdot}0000 \end{bmatrix}$$

It can be seen that the symmetrizer is not unique because if we choose $\mathbf{x}_4 = [1 \ 1 \ 1 \ 1]$ then we get a different X.

2. Leiserson systolic design

The single assignment algorithm [10] for computing a symmetrizer of the Hessenberg matrix B in Equation (1) is as follows.

```
for i:= 1 to n do  for j:= to n do  read B[i,j];
for k:= 1 to n do  read X[n,k];
for i:= n − 1 down to 1 do
begin
  for j:= 1 to n do  Y[i,j,1]:= 0.0;
  for j:= 1 to n do
  begin
    for k:= 1 to n do  Y[i,j,k+1]:= Y[i,j,k] + X[i+1,k]*B[k,j];
    Y[i,n+i+1]:= Y[i,j,n+1];
    for p:= i+1 to n do  Y[i,j,n+p+1]:= Y[i,j,n+p] − B[p,i+1]*X[p,j];
    Y[i,j,2n+2]:= Y[i,j,2n+1]/B[i,i+1];
    X[i,j]:= Y[i,j,2n+2];
    write X[i,j];
  end
end.
```

The implementation [6, 15, 17, 18] of this single assignment code a 4 × 4 matrix on the Leiserson systolic array depicted in figure 1 is straightforward by using the re-ming technique [10]. The allocation of the diagonals of the Hessenberg matrix to the rocessing cells (Type I) of the linear string of processors is as shown in figure 2. The nspecified output of PE_5 in figure 1 is ignored while its unspecified input is zero.

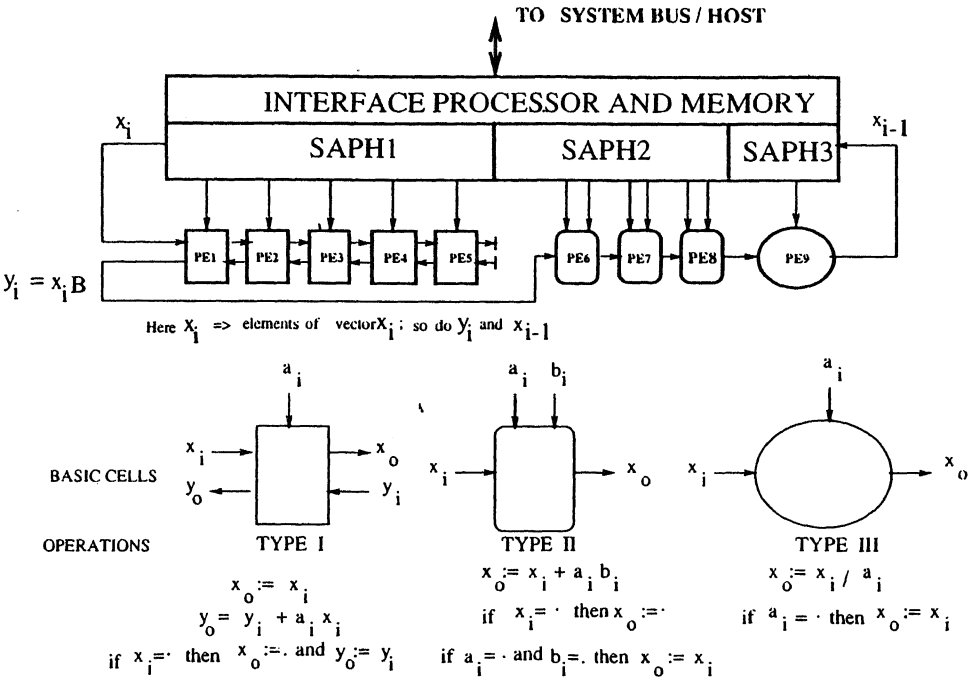


Figure 1. Systolic array cells system for a 4 × 4 matrix symmetrization.

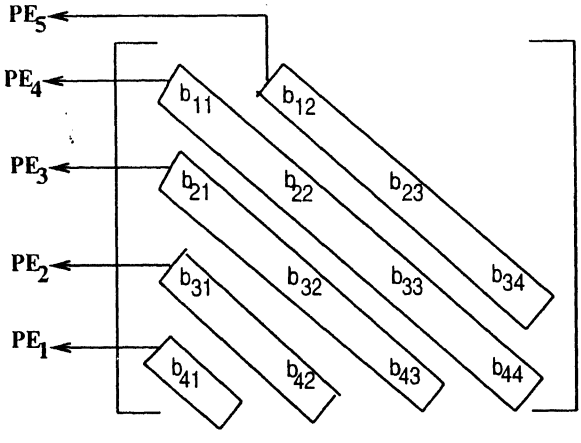
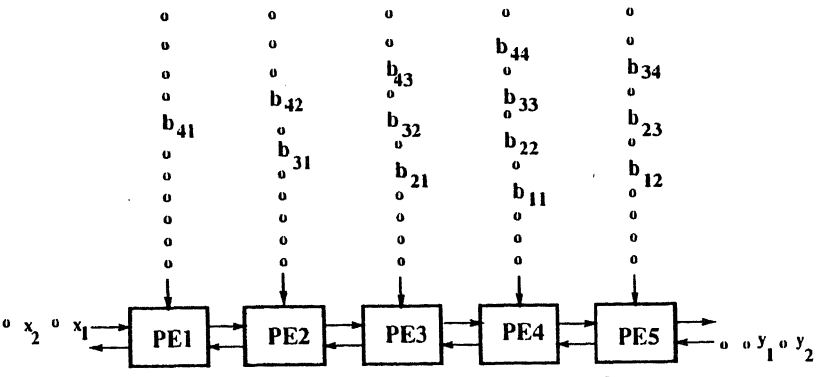


Figure 2. Systolic array cell (Type I) allocation for the diagonals of a 4 × 4 Hessenberg matrix.

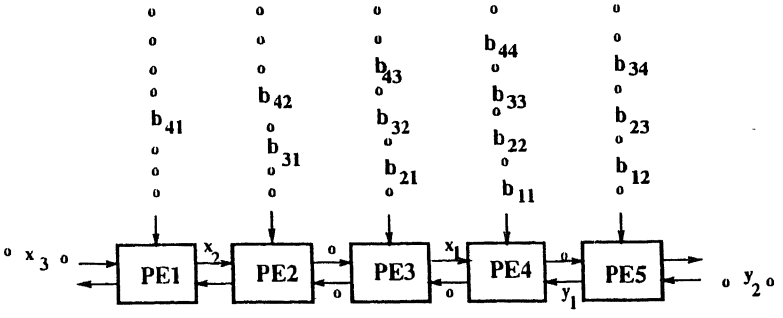
Figure 3 displays how the pumping of the row vector $\mathbf{x}_{i+1}$ and the matrix B into Type I cells is done for the matrix-vector multiplication while figure 4 demonstrates the array consisting of Types II and III cells to generate a symmetrizer row by row. The pumping will be done elementwise in Types II and III cells. The notations $\mathbf{x}_i^n, \mathbf{b}_{ij}, \mathbf{y}_i$, in figure 4, each of which has $2n-1$ elements including tag bits are given as

$$\mathbf{x}_i^n = [x_n^i \; o \; x_{n-1}^i \; o \; \ldots \; o \; x_1^i]^t,$$

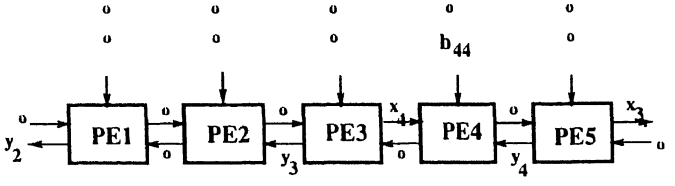
$$\mathbf{b}_{ij} = [b_{ij} \; o \; b_{ij} \; o \; \ldots \; o \; b_{ij}]^t,$$



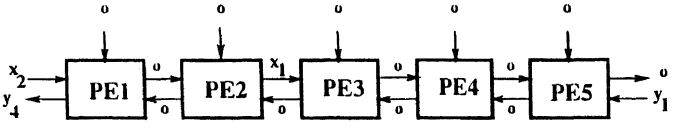
(a) Just before the first time cycle



(b) Just after the third time cycle



(c) Just after the ninth time cycle



(d) Just after thirteenth time cycle

Figure 3. 1-D systolic array for vector Hessenberg matrix multiplication.

and

$$\mathbf{y}_i = \mathbf{x}_i^n B = [y_n^i \; o \; y_{n-1}^i \; o \; \ldots \; o \; y_1^i]$$

This notation is used to conserve space.

A lower (or upper) Hessenberg matrix of order n needs $n+1$ cells of Type I. Denoting these cells $PE_1, PE_2, \ldots, PE_{n+1}$ following the same notation (and connection) as in figure 1, the diagonal consisting of only one element b_{n1} is positioned appropriately

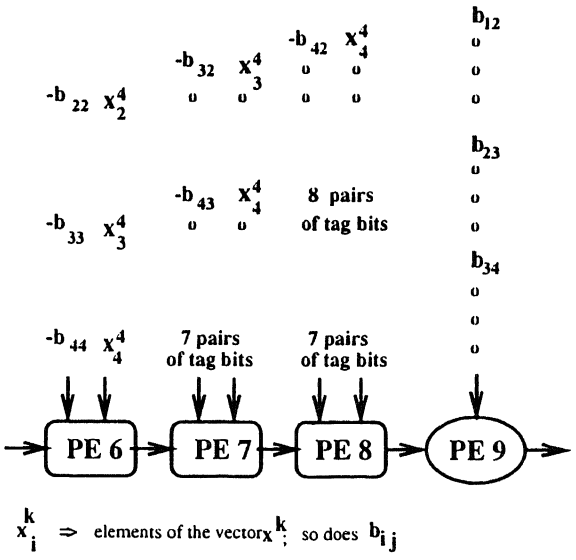

-b 22 x 4 2
-b 32 x 4 3
-b 42 x 4 4
b12
-b 33 x 4 3
-b 43 x 4 4
8 pairs of tag bits
b23
-b 44 x 4 4
7 pairs of tag bits
7 pairs of tag bits
b34
PE 6
PE 7
PE 8
PE 9
x k i => elements of the vector x k ; so does b i j

Figure 4. Systolic array for generating a 4 × 4 symmetrizer row by bow.

to be pumped into PE_1, the next diagonal (just above the foregoing diagonal) consisting of the elements $b_{n-1,1}$, b_{n2} is allocated to $PE2$. The third diagonal with elements $b_{n-2,1}, b_{n-1,2}, b_{n3}$ is assigned to PE_3 and so on. Figure 2 illustrates the allocation of the diagonals of the 4 × 4 Hessenberg (symbolic) matrix $B = [b_{ij}]$ to different cells. The generalization to an $n \times n$ matrix is immediate. However, the diagonals could have been allocated in the reverse order, i.e., the diagonal having the elements $b_{12}, b_{23}, b_{34}, \ldots, b_{n-1,n}$ could have been allocated to PE_1, the principal diagonal to PE_2, and so on. Both the allocations are functionally identical. We, however, use the former allocation.

In a one-dimensional Kung-Leiserson systolic array [4], the elements of the vector $\mathbf{x}$ flow from left to right (figure 3) for row vector-Hessenberg matrix multiplication. This array consists of Type I cells, viz., inner product step (ips) cells. The matrix elements flow into the top and the solution elements appear from the left of the cells. Here half the cells are active at any one time. It is, however, possible to orient the data flow so that the cells are all active simultaneously [8, 11]. Note that the number of cells depends on the bandwidth (number of diagonals) of B and not on the size of B. The summation with a negative sign, viz., the result

$$-\sum_{p=i+1}^{n} b_{p,i+1}\mathbf{x}_p$$

of Step 2 of the algorithm (§ 1) is computed using Type II cells as shown in figure 4. The values $\mathbf{y}_n = \mathbf{x}_n B, \mathbf{y}_{n-1} = \mathbf{x}_{n-1} B, \ldots, \mathbf{y}_1 = \mathbf{x}_1 B$ to which the results is to be added are pumped into the cells from the left while the terms of the summation are pumped into them from the top. The single division by $b_{i,i+1}$ is then carried out in Type III cell, one of which only is needed to be used irrespective of the size of B. The elements of the row-vector of the symmetrizer X which are output rhythmically one after the other by this Type III cell are then fed back as indicated in figure 1. This row-vector is then used in the computation of the remaining row-vectors of X recursively.

3. Double pipe and fitted diagonal designs

The Leiserson systolic model [6, 9, 12] needs $2n+1$ cells and $4n+1$ time cycles to obtain a row of the symmtrizer. Here we discuss two designs – one called the double pipe construction method, based on introducing a second pipe while the other, called the fitted diagonal method, on reducing the number of diagonals of the matrix B. While mapping the single assignment algorithm in § 2, the double pipe design aims at minimizing the time complexity while the fitted diagonal design the number of cells.

3.1 *Double pipe construction method*

This method [18] uses $n+1$ cells comprising two pipes – the first one consisting of odd labelled cells $PE_1, PE_3, \ldots$, while the second one the cells $PE_2, PE_4, \ldots$, where n is the order of B; in addition, it uses one adder and one delay cell (figure 5). It computes a symmetrizer B in $\left\lceil \frac{5n+3}{2} \right\rceil$ time cycles where $\lceil\ \rceil$ indicates the upper integral part. The double pipe concept increases cell efficiency and removes tag bits. It minimizes the hardware delay that exists before the start of actual computation. The data flow and the architecture of the $n+1$ cells are illustrated in figures 6 and 8, respectively.

Split the Hessenberg matrix B i.e., write $B = B1 + B2$. $B1$ contains only odd diagonals of B, where the first diagonal contains only the element b_{n1}, the second the elements $b_{n-1,1}, b_{n2}$, and so on while $B2$ the even diagonals. The remaining elements of $B1$ and $B2$ are zero. Since $\mathbf{x}_i B = \mathbf{y}_i$, we have $\mathbf{x}_i(B1 + B2) = \mathbf{y}_i$. If we allow $\mathbf{x}_i B1 = \mathbf{y}_{B1}$ and $\mathbf{x}_i B2 = \mathbf{y}_{B2}$ then $\mathbf{y}_{B1} + \mathbf{y}_{B2} = \mathbf{y}_i$. Figure 6 depicts the flow and computation of $\mathbf{y}_{B1}$ and $\mathbf{y}_{B2}$. The array needs no dummy elements, viz., the tag bits. Pipe 1 contains $\left\lceil \frac{n+1}{2} \right\rceil$ cells. Pipe 2 requires $(n+1) - \left\lceil \frac{n+1}{2} \right\rceil$ ips cells. The time complexity to obtain a row is $2n + \left\lceil \frac{n+1}{2} \right\rceil + 1$, i.e., $\left\lceil \frac{5n+3}{2} \right\rceil$.

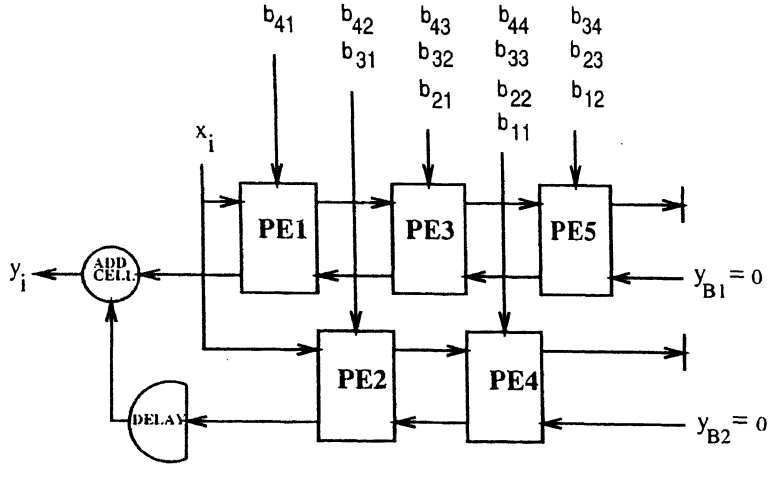


Figure 5. Double pipe array for vector Hessenberg matrix multiplication.

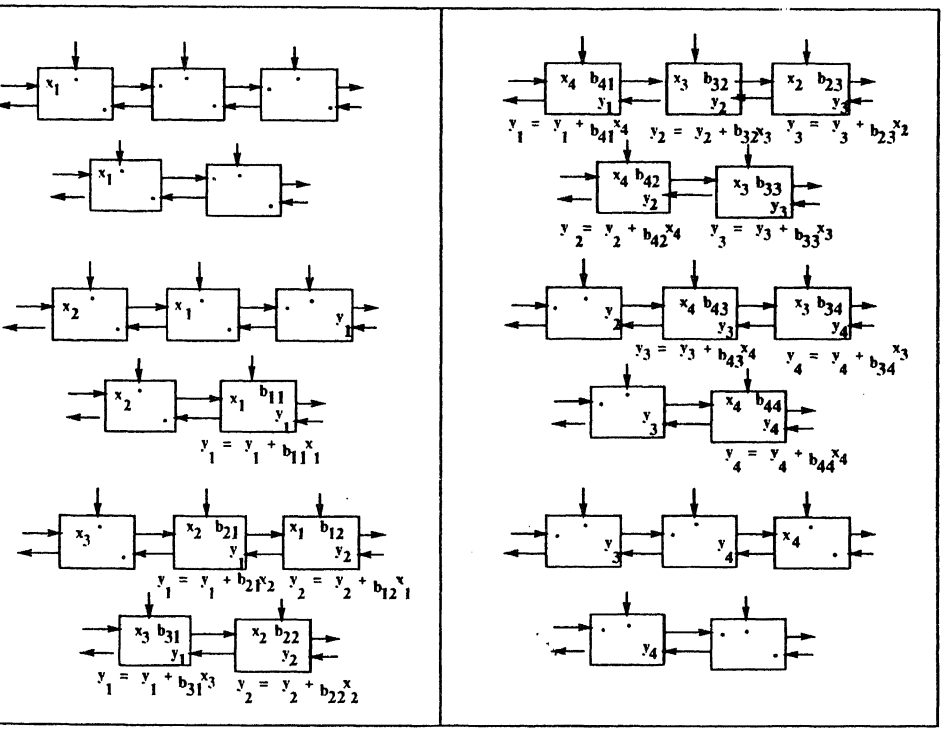


Figure 6. Data flow for double pipe method.

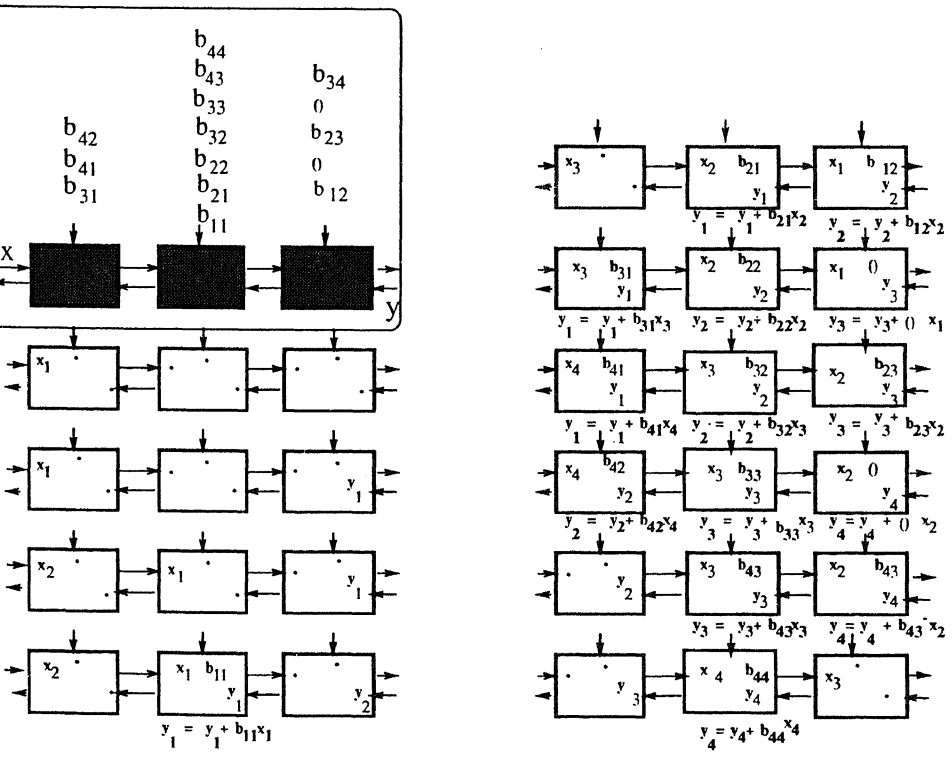


Figure 7. Fitted diagonal method and data flow.

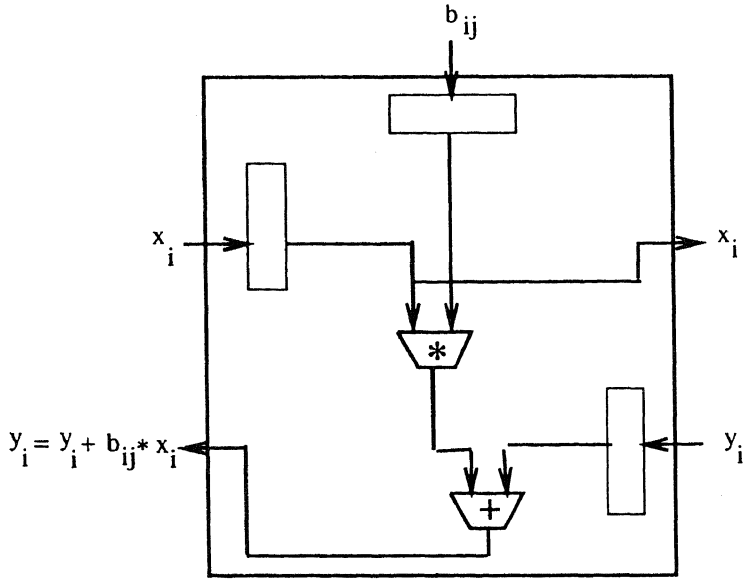


Figure 8. Architecture of ips cells for Leiserson and double pipe methods.

3.2 *Fitted diagonal method*

This method [15] consists in halving the number of diagonals, and hence the number of ips cells used is half of that required in the double pipe method. The number of diagonals can be reduced to $\frac{n+1}{2}$ by fitting two adjacent diagonals into one.

Let $\mathbf{d}_{k+1}$ and $\mathbf{d}_k$ be two adjacent diagonals, of B, of length $k+1$ and k, respectively

$$\mathbf{d}_{k+1} = (b_{n-k,1}, b_{n-k+1,2}, \cdots, b_{n,k+1}),$$
$$\mathbf{d}_k = (b_{n-k+1,1}, b_{n-k+2,2}, \cdots, b_{n,k})$$

A fitted diagonal $\mathbf{d}_f$ is defined by interleaving the elements of $\mathbf{d}_{k+1}$ and $\mathbf{d}_k$, as

$$\mathbf{d}_f = \mathbf{d}_{k+1} \cdot \mathbf{d}_k = (b_{n-k,1}, b_{n-k+1,1}, b_{n-k+1,2}, b_{n-k+2,2}, \cdots, b_{n,k}, b_{n,k+1})$$

where $\text{length}(\mathbf{d}_f) = \text{length}(\mathbf{d}_{k+1}) + \text{length}(\mathbf{d}_k) = 2k+1$.

Therefore, if the bandwidth $n+1$ of B is even then B is transformed to fitted diagonal matrix B_F with bandwith $\frac{n+1}{2}$. For a 5×5 Hessenberg matrix $B = [b_{ij}]$.

$$B_F = \begin{bmatrix} b_{11} & b_{12} & & & & & & & \\ b_{31} & b_{21} & 0 & & & & & & \\ b_{51} & b_{41} & b_{22} & b_{23} & & & & & \\ & 0 & b_{42} & b_{32} & 0 & & & & \\ & & & b_{52} & b_{33} & b_{34} & & & \\ & & & & b_{53} & b_{43} & 0 & & \\ & & & & & & b_{44} & b_{45} & \\ & & & & & & & b_{54} & \\ & & & & & & & & b_{55} \end{bmatrix}$$

the bandwidth is odd then B_F will have $\left\lceil \frac{n-1}{2} \right\rceil + 1$ fitted diagonals where the diagonal is fitted with one additional diagonal of $\frac{n}{2}$ zero elements. For a 4×4 enberg matrix B,

$$B_F = \begin{bmatrix} b_{11} & b_{12} & & & & & & \\ b_{31} & b_{21} & 0 & & & & & \\ & b_{41} & b_{22} & b_{23} & & & & \\ & & b_{42} & b_{32} & 0 & & & \\ & & & & b_{33} & b_{34} & & \\ & & & & & b_{43} & & \\ & & & & & & b_{44} & \end{bmatrix}$$

gure 7 illustrates the fitted diagonal method for a 4×4 Hessenberg matrix. A ction in the number of PEs in this method necessitates some minor modifications e ips cells. Each of the input vectors $\mathbf{x}_{i+1}$ and that of the output vectors $\mathbf{x}_i = \mathbf{y}$ be kept in each of the $\left\lceil \frac{n+1}{2} \right\rceil$ PEs for two time cycles as shown in figure 9. time complexity is the same as that for Leiserson systolic model but the number Es is $\left\lceil \frac{n+1}{2} \right\rceil + n$, where the last n PEs do the same job as the last n PEs in erson systolic model of figure 1. This number is about 75% of those required for conventional Leiserson systolic model. If $(n+1)$ is odd the diagonal left is fitted

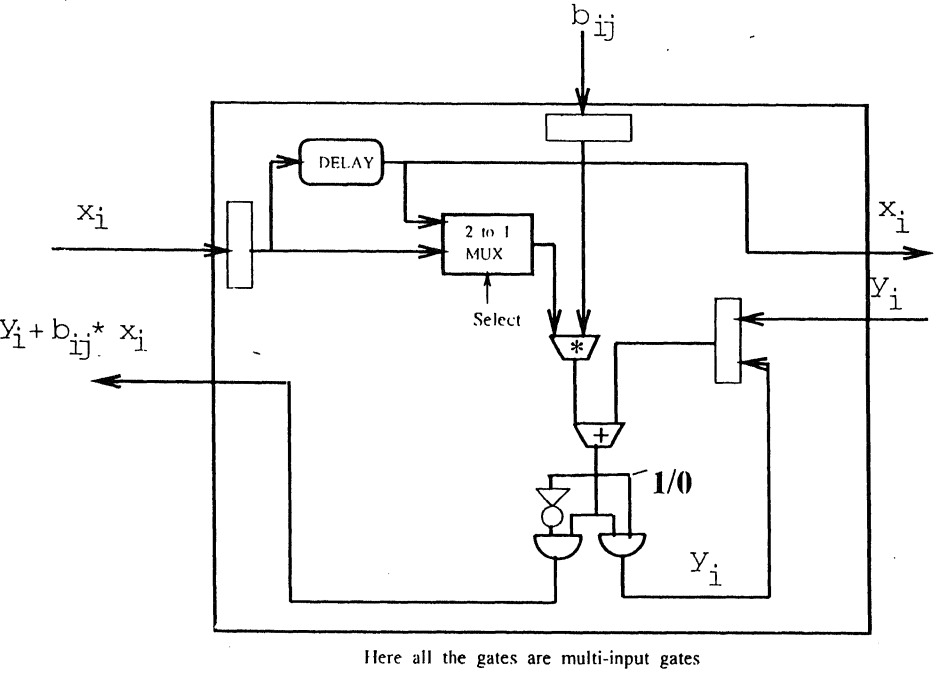


Figure 9. Architecture of ips cell for fitted diagonal method.

Table 1. Time complexity for a row and number of PEs for the designs.

Method	Number of PEs	Time complexity (for computing a row)
1. Leiserson Systolic Method	$w_1 = 2n + 1$	$w_1 + 2n$
2. Double Pipe Construction	$w_2 = 2n + 3$	$w_2 + \left\lceil \frac{n+1}{2} \right\rceil$
3. Fitted Diagonal	$w_3 = \left\lceil \frac{n+1}{2} \right\rceil + n$	$w_3 + 2n$

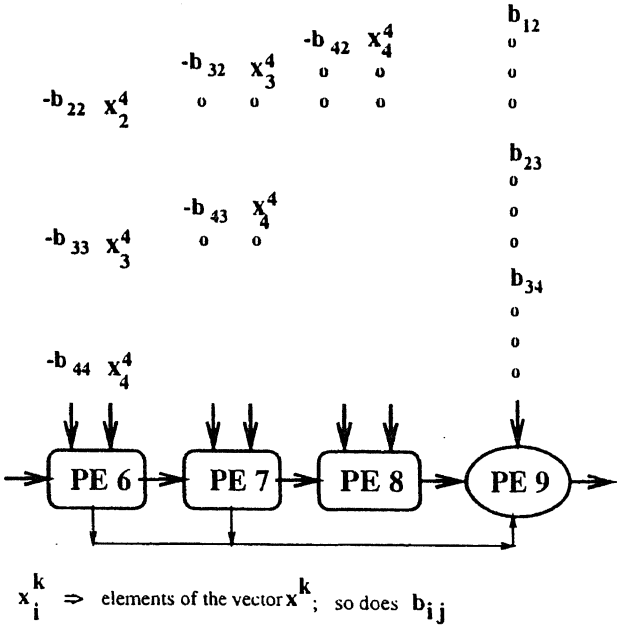


Figure 10. Modified systolic array for generating a 4 x 4 symmetrizer row by row.

with an additional diagonal of null elements. However this reduction in the number of processors needs some minor modification required for the processing elements. The first $(n + 1)$ PEs owing to the elements of vectors **x** and **y** must be kept in each processor for two clock cycles. The time complexity is the same as that of Leiserson systolic model but the number of processors is $\left\lceil \frac{n+1}{2} \right\rceil + n$ which is roughly half of that for conventional Leiserson systolic model.

We present, in table 1, a comparison of time complexity to compute a row of the symmetrizer and number of PEs for the proposed three designs.

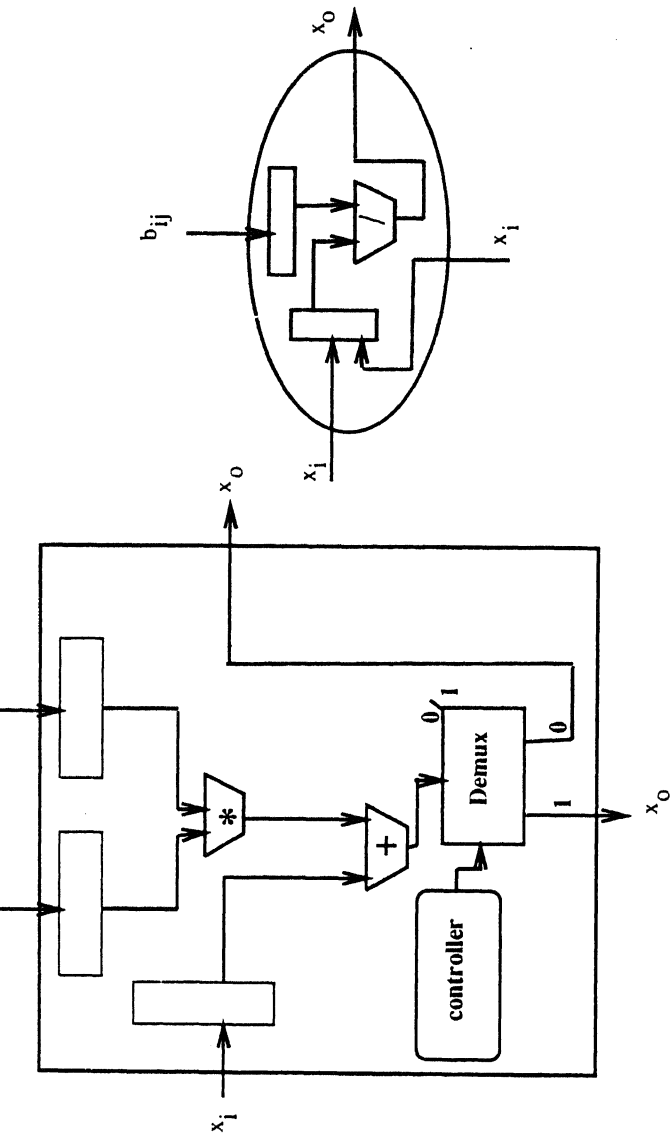


Figure 11. Modified architecture of Types II and III cells.

4. Scheduling and total time complexity

In the Leiserson systolic model, computation of a row of a symmetrizer requires $2n + w_1$ time cycles (where $w_1 = 2n + 1$). Repeating this process for all the rows independently, the total number of time cycles required is $(n-1)(2n + w_1)$. Even though this total number of time cycles is $O(n^2)$, it is still expensive. After $w_1 + n$ time cycles Type I cells (figure 1) are totally idle. A new pumping process is scheduled every $w_1 + n$ time cycles. Therefore, the total number of time cycles to obtain the symmetrizer is $T_1 = (n-1)(n + w_1) + n$ which reduces the number of time cycles by $n^2 - 2n$.

The same number of time cycles T_1 is required in fitted diagonal method even though, in this case, number of PEs is reduced by $\frac{n+1}{2}$, compared to that in Leiserson systolic model. Similarly, in the double pipe construction method, the symmetrizer is computed with the number of time cycles $T_2 = (n-1)\left(\left\lceil \frac{n+1}{2} \right\rceil + (n+3)\right)$.

If we use programmable systolic chip, then Types II and III cells are modified as in figure 10 and the cells architecture is as depicted in figure 11; Type II cells (except the last) have two output gates. The switch value is always assigned zero. The controller sets one for particular clock counter values, e.g., for Leiserson model of 4×4 matrix symmetrization, the processor PE6 controller sets the switch value one from 6th time cycle to 12th time cycle so that the data is pumped to the division cell directly. Type III cell gets input from any one of the gates. This modification reduces the tag bits in Type II and III cells. It also reduces the time complexity by $(n-2)(n-3)/2$.

5. Conclusions

The systolization procedures, i.e., all the three designs can also be easily extended to the general serial algorithm [14] to compute a symmetrizer of an arbitrary square matrix. The bandwidth will, however, be more. We hope that such a systolization will enormously reduce the complexity of computing an error-free symmetrizer [19, 20]. This error free symmetrizer will produce a more accurate equivalent symmetric matrix [14, 19] than what an approximate one does. It can be seen that when a real non-symmetric matrix has one or more pairs of complex eigenvalues then the equivalent symmetric matrix will be a complex one. Jacobi-like methods [1, 2, 5, 13] have been developed for computing eigenvalues, some of which are complex, of a complex symmetric matrix. These methods obviously make use of the "symmetry" property which results in a significant reduction in computation.

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Control of interconnected nonlinear delay differential equations in $W_2^{(1)}$

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Abstract. Our main interest in this paper is the resolution of the problem of controllability of interconnected nonlinear delay systems in function space, from which hopefully the existence of an optimal control law can be deduced later. We insist that each subsystem be controlled by its own variables while taking into account the interacting effects. This is the recent basic insight of [13] on ordinary differential systems. Controllability is deduced for the composite system from the assumption of controllability of each free subsystem and a growth condition of the interconnecting structure. Conditions for a free system's controllability are given. One application is presented. The insight it provides for the growth of global economy has important policy implications.

Keywords. Large-scale systems; delay equations; decentralized control; growth of capital stock; depression.

We motivate the problem with a simple economic system derived by Kalecki [24] and reported in [1]. He argued that the dynamics of capital stock $x(t)$ of a firm is given by

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-h) + bu(t), \tag{1.1}$$

where a_i, $i = 0, 1$, are constants, $bu(t)$ is a sum of two terms—a constant multiple of autonomous consumption and a trend term. The crucial assumption for (1.1) is that the net capital formation $\dot{x}(t)$ is given by $I(t)$, the investment function. To obtain (1.1), Kalecki assumes that the investment decision B is given by

$$B(t) = a(1-c)y(t) - kx(t) + \varepsilon,$$

where a, c, k are constants, ε is a windfall which may be time varying. The income (or output) is y, $x(t)$ denotes the stock of capital, and ε is the trend term. The delay h represents the time lag between the decision to invest and the deliveries of capital equipments. One can interpret (1.1) as a system whose growth can be controlled by autonomous consumption and windfalls. For example one can ask whether it is possible to grow from a 3% growth rate $x(t) = 3t/100 = \phi(t)\, t \in [-h, 0]$, to 10% growth rate, $x(t+T) = 10t/100 \equiv \psi(t)$, $t \in [-h, 0]$ in time T, by using u as a control. To motivate a nonlinear system of the form (1.1) which is interconnected by the so-called "solidarity" function inspired by [14], [20], [21] we argue as follows. Let Z denote aggregate demand consisting of consumption (C), investment (I), net exports (X) and government outlay (G). These differentiable functions are related as follows:

$$Z = \mathrm{C} + \mathrm{I} + \mathrm{X} + \mathrm{G}, \tag{1.2}$$

where

$$C = C_0 + c(y - T), \quad 0 < c < 1, \tag{1.3}$$

and $y - T$ is the current after-tax income,

$$T = T_0 + f_1(y). \tag{1.4}$$

$T_0 > 0$ is the level of non-income taxes, and $f_1(y)$ is the income taxes.

$$X = X_0 - f_2(y) - eR, \quad 0 < m < 1, \quad e_1 > 0. \tag{1.5}$$

$f_2(y)$ the part of income that is spent on other countries' products, X_0 is autonomous net exports, and R is the real rate of interest. Public expenditure is

$$G = f_3(y(t-h)) + v(t), \tag{1.6}$$

where f_3 is public consumption which is dependent on the previous high income, and $v(t)$ public investment. Investment is autonomous, i.e. it does not depend on income, but on "animal spirits" of entrepreneurs:

$$I(t) = I_0(t). \tag{1.7}$$

Thus

$$Z(t) = (Z_0(t) + f_3(y(t-h)) + v(t) + cy(t) - cT_0(t) - cf_1(y) - f_2(y(t)) - e_1 R, \tag{1.8}$$

where $Z_0(t) = I_0 + X_0 + C_0$. From the model equations of money demand and supply [14] we deduce that

$$R = \frac{ky}{r} - \left[\frac{M_0}{P_0} - j\right] \Big/ r, \tag{1.9}$$

since

$$L/P = M/P, \quad L/P = j + ky - rR, \quad M = M_0, \quad P = P_0,$$

where k is a fraction of income, $r > 0$ is measured in dollars. Here M is the nominal value of money supply which is controlled by the Central Bank, P is the price level. The real demand for money is denoted by L/P. The symbol j is autonomous real money demand. With R in (1.9) we deduce that

$$Z(t) = Z_0(t) + y(t)\left(c - \frac{e_1 k}{k}\right) + f_3(y(t-h)) - [cf_1(y(t)) + f_2(y(t))]$$
$$+ \frac{e_1}{r}\left[\frac{M_0}{P_0} - j\right] - cT_0(t) + v(t).$$

Following Allen [1] we postulate that $dy(t)/dt = -\lambda(y(t) - Z(t))$, where λ is a constant. Thus

$$\dot{y}(t) = -\lambda\left(1 + \frac{e_1 k}{r} - c\right) y(t) - \lambda(cf_1(y(t)) + f_2 y(t))$$
$$+ \lambda f_3(y(t-h)) + \frac{e_1}{r}\left[\frac{M_0}{P_0} - j\right] + \lambda v(t) + \lambda Z_0(t) - \lambda c T_0(t). \tag{1.10}$$

Denote "solidarity functions," by

$$q(t) = \lambda\left[\frac{e_1}{r}\frac{M_0}{P_0} - cT_0(t) + v(t)\right]$$

d "private initiative" by

$$p(t) \equiv \lambda\left[Z_0(t) - \frac{e_1}{r}j\right].$$

en the dynamics of income is

$$\dot{y}(t) = -\lambda\left(1 + \frac{e_1 k}{r} - c\right)y(t) + \lambda(cf_1(y(t)) + f_2(y(t)) + \lambda f_3(y(t-h)) + p(t) + q(t). \tag{1.11}$$

It is an interconnected nonlinear system whose controllability is investigated for values of p and q. The type of result which we shall prove in Theorem 2.2 when plied to the special system (1.11) will now be stated.
Suppose

$$q(t) = \lambda\left[\frac{e_1}{r}\frac{M_0}{P_0} - cT_0(t) + v(t)\right]$$

$$p(t) = \lambda\left[\mathrm{I}_0 + \mathrm{X}_0 + \mathrm{C}_0 - \frac{e_1}{r}j\right], \tag{1.11a}$$

d $B(t)u(t) = p(t) + q(t)$; then the dynamics of gross national product is

$$\dot{y}(t) = -\lambda\left(1 + \frac{e_1 k}{r} - c\right)y(t) + \lambda(cf_1(y(t)) + f_2(y(t))) + \lambda f_3(y(t-h)) + B(t)u(t). \tag{1.11b}$$

s possible for national income to be controlled by u, a combination of government d private controls. Using this, for example, we can steer a growth rate of 3% i.e., $(t) = 3t/100$, $t\in[-h, 0]$, to a growth rate of 10%, $\psi_2(t+T) = 10t/100$, $t\in[-h, 0]$, time T *provided*

$B(t) \neq 0$ on $[T-h, T]$, $B(t) \neq 0$, $t\in[0, T-h]$
(Condition (ii), Theorem 2.2),

$$\int_{T-h}^{T} \frac{\mathrm{d}t}{B(t)} < \infty,$$

the combined effect of the coefficient of solidarity and private initiative is "strong", d nontrivial. This is (iii), Theorem 2.2, the condition of the essential uriform undedness of the generalized inverse of $B(t)$.

Also there is a condition of how big q should be compared with $p(t)$: Theorem 2.4 (ii), see Remark 2.4: "private initiative" should dominate "solidarity".

t is proper to consider $x^i(t) = (x_1^i(t), \ldots, x_n^i(t))$ to be the value of n capital stocks

with strategy $u^i = (u^i_1 \ldots u^i_n)$, $Bu^i(t) \equiv p_i(t) - 1 \leqslant u^i_j(t) \leqslant 1$, which is located in an isolated region S_i. They are linked to l other such regional systems in the country and the "interconnection" or "solidarity functions", or government intervention givenby

$$q_i = q_i(x^i_{1t} \ldots x^i_{lt}, u^i_{1l} \ldots u^i_{lt}).$$

Here q_i describes the action of the whole system on its ith interconnected subsystem (S_i),

$$\dot{x}^i(t) - A_{-1}\dot{x}^i(t-h) = A_0 x^i(t) + A_1 x^i(t-h) + Bu^i(t). \qquad (S_i)$$

Thus

$$\dot{x}^i(t) - A_{-1}\dot{x}^i(t-h) = A_0 x^i(t) + A_1 x^i(t-h) + B^i u(t) + q_i. \qquad (1.12)$$

Thus formulated we are interested in using the firms strategy u^i and government interventions q_i to control the growth of capital stock on which the wealth of a nation depends. Theorems 3.1 and 3.2 can be stated loosely as follows. If a regional economy is well behaved, carefully weighted government interventions q_i can maintain the country's economic growth. Even if a regional economy is not controllable the intervention of solidarity function can render the system controllable. (See Remark 3.1 and Theorem 3.2.) Implications of controllability questions for the control of global economy are pursued elsewhere in [5], [8]. The issue of optimality is apparent in [8].

1. Introduction

For linear free systems, criteria for $W^{(1)}_2$ controllability have been provided in [2]. For nonlinear cases a similar investigation was recently carried out in [8]. Recently Sinha [16] treated controllability in Euclidean space of large scale systems in which the base is linear. We extend the scope of the treatment in [16] by treating large scale systems with delay when the state space is $W^{(1)}_2$ and the base system is not necessarily linear. We state criteria for controllability of the free subsystem by defining an L_2 control that does the steering both in the linear and nonlinear case. We prove that such a control exists as a solution of an integral equation in a Banach space. For this, we use Schauder's fixed point theorem. Assuming that the free subsystem is controllable and the interaction function has a certain growth condition we prove the controllability of the interconnected system.

We begin with a simple system. A linear state equation of the ith subsystem of an interconnected control system can be described by,

$$\dot{x}^i(t) = A_{1i}x^i(t) + A_{2i}x^i(t-h) + A_{3i}y^i(t) + A_{4i}y^i(t-h) + B_i u^i(t), \qquad (1.13)$$

where $x^i(t) \in E^{n_i}$ is the n_i-dimensional Euclidean state vector of the ith subsystem, $u_i \in E^{m_i}$ is the control vector and A_{1i} A_{2i} B_i A_{3i} A_{4i} are time invariant matrices of appropriate dimensions. Also, $y^i(t)$ is the supplementary variable of the ith subsystem and is a function of its own Euclidean state vector $x^i(t)$ and other subsystem state vector $x^j(t)$ $j = 1, \ldots, l$. We express this as follows:

$$y^i(t) = M_{ii}x^{(i)}(t) + \sum_{\substack{j=1 \\ j \neq i}}^{l} M_{ij}x^j(t) \qquad (1.14)$$

where M_{ii}, M_{ij} $(j = 1, 2, \ldots, l, j \neq i)$ are constant matrices.

By substituting (1.14) into (1.13) we obtain the state equation of the overall interconnected system,

$$\dot{x}^i(t) = H_i x^i(t) + G_i x^i(t-h) + B_i u^i(t) + m_i(t) + e_i(t-h) \tag{1.15}$$

where

$$H_i = A_{1i} + A_{3i} M_{ii}, \quad G_i = A_{2i} + A_{4i} M_{ii}$$

$$m_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^{l} M_{3i} M_{ij} x^j(t), \quad e_i(t-h) = \sum_{\substack{j=1 \\ j \neq i}}^{l} M_{ij} x^j(t-h).$$

In (1.15), $m_i(t)$ and $e_i(t-h)$ describe the interaction, the effects of other subsystems on the ith subsystem. This can be measured locally. The decomposed system (1.15) can be viewed as an interconnection of l isolated subsystems

$$\dot{x}^i(t) = H_i x^i(t) + G_i x^i(t-h) + B_i u^i(t), \tag{S_i}$$

with interconnection structure characterized by

$$g^i(t, t-h) = m_i(t) + e_i(t-h),$$

which does not depend on the state variables $x^i(t)$.

We now consider the more general free linear subsystem,

$$\begin{aligned} &\dot{x}^i(t) = L_i(t, x_t^i) + B_i u^i(t), \\ &x_\sigma^{(i)} = \phi^i \end{aligned} \tag{L_i}$$

and the decomposed large scale system,

$$\begin{aligned} &\dot{x}^i(t) = L_i(t, x_t^i) + B_i u^i(t) + g_i(t, t-h), \\ &x_\sigma^{(i)} = \phi^i, \; i = 1, \ldots, l, \end{aligned} \tag{I_i}$$

where

$$L_i(t, x_t^i) = \int_{-h}^{0} \mathrm{d}_\theta \eta_i(t, \theta) x^i(t+\theta),$$

$$g_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^{l} \int_{-h}^{0} \mathrm{d}_\theta \eta_{ij}(t, \theta) x^j(t+\theta).$$

We assume $B_i(t)$ is an $n_i \times m_i$ continuous matrix. The linear operator $\phi \to L_i(t, \phi)$ is described by the integral in the Lebesgue-Stieltjes sense, where $(t, \theta) \to \eta_i(t, \theta)$ is an $n_i \times n_i$ matrix function. It is assumed that $t \to \eta_i(t, \theta)$, $t \in E$, is continuous for each fixed $\theta \in [-h, 0]$ and $\theta \to \eta_i(t, \theta)$ is of bounded variation on $[-h, 0]$ for each fixed $t \in E$. Also, $\eta_i(t, \theta) = 0$, $\theta \geqslant 0$, $\eta_i(t, \theta) = \eta_i(t, -h)$, $\theta \geqslant -h$ and $\theta \to \eta_i(t, \theta)$ is left, continuous on $(-h, 0)$. It is assumed that

$$\operatorname*{var}_{\theta \in E} \eta_i(t, \theta) \leqslant \rho_i(t), \quad t \in E,$$

where $\rho_i(t)$ is locally integrable. These conditions also hold for η_{ij}.

Throughout the sequel E^r is the r-dimensional Euclidean space with norm $|\cdot|$. The symbol C denotes the space of continuous functions mapping the interval $[-h, 0]$,

$h>0$, $h\in E$ into E^n with the sup norm $\|\cdot\|$, defined by $\|\phi\| = \sup_{-h\leqslant s\leqslant 0} |\phi(s)|$, $\phi\in C$.

The controls are square integrable functions $u\in L_2([\sigma,t_1],E^m)$, $t_1\in E$, $t_1>\sigma$ and L_2 is the space of measurable functions u defined on finite intervals $[\sigma,t_1]$ for which $|u|^2$ is summable. If $t\in[\sigma,t_1]$, we let $x_t\in C$ be defined by $x_t(s)=x(t+s)$, $-h\leqslant s\leqslant 0$. With L_2 as the space of admissible controls, the state space is either E^n on $W_2^{(1)}$, the Sobolev space of absolutely continuous functions x: $[-h,0]\to E^n$ with the property that $t\to\dot{x}(t)\in L_2([-h,0],E^n)$. Thus if σ, $t\in E$ and $\phi^i\in W_2^{(1)}([-h,0],E^{n_i})$, $u^i\in L_2([\sigma,t_1],E^{m_i})$ there is a unique absolutely continuous function $x^i(\cdot,\sigma,\phi^i,u^i)=x^i:[\sigma-h,t_1]\to E^{n_i}$ which satisfies (L_i) or (I_i) $a\cdot e$ on $[\sigma,t_1]$ and the initial condition $x^i_\sigma=\phi^i$ whenever the earlier conditions in η_i, η_{ij} and B_i are satisfied. Also $x^i_t(\cdot,\sigma,\phi^i,u^i)\in W_2^{(1)}([-h,0],E^{n_i})$ for $t\in[\sigma,t_1]$.

DEFINITION 1.1

The system (L_i) is controllable (respectively Euclidean controllable) on the interval $[\sigma,t_1]$ if for each $\phi^i,\psi^i\in W_2^{(1)}([-h,0],E^{n_i})$ (respectively $\phi^i\in W_2^{(1)}([-h,0],E^{n_i})$), $x_1^i\in E^{n_i}$, there is a controller $u^i\in L_2([\sigma,t_1],E^{m_i})$ such that $x^i_\sigma(\cdot,\sigma,\phi^i,u^i)=\phi^i$ and $x^i_{t_1}(\cdot,\sigma,\phi^i,u^i)=\psi^i$ (resp $x^i(t_1,\sigma,\phi^i,u^i)=x_1^i$). If (L_i) is controllable on every interval $[\sigma,t_1]$, $t_1>\sigma+h$, we say it is controllable. If (L_i) is Euclidean controllable on every interval $[\sigma,t_1]$, $t_1>\sigma$ we say it is Euclidean controllable. For the free subsystem (L_i) the following controllability theorem is available in [2, p. 616].

PROPOSITION 1.1

In (L_i) let $B_i^+(t)$ denote the Moore-Penrose generalized inverse of $B_i(t)$, $t\in E$. Assume that $t\to B_i^+(t)$ is essentially bounded on $[t_1-h,t_1]$. Then (L_i) is controllable on an interval $[\sigma,t_1]$ with $t_1>\sigma+h$ if and only if

$$\operatorname{rank} B_i(t)=n_i \quad on \quad [t_1-h,t_1].$$

An easy adaptation of the argument in [2] yields the following result on the system (I_i).

Theorem 1.1. *Consider the interconnected decomposed system (I_i) in which $B_i^+(t)$ is essentially bounded. Suppose*

$$\operatorname{rank} B_i(t)=n_i \quad on \quad [t_1-h,t_1].$$

Then (I_i) is controllable on $[\sigma,t_1]$, $t_1>\sigma+h$.

Proof. Let $X_i(t,s)$ be the fundamental matrix solution of

$$\dot{x}^i(t)=L_i(t,x_t^i).$$

Then

$$G_i(\sigma,t_1)=\int_\sigma^{t_1-h} X_i(t_1-h,s)B_i(s)B_i(s)^*X_1^*(t_1-h,s)ds,$$

has rank n_i, so that (I_i) is Euclidean, controllable. Here B^* is the algebraic adjoint of B. This is proved by letting $\phi^i\in W_2^{(1)}\equiv W_2^{(1)}([-h,0],E^{n_i})$, $x_1^i\in E^nI$, and by defining a control

$$u^i(t) = [B_i^*(t)X_i^*(t_1 - h, t)]G^{-1}(\sigma, t_1 - \sigma)\Big[x_1^i - x^i(t_1, \sigma, \phi, 0) - \int_\sigma^{t_1 - h} X_i(t_1 - h, s) g_i(s, s - h)\mathrm{d}s \Big],$$

e $x^i(t, \sigma, \phi, 0)$ is the solution of (L_i) with $u^i \equiv 0$. Using the variation of parameter verifies that u^i indeed transfers ϕ^i to x_1^i in time $t_1 - h$. Thus there is a control $_2([\sigma, t_1 - h], E^{m_i})$ such that $x^i(t_1 - h, \sigma, \phi, u^i) = \psi^i(-h)$. We extend u^i and x^i to nterval $[\sigma, t_1]$ so that

$$\psi^i(t - t_1) = x^i(t) \quad t_1 - h \leqslant t \leqslant t_1,$$

$$\dot{\psi}^i(t - t_1) = L_i(t, x_t^i) + B_i(t)u^i(t) + g_i(t),$$

on $[t_1 - h, t_1]$. To do this note that

$$L_i(t, \phi^i) = -\eta_i(t, -h)\phi^i(-h) - \int_{t-h}^{t_1 - h} \eta_i(t_1, \alpha - t)\dot{\phi}^i(\alpha)\mathrm{d}\alpha - \int_{t_1 - h}^{t} \eta_i(t, \alpha - t)\dot{\phi}^i(\alpha)\mathrm{d}\alpha, \quad t_{1-h} \leqslant t \leqslant t_1;$$

$$g_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^{l} \int_{-h}^{0} \mathrm{d}_\theta \eta_{ij}(t, \theta)x^j(t + \theta)$$

$$= -\sum_{\substack{j=1 \\ j \neq i}}^{l} \Big\{ \eta_{ij}(t, -h)x^j(t - h) + \int_{t-h}^{t_1 - h} \eta_{ij}(t, \alpha - t)\dot{x}^j(\alpha)\mathrm{d}\alpha + \int_{t_1 - h}^{t} \eta_{ij}(t, \alpha - t)\dot{x}^j(\alpha)\mathrm{d}\alpha \Big\}.$$

ow define

$$u^i(t) = B_i^+(t)\Big\{ \dot{\psi}^i(t - t_1) + \eta_i(t, -h)x^i(t - h) + \int_{t-h}^{t_1 - h} \eta_i(t, \alpha - t)\dot{x}^i(\alpha)\mathrm{d}\alpha + \int_{t_1 - h}^{t} \eta_i(t, \alpha - t)\dot{\psi}^i(\alpha - t)\mathrm{d}\alpha + \sum_{\substack{j=1 \\ j \neq i}}^{l} \Big[\eta_{ij}(t, -h)x^j(t - h) + \int_{t-h}^{t_1 - h} \eta_{ij}(t, \alpha - t)\dot{x}^j(\alpha)\mathrm{d}\alpha + \int_{t_1 - h}^{t} \eta_{ij}(t, \alpha - t)\dot{x}^j(\alpha)\mathrm{d}\alpha \Big] \Big\},$$

$_1 - h \leqslant t \leqslant t_1$. Because of the smoothness properties of x^i, x^j and ψ^i, u^i is indeed ropriate. Thus the controllability of the composite system can be deduced from of the subsystems so long as the interconnection is as proposed. We now turn attention to the nonlinear situation.

2. Nonlinear systems

Consider the general nonlinear large scale system,

$$\dot{x}(t) = f(t, x_t, u(t)) + B(t, x_t)u(t) + g(t, x_t, v(t)), \tag{2.1}$$

where $f: E \times C \times E^m \to E^n$ is a nonlinear function $g: E \times C \times E^m \to E^n$ is a nonlinear interconnection and the $n \times m$ matrix function $B: E \times C \to E^{n \times m}$ is possibly nonlinear. Conditions for the existence of a unique solution $x(\cdot, \sigma, \phi, u)$, when $u \in L_2, \phi \in C([-h, 0], E^n)$ are given in Underwood and Young [18]. It is shown there that $(\phi, u) \to x_t(\cdot, \sigma, u) \in C$ is continuously differentiable. These conditions are assumed to prevail here. Indeed we have,

Lemma 1.1. *For the system* $\dot{x}(t) = f(t, x_t, u(t)) + B(t, x_t)u(t)$ *assume that*

(i) $B: E \times C \to E^{n \times m}$ *is continuously differentiable.*
(ii) *There exist integrable functions* $N_i, N_i: E \to [0, \infty)$, $i = 1, 2$, *such that*

$$\|D_2 B(t, \phi)\| \leqslant N_1(t), \quad \|B(t, \phi)\| \leqslant N_2(t),$$

for $t \in E$ *and* $\phi \in C([-h, 0], E^n)$. *Here and in the sequel* $D_i g(\,)$ *is the Fréchet derivative of* g *with respect to the ith variable.*
(iii) $f(t, \cdot, \cdot)$ *is continuously differentiable for each* t.
(iv) $f(\cdot, \phi, \omega)$ *is measurable for each* ϕ *and* ω.
(v) *For each compact set* $K \subset E^n$ *there exists an integrable function* $M_i: E \to [0, \infty)$, $i = 2, 3$ *such that*

$$\|D_2 f(t, \phi, \omega)\| \leqslant M_1(t) + M_2(t)|\omega|,$$
$$\|D_2 f(t, \phi, \omega)\| \leqslant M_3(t) \quad \forall t \in E, \quad \omega \in E^m, \quad \phi \in C([-h, 0], K).$$

Then for each $u \in L_2$ *there exists a unique solution* x *to*

$$\dot{x}(t) = f(t, x_t, u(t)) + B(t, x_t)u(t).$$

Remark 1.1 Note that

$$\forall t \in [\sigma, t_1], \quad \phi \in C([-h, 0], K), \quad \omega \in E^m$$

we have

$$|f(t, \phi, \omega)| \leqslant M_1(t)\|\phi\| + M_2(t)|\omega|.$$

The system (2.1) may be decomposed as

$$\dot{x}^i(t) = f_i(t, x_t^i, u^i(t)) + B_i(t, x_t^i)u^i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{l} g_{ij}(t, x_t^j, v^i(t)), \quad i = 1, \ldots, l, \tag{2.2}$$

where

$$f_i: E \times C([-h, 0], E^{n_i}) \times E^{m_i} \to E^{n_i} g_{ij}: E \times C([-h, 0], E^{n_i}) \times E^{m_i} \to E^{n_i},$$
$$B_i: E \times C([-h, 0], E^{n_i}) \to E^{n_i \times m_i}.$$

Let

$$\sum_{i=1}^{l} n_i = n, \quad \sum_{l=1}^{l} m_i = m, x^{\mathrm{T}} = [(x^1)^{\mathrm{T}}, \ldots, (x^l)^{\mathrm{T}}] \in E^n, \; u^{\mathrm{T}} = [(u^1)^{\mathrm{T}}, \ldots, (u^l)^{\mathrm{T}}] \in E^m,$$

$$[f(t,x_t,u)]^{\mathrm{T}} = [(f_1(t,x_t^1,u^1))^{\mathrm{T}},\ldots,(f_l(t,x^l,u^l)^{\mathrm{T}}],$$

$$g_i(t,x_t,v^i(t)) = \sum_{\substack{j=1\\ j\neq i}}^{l} g_{ij}(t,x_t^j,v^i(t));$$

$$g(t,x_t,v^i(t)) = [g_1(t,x_t^1,v^1(t))^{\mathrm{T}},\ldots,(g_l(t,x_t^l,v^l(t))^{\mathrm{T}}],$$

$$B(t,x_t) = \mathrm{diag}[B_1(t,x^1),\ldots,B_l(t,x^l)].$$

ıen we can view (2.1) with decomposition (2.2) as an interconnection of l isolated bsystems (S_i) described by the equations

$$\dot{x}^i(t) = f_i(t,x_t^i,u^i(t)) + B_i(t,x_t^i)u^i(t), \qquad (S_i)$$

th interconnecting structure characterized by

$$g_i(t,x_t,v^i(t)) = \sum_{\substack{j=1\\ j\neq i}}^{l} g_{ij}(t,x_t^j,v^i(t)) \equiv g_i(t,x_t,v^i(t)).$$

ɔnditions for the existence and uniqueness of solutions are assumed. In particular $g_i(t,x_t,v^i(t))$ is assumed integrable.

First we shall state the conditions for controllability of each isolated subsystem To do this we define a matrix H_i:

$$H_i = \int_{\sigma}^{t_1} B_i(s,\phi^i)B_i^*(s,\phi^i)\mathrm{d}s, \quad t_1 > \sigma, \qquad (\mathrm{L})$$

r each $\phi^i \in C([-h,0],E^{n_i}) \equiv C^{n_i}$. Here B_i^* is the transpose of B_i.

ıeorem 2.1. *In* (S_i) *assume that*

i) *there is a continuous function* $N_{2i}^*(t)$ *such that* $\|B_i^*(t,\phi^i)\| \leqslant N_{2i}^*(t)\ \forall \phi^i \in C^{n_i}$;
i) H_i *in* (L) *has a bounded inverse*;
i) *there exist continuous functions* $G_{ij}: C^{n_i} \times E^m \to E^+$ *and integrable functions* $: E \to E^+$ $j = 1,\ldots,q$ *such that*

$$|f_i(t,\phi,u^i(t))| \leqslant \sum_{j=1}^{q} \alpha_j(t)G_{ij}(\phi,u^i(t)),$$

r *all* $(t,\phi,u(t)) \in E \times C^{n_i} \times E^{m_i}$, *where the following growth condition is satisfied*:

$$\limsup_{r\to\infty}\left(r - \sum_{j=1}^{q} c_i \sup\{G_{ij}(\phi^i,u^i): \|(\phi^i,u^i)\| \leqslant r\}\right) = \infty.$$

hen (Si) *is Euclidean controllable on* $[\sigma,t_1]$.

emark 2.1 Condition (iii) is a growth condition which should be compared to a ıiform bound imposed on f by Mirza and Womack [15, Theorem C] when treating ɛlay equations. Such growth conditions have a long history: see [9, 4, 3, 22]. In 22] one sees the consequences of the growth condition.

Proof. Let $\phi^i \in W_2^{(1)}$, $x_1^i \in E^{n_i}$. Then the solution of (S_i) is given by

$$x^i(t+\sigma) = \phi^i(t), \quad t \in [-h, 0];$$

$$x^i(t) = \phi^i(0) + \int_\sigma^t f_i(x, x_s^i, u^i(s))\mathrm{d}s + \int_\sigma^t B_i(s, x_s^i)u^i(s)\mathrm{d}s, \quad t \geqslant \sigma. \tag{2.3}$$

Now define a function u^i on $[\sigma, t_1]$ as follows:

$$u^i(t) = B_i^*(t, x_t)H_i^{-1}\left[x_1^i - \phi^i(0) - \int_\sigma^{t_1} f_i(s, x_s^i, u^i(s))\mathrm{d}s \right], \tag{2.4}$$

where $x^i(\,)$ is a solution of (S_i) corresponding to u^i with initial function ϕ^i. Such a solution exists as earlier remarked if u^i exists as an L_2 function. Since $t \to B_i(t, \phi)$ is continuous, u^i, as defined, is L_2. Introduce the following space

$$\mathrm{X} = \mathrm{C}([-h, t_1], E^{m_i}),$$

with norm

$$\|(\phi, u)\| = \|\phi\| + \|u\|_2, \quad (\phi, u) \in \mathrm{X},$$

where

$$\|\phi\| = \sup|\phi(s)|, \quad s \in [-h, t_1], \quad \|u\|_2 = \left(\int_\sigma^{t_1} |u(s)|^2 \mathrm{d}s \right)^{1/2}.$$

We show the existence of a positive constant r_0, and a subset $A(r_0)$ of X such that

$$A(r_0) = A_1(t_1, r_0) \times A_2(t_1, r_0),$$

where

$$A_1(t_1, r_0) = \{\xi : [-h, t_1] \to E^{n_i} \text{ continuous } \xi_\sigma = \phi, \ \|\xi_t\| \leqslant r_0, t \in [\sigma, t_1]\},$$

$$A_2(t_1, r_0) = \{u \in L_2(0, t_1[, E^{m_i}) : \text{(i)} |u(t)| \leqslant r_0 \text{ a.e. in } t \in [\sigma, t_1] \text{ and}$$

$$\text{(ii)} \int_\sigma^{t_1} |u(t+s) - u(t)|^2 \mathrm{d}t \to 0 \text{ as } s \to 0 \text{ uniformly with}$$

$$\text{respect to } u \in A_2(t_1, r_0)\}.$$

It is obvious that the two conditions for A_2 ensures that A_2 is a compact convex subset of the Banach space L_2 ([11, p. 297]). Define the operator T on X as follows:

$$T(x^i, u^i) = (z, v),$$

where

$$z(t+\sigma) = \phi^i(t), \quad t \in [-h, 0];$$

$$z(t) = \phi^i(0) + \int_\sigma^t f_i(s, x_s^i, u^i(s))\mathrm{d}s + \int_\sigma^t B_i(s, x_s^i)v(s)\mathrm{d}s, \quad t \geqslant \sigma; \tag{2.5}$$

$$v(t) = B_i^*(t, x_t^i)H_i^{-1}\left[x_1^i - \phi^i(0) - \int_\sigma^{t_1} f_i(s, x_s^i, u^i(s))\mathrm{d}s \right]. \tag{2.6}$$

Obviously the solutions $x^i(\cdot)$ and $u^i(\cdot)$ of (2.3) and (2.4) are fixed points of T; i.e., $T(x^i, u^i) = (x^i, u^i)$. Using Schauder's fixed-point theorem we shall prove the existence of such a fixed point in A. Let

$$\mathrm{G}_{ij}(r) = \sup\{G_{ij}(\phi^i, u^i) : \|(\phi^i, u^i)\| \leqslant r\},$$

where G_{ij} are defined in (iii). Because the growth condition of (iii) is valid there exists a constant $r_0 > 0$ such that

$$r_0 - \sum_{i=1}^{q} c_i G_{ij}(r_0) \geqslant d \quad \text{or} \quad \sum_{i=1}^{q} c_i G_{ij}(r_0) + d \leqslant r_0.$$

See a recent paper by Do [22, p. 44]. With this r_0 define $A(r_0)$ as described above. To simplify our argument we introduce the following notation:

$$\beta = \max\{\|B_i(t,\phi)\| : \sigma \leqslant t \leqslant t_1\}, \quad k = \max\{\beta(t_1 - \sigma), 1\},$$

$$\lambda = \max\{\|B_i^*(t,\phi)\| \cdot \|H_i^{-1}\| : t \in [\sigma, t_1]\}, \quad \|\alpha_i\| = \int_\sigma^{t_1} |\alpha_i(s)| \mathrm{d}s, \quad a_i = 3k\lambda\|\alpha_i\|,$$

$$b_i = 3\|\alpha_i\|, \quad c_i = \max\{a_i, b_i\}, \quad d_1 = 3k\lambda[|x_1^i| + |\phi^i(0)|],$$

$$d_2 = 3|\phi^i(0)|, \quad d = \max\{d_1, d_2\}.$$

If $(x^i, u^i) \in A(r_0)$, from (2.5) and (2.6), we have that

$$|v(t)| \leqslant \lambda\left[|x_1^i| + |\phi^i(0)| + \int_\sigma^{t_1} \sum_{j=1}^{q} \alpha_j(s) G_{ij}(x_s^i, u^i(s)) \mathrm{d}s\right]$$

$$\leqslant \lambda\left[|x_1^i| + |\phi^i(0)| + \sum_{j=1}^{q} \int_\sigma^{t_1} \alpha_i(s) G_{ij}(r_0) \mathrm{d}s\right]$$

$$\leqslant \frac{1}{3k}\left(d + \sum_{j=1}^{q} c_j G_{ij}(r_0)\right) \leqslant \left(\frac{1}{3k}\right) r_0 \leqslant \frac{r_0}{3}.$$

Also

$$\|z\| \leqslant |\phi^i(0)| + \beta(t_1 - \sigma) r_0/3k + \int_\sigma^{t_1} \sum_{j=1}^{q} \alpha_j(s) G_{ij}(x_s, u(s)) \mathrm{d}s$$

$$\leqslant \frac{\mathrm{d}}{3} + \frac{r_0}{3} + \sum_{j=1}^{q} \int_\sigma^{t_1} \alpha_j(s) G_{ij}(r_0) \mathrm{d}s$$

$$\leqslant \frac{\mathrm{d}}{3} + \frac{r_0}{3} + \sum_{i=1}^{q} \frac{c_j}{3} G_{ij}(r_0)$$

$$\leqslant \frac{r_0}{3} + \frac{r_0}{3} = \frac{2r_0}{3}.$$

We now verify that

$$\int_\sigma^{t_1} |v(t+s) - v(t)|^2 \mathrm{d}t \to 0 \quad \text{as} \quad s \to 0$$

uniformly with respect to $v \in A_2(t_1, r_0)$. Indeed

$$\int_\sigma^{t_1} |v(t+s) - v(t)|^2 \mathrm{d}t \leqslant \int_\sigma^{t_1} |[B_i^*(t+s, x_{t+s}) - B_i^*(t, x_t)] H_i^{-1} \xi|^2 \mathrm{d}t$$

$$\leqslant \|H_i^{-1}\xi\|^2 \int_\sigma^{t_1} \|B_i(t+s, x_{t+s}) - B_i^*(t, x_t)\|^2 \mathrm{d}t,$$

where

$$\xi = \left[x_1 - \phi(0) - \int_\sigma^{t_1} f_i(s, x_s, u(s)) \mathrm{d}s\right].$$

Because $t \to B_i^*(t, x_t)$ and $t \to x_t$ are continuous, we assert that indeed $\int_\sigma^{t_1} |v(t+s) - v(t)|^2 \, dt \to 0$ as $s \to 0$. This proves that $v \in A_2$ and we have completed the proof that T maps $A(r_0)$ into itself. We next prove that T is a continuous operator. This is obvious if

$$(t, \phi) \to B_i(t, \phi), \quad (t, \phi, u) \to f_i(t, \phi, u),$$

are continuous since $u \to x(\cdot, u)$ is continuous. To prove continuity in the general situation we argue as follows. Let (x^i, u^i), $(x'^i, u'^i) \in A(r_0)$ and $T(x^i, u^i) = (z(t), v(t))$, $T(x'^i, w'^i) = (z'(t), v'(t))$, where $v(t)$ is as given in (2.6) corresponding to u^i and $v'(t)$ is also given by (2.6) corresponding to w^i. Also $z(t)$ is given in (2.5) corresponding to u^i and $z'(t)$ corresponds to w. Then

$$\begin{aligned}
|v(t) - v'(t)|^2 &= \left| B_i^*(t, x_t^{iu}) \left[x_1^i - \phi^i(0) - \int_\sigma^{t_1} f_i(s, x_s^u, u(s)) \mathrm{d}s \right] \right. \\
&\quad \left. - B_i^*(t, x_t^{iw}) \left[x_1^i - \phi^i(0) - \int_\sigma^{t_1} f_i(s, x_s^{iw}, w(s)) \mathrm{d}s \right] \right|^2 \\
&\leqslant \| B_i^*(t, x_t^u) - B_i^*(t, x_t^w) \|^2 [|x_1^i| + |\phi^i(0)|]^2 \\
&\quad + \left| B_i^*(t, x_t^{iw}) \int_\sigma^{t_1} f_i(s, x_s^{iw}, w(s)) \mathrm{d}s - B_i^*(t, x_t^{iu}) \right. \\
&\quad \left. \times \int_\sigma^{t_1} f_i(s, x_s^{iu}, u(s)) \mathrm{d}s \right|^2 \qquad (2.7) \\
&\leqslant \| B_i^*(t, x_t^u) - B_i^*(t, x_t^{iw}) \|^2 [|x_1^i| + |\phi^i(0)|]^2 \\
&\quad + \| B_i^*(t, x_t^{iw}) - B_i^*(t, x_t^{iu}) \|^2 \int_\sigma^{t_1} |f_i(s, x_s^{iw}, w(s))|^2 \mathrm{d}s \\
&\quad + \| B_i^*(t, x_t^u) \|^2 \int_\sigma^{t_1} |f_i(s, x_s^{iw}, w(s)) - f_i(s, x_s^{iu}, u(s))|^2 \mathrm{d}s.
\end{aligned}$$

Since $u \to B_i^*(t, x_t^u)$ and $u \to f_i(t, x_t^u, u)$ are continuous given $\varepsilon > 0$, there is a $\eta > 0$ such that if $|u - w| < \eta$ then $|B_i^*(t, x_t^{iu} - B_i^*(t, x_t^{iw})| < \varepsilon$, and $|f_i(t, x_t^{iu}, u) - f_i(t, x_t^{iw}, w)| < \varepsilon$, $t \in [\sigma, t_1]$. Divide the interval $[\sigma, t_1]$ into two sets e_1 and e_2; put the points at which $|u(t) - w(t)| < \eta$ to be e_1 and the remainder e_2. If we write $\|u - w\|_{L_2} = \gamma$, then $\gamma^2 = \int_\sigma^{t_1} |u(t) - w(t)|^2 \mathrm{d}t \geqslant \int_{e_2} |u(t) - \omega(t)|^2 \mathrm{d}t \geqslant \eta^2 \operatorname{mes} e_2$ so that $\operatorname{mes} e_2 \leqslant \gamma^2/\eta^2$. Consider the integral

$$I = \int_\sigma^{t_1} |f_i(s, x_s^{iw}, w(s)) - f_i(s, x_s^{iu}, u(s))|^2 \mathrm{d}s.$$

Then

$$\begin{aligned}
I &= \int_{e_1} + \int_{e_2} |f_i(s, x_s^{iw}, w(s)) - f_i(s, x_s^{iu} u(s))|^2 \mathrm{d}s \\
&\leqslant \varepsilon^2 \operatorname{mes} e_1 + \frac{4\gamma^2}{\eta^2} \{\sup f_i(\,)\}^2 \\
&\leqslant \varepsilon^2 \operatorname{mes} e_1 + \frac{4\gamma^2}{\eta^2} R^2,
\end{aligned}$$

for some R. This last estimate is deduced from the fact that if (x,u), $(x',u')\in A(r_0)$ this implies that $\sup\{f_i(\,)\}\leqslant R$, for some R.

On using this last estimate in (2.7) we deduce that if $|u-w|<\eta$; then

$$|v(t)-v'(t)|^2\leqslant\varepsilon^2[|x_1^i|+|\phi^i(0)|]^2 + \varepsilon^2(t_1-\sigma)R^2+N_2^*(t)\left[\varepsilon^2(t_1-\sigma)+\frac{4\gamma^2R^2}{\eta^2}\right].$$

Thus

$$\|v-v'\|^2=\int_\sigma^{t_1}|v(s)-v'(s)|^2\,\mathrm{d}s$$

$$\leqslant(t_1-\sigma)\left[\varepsilon^2\{|x_1^i|+|\phi^i(0)|\}^2+\varepsilon^2(t_1-\sigma)R^2 + \varepsilon^2(t_1-\sigma)+\frac{4\gamma^2R^2}{\eta^2}\int_\sigma^{t_1}N_{2i}^*(s)\mathrm{d}s\right].$$

Because $\gamma^2=\|u-w\|^2$ and N_{2i}^* is integrable v and v' can be made as close as possible if u and w are sufficiently close. We next consider the term $|z(t)-z'(t)|$. We have

$$|z(t)-z'(t)|^2\leqslant\int_\sigma^t|f_i(s,x_s,u(s))-f(s,x_s',w(s))|^2\mathrm{d}s + \int_\sigma^t|B_i(s,x_s)v(s)-B_i(s,x_s')v'(s)|^2\mathrm{d}s$$

$$\leqslant\int_\sigma^t\|B_i(s,x_s)-B_i(s,x_s')\|\,|v(s)|\mathrm{d}s+\int_\sigma^t\|B_i(s,x_s')\|\,|v(s)-v'(s)|\mathrm{d}s + \int_\sigma^t|f_i(s,x_s,u(s))-f_i(s,x_s',w(s))|\mathrm{d}s.$$

Because of this inequality and an argument similar to the above, z, z' can be made as close as possible in A_1 if u, w are sufficiently close. We have proved that T is continuous in u. It is easy to see that T is continuous in x, the first argument, and thus, by a little reasoning based on the continuity hypothesis on f_i and B_i, that $T(x,u)$ is continuous on both arguments.

To be able to use Schauder's fixed point theorem, we need to verify that $T(A(r_0))$ is compact. Since $A_2(t_1,r_0)$ is compact we need only verify that if $(x,u)\in A(r_0)$ and $(z,v)=T(x,u)$ then z as defined in (2.5) is equicontinuous for each r_0. To see this we observe that for each $(x,u)\in A(r_0)$ and s_1, $s_2\in[\sigma,t_1]$, $s_1<s_2$, we have

$$|z(s_2)-z(s_1)|\leqslant\int_{s_1}^{s_2}|f_i(s,x_s,u(s))|\mathrm{d}s+\int_{s_1}^{s_2}\|B_i(s,x_s)\|\,|v(s)|\mathrm{d}s$$

$$\leqslant\beta\frac{r_0}{3}|s_2-s_1|+\int_{s_1}^{s_2}\sum_{j=1}^{q}\alpha_j(s)\mathrm{G}_{ij}(r_0)\mathrm{d}s \tag{2.8}$$

$$\leqslant\beta\frac{r_0}{3}|s_2-s_1|+\sum_{j=1}^{q}\|\alpha_i\|\,\mathrm{G}_{ij}(r_0)|s_2-s_1|$$

In the above estimate we have used the fact that $\beta = \max_{s\in[\sigma,t_1]} \|B_i(s,\phi)\|$, and

$$|v(t)| \leqslant \lambda[|x_1^i| + |\phi^i(0)| + \sum_{j=1}^{q} |(\alpha_i)|G_{ij}(r_0) \leqslant r_0/3.$$

It now follows that the right hand of (2.8) does not depend on particular choices of (x,u). Hence, the set of the first components of $T(A(r_0))$ is relatively compact. Thus $\overline{T(A(r_0))}$ is compact which by an earlier remark proves that T is a compact operator. Gathering results we have proved that $T{:}A(r_0)\to A(r_0)$ is a continuous compact operator from a closed convex subset into itself. By Schauder's fixed point theorem there exists a fixed point $(x,u) = T(x,u)$, given by (2.3) and (2.4).

$$x_\sigma = \phi x^i(t_1) = \phi^i(0) + \int_\sigma^{t_1} f_i(s,x_s^i,u^i(s))\mathrm{d}s + \int_\sigma^{t_1} B_i(s,x_s^i)B_i^*(s,x_s^i)\mathrm{d}sH_i^{-1}\xi = x_1.$$

Euclidean controllability is proved.

We now consider the criteria for controllability in W_2^1 for the system (Si).

It is well known that with L_2 controls the natural state space of (2.1) is $W_2^{(1)}$. Conditions on existence and uniqueness of solutions of variants of (2.1) are treated by Melvin [27, 28] and recently by Chukwu and Simpson [29]. Since the optimal control of the linear system has been extensively studied in $W_2^{(1)}$ it seems appropriate to treat the nonlinear case. The growth rate we desire in economics is a function.

Theorem 2.2. *In* (S_i) *assume*

(i) *Conditions* (i) *and* (iii) *of Theorem* 2.1.
(ii) *rank* $[B_i(t,\xi] = n_i$ *on* $[t_1-h,t_1]$ *for each* $\xi\in C([-h,0],E^{n_i})$, $t\in[t_1-h,t_1]$.
(iii) *The Moore Penrose generalized inverse of* B_i, $B_i^+(t,\xi)$*–is essentially uniformly bounded on* $[t_1-h,t_1]$, *for each* $\xi\in C$ $t\in[t_1-h,t_1]$;
(iv) $\xi\to B_i^+(t,\xi)$ *is continuous.*

Then (S_i) *is controllable on* $[\sigma,t_1]$, *with* $t_1>\sigma+h$.

Proof. First we show that (S_i) is Euclidean-controllable on $[\sigma,t_1-h]$. For this we let $\phi^i\in W_2^{(1)}$, $x_1^i\in E^n$. The solution $x^{(i)}$ of (S_i) with $x_\sigma^i = \phi^i$ is given by (2.3). Since hypothesis (ii) is valid, the matrix $B_i(t_1-h,x_{t_1-h}^i)B_i^*(t_1-h,x_{t_1-h}^i)$ (where $B_i^*(\,)$ is the algebraic adjoint of $B_i(\,)$) has rank n_i. Since $t\to B_i(t,x_t^i)$ is continuous, there exists some $\varepsilon>0$ such that for each s, $0\leqslant s<\varepsilon$, $B_i(t_1-h-s,x_{t_1-h-s}^i)B_i^*(t_1-h-s,x_{t_1-h-s}^i)$ has rank n_i. As a consequence of this

$$H_i(t_1-h) = \int_\sigma^{t_1-h} B_i(s,x_s^i)B_i^*(s,x_s^i)\mathrm{d}s$$

$$= \int_\sigma^{t_1-h-\varepsilon} B(s,x_s)B^*(s,x_s)\mathrm{d}s + \int_{t_1-h-\varepsilon}^{t_1-h} B(s,x_s)B^*(s,x_s)\mathrm{d}s$$

has rank n_i, since the last integral is positive definite and $H_i(t_1-h)$ is positive semidefinite. By Theorem 2.1, (S_i) is Euclidean controllable on $[\sigma,t_1-h]$, $t_1>\sigma+h$, so that given any ϕ^i, $\psi^i\in W_2^{(1)}$ there exists a $u^i\in L_2([\sigma,t_1-h],E^{m_i})$ such that the solution of (S_i) satisfies $x_\sigma^i = \phi^i$, $x^i(t_1-h,\sigma,\phi^i,u^i) = \psi^i(-h)$. We conclude the proof by extending u^i and $x^i(\cdot,\sigma,\phi^i,u^i) = x^i(\,)$ to the interval $[\sigma,t_1]$ $t_1>\sigma+h$ so that

$$\dot{\psi}^i(t-t_1) = f_i(t, x_t^i, u^i(t)) + B_i(t, x_t^i)u^i(t), \tag{2.9a}$$

e. on $[t_1 - h, t_1]$, where $x^i(\tau) = \psi^i(\tau - t_1)$, $t_1 - h \leqslant \tau \leqslant t_1$ on the right hand side of .9a). Because of the rank condition (ii) we may define a control function u^i as follows:

$$u^i(t) = B_i^+(t, x_t^{iu})\left[\frac{\mathrm{d}}{\mathrm{d}t}\psi^i(t-t_1) - f_i(t, x_t^{iu}, u^i(t))\right], \tag{2.9b}$$

r $t_1 - h \leqslant t \leqslant t_1$. That such a u exists can be proved as follows: We define the llowing set

$$A_1(r_0) = \left\{u \in L_2([t_1 - h, t_1], E^{m_i}) : \|u(\,)\|_{L_2} \leqslant r_0 \text{ and with } \int_{t_1-h}^{t_1} |u(t+s) - u(t)|\,\mathrm{d}t \to 0 \text{ as } s \to 0\right\}. \tag{2.10}$$

follows from [7, p. 297] that $A = A_1(r_0)$ is compact. Let T be a map on A defined s follows

$$T(u)(t) = v(t),$$

here

$$v(t) = B_i^+(t, x_t^u)[\dot{\psi}(t-t_1) - f_i(t, x_t^u, u(t))]. \tag{2.11}$$

/e shall prove that there is a constant r_0 such that with

$$A = A_1(r_0), \quad T: A \to A, \quad \text{where } T \text{ is continuous.}$$

ecause of [7, p. 297] and [7, p. 645], T is guaranteed a fixed point, that is

$$T(u) = u \in A,$$

hich implies that (2.9a) and (2.9b) hold. Observe that A is a compact and convex ubset of the Banach space L_2. Because of a result of Campbell and Meyer [25, . 225], and hypotheses (i) and (ii) of the theorem, the generalized inverse $t \to B_i^+(t, \xi)$ is ontinuous and therefore uniformly bounded on $[t_1 - h, t_1]$. Since the growth ondition (iii) is valid there exists a $r_0 > 0$ such that

$$\sum_{j=1}^{q} c_j G_{ij}(r_0) + d \leqslant r_0,$$

or some d. With this r_0 define $A = A_1(r_0)$. Now introduce the following notations

$$\beta = \max\{\|B_i(t,\phi)\| \sigma \leqslant t \leqslant t_1\}, \quad \beta^+ = \max\{\|B_i^+(t,\phi)\| \sigma \leqslant t \leqslant t_1\},$$

$$b = \max[\beta, \beta^+], \quad k = \max\{\beta(t_1 - \sigma), 1, b\}, \quad \lambda = \max\{\beta^+, \beta^+ \|H_i^{-1}\|\},$$

$$\|\alpha_i\| = \max\left\{\int_\sigma^{t_1} |\alpha_i(s)|\,\mathrm{d}s, \|\alpha_i\|, \sup_{\sigma \leqslant t \leqslant t_1} |\alpha_j(t)|\right\}, \quad a_i = 3k\lambda\|\alpha_i\|;$$

$$b_i = 3\|\alpha_i\|, \quad c_i = \max\{a_i, b_i\},$$

$$d_1 = \max 3k\lambda\left[|x_1^i| + |\phi^i(0)|, \|\dot{\psi}\|_{L_2}, \sup_{\sigma \leqslant t \leqslant t_1} |\dot{\psi}^i(t-t_1)|\right], \quad d_2 = 3|\phi(0)|,$$

$$d = \max[d_1, d_2].$$

Let $u \in A$. Then

$$|T(u)(t)| \leqslant |v(t)|,$$

where

$$|v(t)| \leqslant |B_i^+(t, x_t^i)|[|\dot{\psi}^i(t-t_1) - g_i(t, x_t^\mu, u] \leqslant \beta^+[|\dot{\psi}^i(t-t_1)| + |f_i(t, x_t^i, u(t))|]$$

$$\leqslant \beta^+\left[|\dot{\psi}(t-t_1)| + \sum_{j=1}^{q} \alpha_j(t) G_{ij}(x_t^i, u(t))\right]$$

$$\leqslant \beta^+\left[|\dot{\psi}(t-t_1)| + \sum_{j=1}^{q} \alpha_j(t) G_{ij}(r_0)\right]$$

$$\leqslant \frac{d_1}{3k} + \sum_{j=1}^{q} \frac{\|\alpha_j\|}{3k} G_{ij}(r_0) \leqslant \frac{1}{3k}\left[d_1 + \sum_{j=1}^{q} c_i G_{ij}(r_0)\right].$$

Therefore

$$\|v\| \leqslant \beta^+\left[\|\dot{\psi}^i\|_{L_2} + \sum_{j_1}^{q} \|\alpha_j\| G_{ij}(r_0)\right] \leqslant \lambda\left[\|\dot{\psi}\|_{L_2} + \sum_{j=1}^{q} \|\alpha_j\| G_{ij}(r_0)\right]$$

$$\leqslant \lambda\|\dot{\psi}\|_2 + \sum_{j=1}^{q} \frac{a_j}{3k} G_{ij}(r_0) \leqslant \frac{d_i}{3k} + \sum_{j=1}^{q} \frac{a_j}{3k} G_{ij}(r_0)$$

$$\leqslant \frac{1}{3k}\left[d + \sum_{j=1}^{q} c_j G_{ij}(r_0)\right] \leqslant \frac{1}{3k} r_0.$$

Therefore

$$\|T(u)\| \leqslant r_0.$$

We have proved that $T: A \to A$, if we can verify the second condition. Now

$$\int_{t_1-h}^{t_1} |v(t+s) - v(t)|^2 \, dt = \int_{t_1-h}^{t_1} B_i^+(t+s, x_{t+s}^i)\xi_i(t+s) - B_i^+(t, x_t^i)\xi_i(t)|^2 \, dt,$$

where $\xi_i(t) = \dot{\psi}^i(t-t_1) - f_i(t, x_t^i, u(t))$.

The function $k(t) \equiv B_i^+(t, x_t^i)\xi_i(t)$ is measurable in t, and is in L_2. We can therefore choose a sequence $\{k_n(t)\}$ of continuous functions such that

$$\int_{t_1-h}^{t_1} |k(t) - k_n(t)|^2 \, dt \to 0 \quad \text{as } n \to \infty.$$

Therefore

$$\left[\int_{t_1-h}^{t_1} |k(t+s) - k(t)|^2 \, dt\right]^{1/2} \leqslant \left[\int_{t_1-h}^{t_1} |k(t+s) - k_n(t+s)|^2 \, dt\right]^{1/2}$$

$$+ \left[\int_{t_1-h}^{t_1} |k_n(t+s) - k_n(t)|^2 \, dt\right]^{1/2}$$

$$+ \left[\int_{t_1-h}^{t_1} |k_n(t) - k(t)|^2 \, dt\right]^{1/2}.$$

We choose n large so that the last and first integral on the right hand side of this inequality are less than an arbitrary $\varepsilon > 0$. Also s can be made small enough for the second integral to be less than $\varepsilon > 0$. This verifies the first part of the assertion $T: A \to A$.

We now turn to the problem of continuity. Let $(u), (u') \in A(r_0)$, $T(u) = (v)$, $T(u') = (v')$.

Then

$$\begin{aligned}|v(t)-v'(t)| &\leqslant B_i^+(t,x_t^u)[\dot{\psi}^i(t-t_1)-f_i(t,x_t^u)\\ &\quad -B_i^+(t,x_t^{u'})[\dot{\psi}(t-t_1)-f_i(t,x_t^{u'},u'(t)]\\ &\leqslant \|B_i^+(t,x_t^i)-B_i^+(t,x_t')\|\,|\dot{\psi}^i(t-t_1)|\\ &\quad +\|B_i^+(t,x_t'^i)-B_i^+(t,x_t^i)\|\,|f_i(t,x_t'^i,u'(t)|\\ &\quad +\|B_i^+(t,x_t^i)\|\,[|f_i(t,x_t'^i,u'(t))-f_i(t,x_t^i,u(t))|].\end{aligned}$$

Since $u\to B_i^+(t,x_t^u)$ and $u(t)\to f_i(t,x_t,u(t))$ is continuous given $\varepsilon>0$ there exists an $\eta>0$ such that if $|u(t)-u'(t)|<\eta$ then

$$\begin{aligned}&|B_i^+(t,x_t^{iu})-B_i^+(t,x_t^{iu'})|<\varepsilon\\ &|f_i(t,x_t^{iu},u(t))-f_i(t,x_t^{iu'},u'(t))|<\varepsilon \quad \forall t\in[t_1-h,t_1]\equiv \mathrm{I}.\end{aligned}$$

Divide I into two sets e_1 and e_2 and put the points at which $|u(t)-u'(t)|<\eta$ to be e_1 and the other to be e_2. If we set $\|u-u'\|_2=\gamma$, then

$$\begin{aligned}\gamma^2 &= \int_{t_1-h}^{t_1}|u(t)-u'(t)|^2\,\mathrm{d}t\\ &\geqslant \int_{e_2}|u(t)-u'(t)|^2\,\mathrm{d}t\\ &\geqslant \eta^2\,\mathrm{mes}\,e_2\end{aligned}$$

so that $\mathrm{mes}\,e_2\leqslant\gamma^2/\eta^2$. A simple analysis shows that

$$\begin{aligned}\|v-v'\|^2 &\leqslant \varepsilon^2\,\mathrm{mes}\,e_1\,\|\dot{\psi}\|_2+2\beta^{+2}\frac{\gamma^2}{\eta^2}\|\dot{\psi}\|_2\\ &\quad +\varepsilon^2\,\mathrm{mes}\,e_2\{\sup|f(\,)|\}^2+4\beta^{+2}\frac{\gamma^2}{\eta^2}\sup\{|f_i(\,)|\}^2\\ &\quad +\beta^{+2}\varepsilon^2\,\mathrm{mes}\,e_1+\beta^{+2}\frac{4\gamma^2}{\eta^2}\{\sup|f_i(\,)|\}.\end{aligned}$$

It follows from these estimates that $\|v-v'\|^2$ can be made arbitrarily small if $\|u-u'\|^2$ is small. This proves that $T:A\to A$ is a continuous mapping of a compact convex subset of L_2 with itself. By Schauder's fixed point theorem [11, p. 645] T has a fixed point:

$$u=B_i^+(t,x_t^{iu})[\dot{\psi}^i(t-t_1)-f_i(t,x_t^{iu},u(t)].$$

With this u in

$$\dot{\psi}^i(t-t_1)=g_i(t,x_t,u(t))+B_i(t,x_t)u(t)$$

(2.9a) is satisfied. The proof is complete.

Remark. Condition (iv) can be removed by employing an argument similar to the earlier proof.

In Theorem 2.2 we have stated conditions which guarantee the controllability of each isolated free subsystem (S_i). Next we assume these conditions and give an additional condition on the interconnection g_i which will ensure that the composite system (2.2) is controllable. It should be carefully noted that

$$g_i(t, x_t, v^i(t)) = \sum_{\substack{j=1 \\ j \neq i}}^{l} g_{ij}(t, x_t^j, v^i(t))$$

is independent of x^i, the state of the ith subsystem, though it is measured locally in the (S_i) system.

Theorem 2.3. *Consider the interconnected system in* (2.2). *Assume that*

(i) *Conditions* (i)–(iii) *of Theorem* 2.1 *are valid: Thus each isolated subsystem is Euclidean controllable on* $[\sigma, t_1]$.
(ii) *For each* $i, j = 1, \ldots, l$, $i \neq j$

$$g_i(t, x_t, v^i(t)) = \sum_{\substack{j=1 \\ j \neq i}}^{l} g_{ij}(t, x_t^j, v^j(t))$$

satisfies the following growth condition: There are continuous functions

$$G_{ij}: E^{n_i} + E^{n_i} \to E^+,$$

and L^1 *functions* $\beta_j: E \to E^+ j = 1, \ldots, q$ *such that*

$$|g_i(t, x_t, v^i(t))| \leqslant \sum_{j=1}^{q} \beta_j(t) G_{ij}(x_t, u_t^i) \quad \text{for all } (t, x_t, u^i), \quad \beta_i < \alpha_i$$

where for some constants c

$$\limsup_{r \to \infty} \left(r - \sum_{j=1}^{q} c_j \sup\{G_{ij}(x_t, u^i(t)) : \|(x_t, u^i(t))\| \leqslant r\} \right) = +\infty.$$

Then (2.2) *is Euclidean controllable on* $[\alpha, t_1]$.

Theorem 2.4. *In* (2.1) *and* (2.2), *assume that*

(i) *Conditions* (i) – (iv) *of Theorem* 2.2 *hold.*
(ii) *For each* $i, j = 1, \ldots, l$, $i \neq j$, g_i *satisfies the growth condition: there are continuous functions*

$$G_{ij}: C^{n_i} \times E^{m_i} \to E^+,$$

and L^1 *functions* $a_{ij}: E \to E^+$, $j = 1, \ldots, q$ *such that*

$$|g_i(t, \phi, v^i(t))| \leqslant \sum_{j=1}^{q} \beta_j(t) G_{ij}(\phi, u^i(t)),$$

for all (t, ϕ, u^i, v^i), *where* $\beta_j < \alpha_j$, *and for some constants* c_j,

$$\limsup_{r \to \infty} \left(r - \sum_{j=1}^{q} c_j \sup\{G_{ij}(\phi, u^i) : \|(\phi, u^i)\| \leqslant r\} \right) = +\infty.$$

Then (2.2) *is controllable on* $[\sigma, t_1]$, $t_1 > \sigma + h$.

Remark 2.4 The condition (ii) of Theorem 2.3 and Theorem 2.4 is similar to the growth condition of Michel and Miller in [23, Theorem 5.8.4 (ii), Theorem 3.3.5 (iii), Theorem 3.3.2 (iii), Theorem 2.4.20 (iii)]. The condition states that the external (government intervention g_i on (S_i) (in forms of taxation, money supply, investment, etc., i.e. $g_i \equiv g_i(t, M_0/P, T, V)$) should be dominated by some "power" $\beta_j F_{ij}$ of the firm, "power" measured as a function of (I_0, C_0, X_0, j). This condition that g_i is sufficiently "small" is a nonlinear generalization of the requirement in the linear pursuit game,

$$\dot{y}(t) = a_0 y(t) + a_1 y(t-h) + p(t) + q(t), \quad p(t) \in P, \quad q(t) \in Q,$$

that

$$\operatorname{Int} P \supset Q.$$

The firm's control set (or initiative) should dominate the government's. This is a necessary and sufficient condition (on the control sets) for controllability. See Hájek [10, p. 61] for the genesis of this idea. It settles this century's basic problem: How much (in comparison to private effort (i.e. autonomous consumption, investment, export, money holding) should government intervention (i.e. $q(M_0/P_0, T_0, v)$) be in the economy. The nonlinearity of (2.2) has been well motivated in our introduction. The interconnectedness is natural and essential in the economic application. As a control action of government, $q(t) = \lambda[(e_1/r)(M_0/P_0) - cT_0(t) + v(t)]$, in (1.10) and (1.11) is (realistically) not linear in $w(t) = (w^1, w^2, w^3, w^4) \equiv (M_0, P_0, T_0, V)$. We combine the fiscal and the monetarist views. The modern debate of macroeconomics, particularly of Lucas critique [12] makes the incorporation of $q(t)$ very reasonable. (see [17, Macroeconomics in the Global Economy, Chapter 10]. The argument demands a game theoretic formulation for the dynamics of income. This is well spelled out in Mullinex [26, p. 91]. We are therefore compelled to insert a nontrivial g_i. Mathematicians often object to and scoff at the full rank of B, but the economic insight of Tinbergen in [17, p. 5, 90] shows how essential this "classical non-degeneracy assumption is in executing monetary and fiscal policies to achieve a target with several dimensions.

Proof of Theorem 2.3. The proof parallels that of Theorem 2.1. The integral equation of (2.2) corresponding to (2.3) is

$$x^i(t) = \phi^i(0) + \int_\sigma^t f_i(x, x_s^i, u^i(s))\mathrm{d}s + \int_\sigma^t g_i(s, x_s^i, v^i(s))\mathrm{d}s + \int_\sigma^t B_i(s, x_s^i)u^i(s)\mathrm{d}s,$$

$$t \geqslant \sigma.$$

The control function corresponding to (2.4) is defined by

$$u^i(t) = B_i^*(t, x_t^i)H_i^{-1}\left[x_1^i - \phi^i(0) - \int_\sigma^{t_1} f_i(s, x_s^i, u^i(s))\mathrm{d}s - \int_\sigma^{t_1} g_i(s, x_s^i, v^i(s))\mathrm{d}s\right].$$

This control steers ϕ^i to x_1^i in time t_1. The additional sum $\int_\sigma^t g_i(s, x_s^i, v^i(s))\mathrm{d}s$ is utilized in the estimates by using condition (ii), noting that $\beta_i < \alpha_i$. Just as in the proofs of Theorems 2.1 and 2.2 under the conditions of Theorem 2.3 the system (2.2) is Euclidean controllable. The operator T is defined as in the proof of Theorem 2.1 with the

modification that

$$T(x^i, u^i) = (z, v),$$

where

$$z(t+\sigma) = \phi^i(t), \quad t \in [-h, 0],$$

$$z(t) = \phi^i(0) + \int_\sigma^t f_i(s, x_s^i, u^i(s))\mathrm{d}s + \int_\sigma^t B_i(s, x_s^i)\mathbf{v}(s) + \int_\sigma^t g_i(s, x_s^i, v(s))\mathrm{d}s,$$

$$\mathbf{v}(t) = B_i^*(t, x_t^i)H_i^{-1}\left[x_1^i - \phi^i(0) - \int_\sigma^{t_1} f_i(s, x_s^i, u^i(s))\mathrm{d}s \right.$$

$$\left. - \int_\sigma^{t_1} g_i(s, x_s^i, v(s))\mathrm{d}s \right].$$

Just as before we prove that T has a fixed point: $T(x^i, u^i) = (x^i, u^i)$, so that (2.2) is Euclidean controllable. To prove this we suppose that

$$\mathrm{G}_{ij}(r) = \sup\{\mathrm{G}_{ij}(\phi, u) : \|(\phi, u)\| \leqslant r\}.$$

Because of the growth condition in (ii), there is some r_0 such that $\sum_{j=1}^{q} c_i \mathrm{G}_{ij}(r_0) + d \leqslant r_0$ for some c_i, d. With this we deduce the estimate

$$|\mathbf{v}(t)| \leqslant \lambda\left[|x_1^i| + |\phi^i(0)| \sum_{j=1}^{q} \int_\sigma^{t_1} (\alpha_j(s) + \beta_j(s))\mathrm{G}_{ij}(r_0)\mathrm{d}s \right]$$

$$\leqslant \lambda\left[|x_1^i| + |\phi^i(0)| + \sum_{j=1}^{q} 2\int_\sigma^{t_1} \alpha_i(s)\mathrm{G}_{ij}(r_0)\mathrm{d}s \right],$$

since $\beta_j < \alpha_j$ by condition (ii). With this,

$$|\mathbf{v}(t)| \leqslant \frac{1}{3k}\left(d_1 + \sum_{j=1}^{q} 2|\alpha_j|\mathrm{G}_{ij}(r_0) \right) \leqslant \frac{1}{3k}\left(d + \sum_{j=1}^{q} c_i \mathrm{G}_{ij}(r_0) \right) \leqslant \frac{1}{3k} \leqslant \frac{r_0}{3}.$$

In the same way, we have $\|z\| \leqslant \dfrac{2r_0}{3}$. Thus

$$T : A(r_0) \to A(r_0).$$

T is a continuous operator since $(t, \phi) \to B_i(t, \phi), (t, \phi, v) \to f_i(t, \phi, v)$ and $(t, \phi, u) \to f_i(t, \phi, u)$ are continuous and $u \to x(\cdot, u)$ is continuous. The general situation follows an argument that yielded (2.7) and the subsequent inequalities. For equicontinuity the inequality (2.8) has an extra term $\int_{s_1}^{s_2} \sum_{j=1}^{q} \alpha_j(s)\mathrm{G}_{ij}(r_0)\mathrm{d}s$ due to g_i. The reasoning is as before. The interconnected system is Euclidean controllable.

Proof of Theorem 2.4. Our proof here parallels that of Theorem 2.2. From Theorem 2.3 we conclude that (2.2) is Euclidean controllable on $[\sigma, t_1 - h]$, so that given any ϕ^i, $\psi^i \in W_2^{(1)}$ there exists a $u^i \in L_2([\alpha, t_1 - h], E^{m_i})$, such that the solution x of (2.2) satisfies

$$x_\sigma^i = \phi^i \quad \text{and} \quad x^i(t_1 - h, \sigma, \phi^i, u) = \psi^i(-h).$$

The control u and the solution $x^i(\cdot, \sigma, \phi^i, u)$ are extended on the interval $[\sigma, t_1], t_1 > \sigma + h$ so that

$$\dot{x}(t) = f_i(t, x_t^i, u(t)) + g_i(t, x_t^i, v^i(t)) + B_i(t)u(t)$$

for $t_1 - h \leqslant t \leqslant t_1$, where $x(t) = \psi(t - t_1)$, $t_1 - h \leqslant t \leqslant t_1$. Define a control

$$\mathbf{v}(t) = B_i^+(t, x_t^u)[\dot{\psi}(t - t_1) - f_i(t, x_t^u, u(t)) - g_i(t, x_t^u u(t), v^i(t))]. \tag{2.12}$$

The various estimates that lead to the proof of the existence of a fixed point carry through with α_j replaced by $2\alpha_j$ (since $\beta_j < \alpha_j$) and $\xi_i(t)$ defined by

$$\xi_i(t) = \dot{\psi}(t - t_1) - f_i(t, x_t^i, u(t) - g_i(t, x_t^u, v(t)).$$

In all the calculations one remembers that once v is chosen and fixed, u is allowed to vary with its constraints. With minor modification caused by adding $\sum_{j=1}^{q} \beta_j(t)\mathrm{G}_{ij}(x_t^i, u^i(t))$ the rest of the proof is completed as in the case when $g_i \equiv 0$.

Remark. An economic interpretation may define β_i as a measure of government intervention while α_i is a measure of the firm's reaction. To ensure controllability $\alpha_i > \beta_j$.

3. General nonlinear systems

In (2.2) it is very important that the system is of the form in which some term is linear in u. Here we consider the more general situation

$$\dot{x}(t) = f(t, x_t, u(t)), \tag{3.1}$$

where $f: E \times C \times E^m \to E^n$ is continuously differentiable in the second and third arguments, and is continuous, and it also satisfies all the conditions of Lemma 1.1. Details of the proof of the following is contained in Chukwu [8].

Theorem 3.1. *In* (3.1) *assume that:*

(i) $f(t, 0, 0) = 0 \; \forall t \geqslant \sigma$.
(ii) *The system*

$$\dot{z}(t) = L(t, z_t) + B(t)v(t), \tag{3.2}$$

is controllable on $[\sigma, t_1]$, *where* $t_1 \geqslant \sigma + h$, *and where*

$$D_2 f(t, 0, 0)z_t = L(t, z_t), \quad D_3 f(t, 0, 0)v = B(t)v.$$

Then

$$0 \in \mathrm{Int}\, \mathscr{A}(t, \sigma), \tag{3.3}$$

where

$$\mathscr{A}(t, \sigma) = \{x_t(\sigma, 0, u): u \in L_2([\sigma, t], E^m) \, \|u\|_{L_2} \leqslant 1 \, x(u) \text{ is a solution of (3.1) with } x_\sigma = 0\} \tag{3.4}$$

is the attainable set associated with (3.1).

Remark. The argument in the proof is as follows. The solution of (3.1) with $x_\sigma = 0$ is

$$x(t) = \int_\sigma^t f(s, x_s(u), u(s))\mathrm{d}s.$$

The mapping

$$(Fu)(t) = \int_\sigma^t f(s, x_s(u), u(s))\mathrm{d}s, \quad F: L_2([\sigma, t_1], E^m) \to W_2^{(1)}$$

can be demonstrated to be Gateaux differentiable with Gateaux derivative

$$\delta F(0, v) = F'(0)v = z_t(0, v),$$

where $z(t, 0, v)$ is a solution of (3.2). Because $F'(0): L_2 \to W_2^{(1)}$ is a surjection because of Condition (ii), all the requirements of Corollary 15.2 of [19, p. 155] are met. Therefore F is locally open which implies (3.3).

We shall next investigate the large scale system

$$\dot{x}^i(t) = f_i(t, x_t^i, u^i(t)) + \sum_{\substack{j=1 \\ i \neq j}}^{l} g_{ij}(t, x_t^j, v^i(t)) \tag{3.5}$$

where f_i and g_{ij} are as defined following (2.2). Thus we investigate the interconnected system (2.1)

$$\dot{x}(t) = f(t, x_t, u(t)) + g(t, x_t, v(t)) \tag{3.6}$$

where f and g are identified following (2.2). We state the following result.

Theorem 3.2. *Consider the large scale system* (3.6) *with its decomposition* (3.5) *where*

(i) $f_i(t, 0, 0) = 0, \quad g_{ij}(t, \phi, 0) = 0.$
(ii) f_i, g_i *satisfy all the requirements of Lemma* 1.1.
(iii) *Assume that the linear variational system*

$$\dot{z}^i(t) = L_i(t, z_t^i) + B^i(t)v(t) \tag{3.7}$$

of

$$\dot{x}^i(t) = f_i(t, x_t^i, u^i(t)), \tag{3.8}$$

where

$$D_2 f_i(t, 0, 0) z_t^i = L_i(t, z_t^i), \quad D_3 f_i(t, 0, 0)v = B^i(t)v,$$

is controllable on $[\sigma, t_1]$, $t_1 > \sigma + h$.
Then the interconnected system (3.5) *is locally null controllable with constraints.*

COROLLARY 3.2

Assume

(i) *Conditions* (i)–(iii) *of Theorem* 3.2.
(ii) *The system*

$$\dot{x}^i(t) = f_i(t, x_t^i, 0) \tag{3.9}$$

is globally exponentially stable.

Then the composite system is (globally) null controllable with controls in

$$U_i = \{u^i \in L_2([\sigma, t_1], E^{m_i}), \quad \|u^i\|_{L_2} \leqslant 1\}.$$

Proof of Theorem. By Theorem 3.1,

$$0 \in \operatorname{Int} \mathscr{A}_i(t, \sigma) \quad \text{for} \quad t > \sigma + h, \tag{3.10}$$

where $\mathscr{A}_i$ is the attainable set associated with (3.8). Let x^i be the solution of (3.5), with $x_\sigma^i = 0$. Then

$$x^i(\sigma, 0, u^i, u^j)(t) = \int_\sigma^t f_i(s, x_s^i, u^i(s))\mathrm{d}s + \int_\sigma^t \sum_{\substack{j=1 \\ i \neq j}}^{l} g_{ij}(s, x_s^j, v^j(s))\mathrm{d}s.$$

Thus, if we define the set

$$H_i(t_1, \sigma) = \{x^i(\sigma, 0, u^i, v^i) \in W_2^{(1)}([0, t_1], E^{n_i}) : u^i \in U_i\, v^i \in U_j\},$$

we deduce that

$$\mathscr{A}_i(t, \sigma) \subset H_i(t_1, \sigma).$$

Because $f_i(t, 0, 0) = g_{ij}(t, \phi, 0) = 0$ and because $x^i(t, 0, 0, 0) = 0$ is a solution of (3.5), $0 \in H_i(t_1, \sigma)$. As a result of this and (3.10) we deduce

$$0 \in \operatorname{Int} \mathscr{A}_i(t_1, \sigma) \subset H_i(t_1, \sigma). \tag{3.11}$$

There is an open ball $\mathbb{B}(0, r)$ center zero, radius r such that

$$0 \in \mathbb{B}(0, r) \subset \mathbb{R}_i(t_1, \sigma) \subset H_i(t_1, \sigma).$$

The conclusion

$$0 \in \operatorname{Int} H_i(t_1, \sigma), \tag{3.12}$$

follows at once. Using this one deduces readily that $0 \in \operatorname{Int} \mathscr{D}$, the interior of the domain of null controllability of (3.5), proving local null controllability with constraints.

Proof of Corollary 3.2. One uses the control $u^i = 0 \in U_i$ $v^i = 0 \in U_i$ to glide along the system (3.5) and approach an arbitrary neighborhood $\mathscr{O}$ of the origin in $W_2^{(i)}([-h, 0], E^{n_i})$. Note that

$$0 \in U_i = \{u^i \in L_2([0, \sigma], E^{m_i}), \quad |\|u^i\|_{L_2} \leqslant 1\}.$$

Because of stability in hypothesis (ii) of (3.9) every solution with $u^i = 0$ is entrapped in $\mathscr{O}$ in time $\sigma \geqslant 0$. Since (i) guarantees that all initial states in this neighborhood $\mathscr{O}$ can be driven to zero in finite time, the proof is complete.

Remark. Conditions for global stability of hypothesis (ii) are available in Chukwu [6, Theorem 4.2].

Remark 3.1. From the condition

$$0 \in \operatorname{Int} \mathscr{A}_i(t, \sigma) \subset H_i(t, \sigma), \tag{3.11}$$

we deduced that

$$0 \in \operatorname{Int} H_i(t, \sigma), \tag{3.12}$$

is of fundamental importance. If the condition

$$0 \in \operatorname{Int} \mathscr{A}_i(t, \sigma), \tag{3.13}$$

fails, the isolated system is "not well-behaved" and cannot be controlled (3.12) may still prevail and the composite system may be locally controllable. To have this situation we require

$$0 \in \operatorname{Int} G_i(t, \sigma), \tag{3.14}$$

where

$$G_i(t, \sigma) = \left\{ \int_\sigma^t \sum_{\substack{j=1 \\ i \neq j}}^{l} g_{ij}(s, x_s^j, v^i(s)) \mathrm{d}s : v^i \in U_i \right\}.$$

In words, we require a sufficient amount of control impact (i.e., (3.14)) to be brought to bear on (S_i), which is not an integral part of S_i. Thus knowing the limitations of the control $u^i \in U_i$ a sufficient signal $g_{ij}(t, x_t^j, v^i)$ is despatched to make (3.14) hold. And (3.12) will follow.

Remark 3.2. The same type of reasoning yields a result similar to Theorem 3.2 if we consider the system

$$\dot{x}_i(t) = f_i(t, x_t^i, u^i(t)) + g_i(t, x_t^1 \cdots x_t^l, u^1(t) \cdots u^l(t)), \tag{3.15}$$

where

(i) $f(t, 0, 0) = 0$

(ii) $g_i(t, x_t^1, \ldots, x_t^{i-1}, 0, x_t^{i+1}, \ldots, x_t^l, u^1(t), \ldots, u^{i-1}(t), 0, u^{i+1}(t), \ldots, u(t)) = 0,$
$g_i(t, x_t, \ldots, x_t^l, 0, 0 \ldots 0) = 0.$

Also conditions (ii) and (iii) of Theorem 3.2 are satisfied.
If we consider

$$\dot{x}^i(t) = f_i(t, x_t^i, u^i(t)) + \sum_{\substack{j=1 \\ i \neq j}}^{l} g_{ij}(t, x_t^j, u^i(t)), \tag{3.16}$$

instead of (3.5) we can obtain the following result.

Theorem 3.3. *In* (3.16) *assume* (i)→(iii) *of Theorem* 3.2. *But in* (3.7) $L_i(t, z_t^i)$ *and* $B^i(t)$ *are defined as*

$$D_2 f_i(t, 0, 0) z_t^i = L_i(t, z_t^i), \quad D_3(f_i(t, 0, 0) + g_i(t, x_t, 0)) = B^i(t).$$

Then (3.16) *is locally null-controllable with constraints.*

The proof is essentially the same as that of Theorem 3.1. We note that the essential

uirement for (3.16) to be locally null-controllable is the controllability of

$$\dot{x}(t) = L(t, x_t) + (B_1(t) + B_2(t))u(t), \tag{3.17}$$

ere

$$B_1(t) = D_3 f(t, 0, 0), \quad B_2(t) = D_3 g(t, x_t, 0).$$

he isolated system (3.8) is not "proper" (and this may happen where $B_1(t)$ does have full rank on $[\sigma, t_1]$, $t > \sigma + h$, the "solidarity function" g_i can be brought bear to force the full rank of $B = B_1 + B_2$, from which (3.16) will be "proper" ause (3.17) is controllable. Even if B_1 has full rank and (3.8) is proper, the large le system need not be locally null controllable. The function has to be so nice that $+ B_1$ has full rank. An adequate "proper" amount of "regulation" is needed in the m of a "solidarity function" g_i.

n applications it is important to know something about g_i and to decide its equacy. It is possible to consider g_i as a control and view

$$\dot{x}_i(t) = f_i(t, x_t^i, u^i(t)) + g_i(t),$$

a differential game. Considered in this way the control set for g_i can be described. the linear case see Chukwu [7].

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ıote on integrable solutions of Hammerstein integral equations

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Abstract. We derive a set of sufficient conditions for the existence of solutions of a Hammerstein integral equation.

Keywords. Hammerstein integral equation; Caratheodory condition; Lusin theorem; Scorza Dragoni theorem; Schauder fixed point theorem.

ntroduction

e of the most frequently investigated integral equations in nonlinear functional lysis is the Hammerstein equation

$$x(t) = a(t) + \int_0^1 k(t,s) f(t, x(s))\,\mathrm{d}s \quad t \in [0,1]. \tag{1}$$

h an equation has been studied in several papers and monographs [1–6]. Existence orems for eq. (1) can be obtained by applying various fixed point principles. In Banas proved an existence theorem for (1) using the measure of weak non-ıpactness. On the other hand Emmanuele [5] established an existence theorem the same equation using Schauder's fixed point theorem. In this paper we shall ve the existence of solutions of the following nonlinear Hammerstein equation

$$x(t) = g(t, x(t)) + \int_0^1 k(t,s) f(t, x(\sigma(s)))\,\mathrm{d}s \quad t \in [0,1] \tag{2}$$

suitably adopting the technique of [5]. The result generalizes the result of [5].

Existence theorem

ɔrder to prove existence theorem for (2) we shall first prove the following theorem:

eorem 1. *Assume that*

$a_1 \in L^1[0,1]$ *and* $a_1(t) \geqslant 0$ *for all* $t \in [0,1]$.
$f:[0,1] \times R \to R$ *satisfies Caratheodory condition and there exist* $a_2 \in L^1[0,1]$ *and* $b_2 > 0$ *such that*

$$|f(t, x(t))| \leqslant a_2(t) + b_2 |x(t)|$$

for a.e. $t \in [0,1]$ *and all* $x \in R$.

(iii) $k:[0,1]\times[0,1]\to R^+$ *is measurable with respect to both variables and is such that the integral* operator

$$Kx(t)=\int_0^1 k(t,s)x(s)\,\mathrm{d}s \text{ maps } L^1[0,1] \text{ into itself.}$$

(iv) $\sigma:[0,1]\to[0,1]$ *is absolutely continuous and there exists a constant* $M>0$ *such that* $\sigma'(t)\geqslant M$ *for all* $t\in[0,1]$.

(v) $b_1+\dfrac{b_2\|K\|}{M}<1.$

Then there exists a unique a.e. non-negative function $\varphi\in L^1[0,1]$ *such that*

$$\varphi(t)=\frac{a_1(t)}{1-b_1}+\frac{1}{1-b_1}\int_0^1 k(t,s)[a_2(s)+b_2\varphi(\sigma(s))]\,\mathrm{d}s.$$

Proof. Define a function $\psi:[0,1]\to R$ by

$$\psi(t)=a_1(t)+\int_0^1 k(t,s)a_2(s)\,\mathrm{d}s.$$

Put $B_r=\{x\in L^1[0,1]:\|x\|\leqslant r\}$ where $r=\dfrac{M\|\psi\|}{M-b_1M-b_2\|K\|}$.

Define an operator $F:L^1[0,1]\to L^1[0,1]$ by

$$Fx(t)=\frac{a_1(t)}{1-b_1}+\frac{1}{1-b_1}\int_0^1 k(t,s)[a_2(s)+b_2x(\sigma(s))]\,\mathrm{d}s.$$

From our assumptions for $x\in B_r$ we have

$$\begin{aligned}\|Fx\|&=\int_0^1|Fx(t)|\,\mathrm{d}t\\
&\leqslant\frac{1}{1-b_1}\int_0^1 a_1(t)\,\mathrm{d}t+\frac{1}{1-b_1}\int_0^1\left|\int_0^1 k(t,s)[a_2(s)+b_2x(\sigma(s))]\,\mathrm{d}s\right|\mathrm{d}t\\
&\leqslant\frac{1}{1-b_1}\int_0^1\left[a_1(t)+\int_0^1 k(t,s)a_2(s)\,\mathrm{d}s\right]\mathrm{d}t\\
&\quad+\frac{1}{(1-b_1)M}\int_0^1\int_0^1 k(t,s)b_2|x(\sigma(s))|\sigma'(s)\,\mathrm{d}s\,\mathrm{d}t\\
&\leqslant\frac{1}{1-b_1}\|\psi\|+\frac{1}{(1-b_1)M}b_2\|K\|\|x\|\\
&\leqslant\frac{1}{1-b_1}\|\psi\|+\frac{1}{(1-b_1)M}b_2\|K\|r=r.\end{aligned}$$

Thus we have $F(B_r)\subset B_r$. If we define $B_r^+=\{x\in B_r:x(t)\geqslant 0 \text{ a.e.}\}$ then $F(B_r^+)\subset B_r^+$. Also B_r^+ is a complete metric space, since B_r^+ is a closed subset of $L^1[0,1]$.

Now for any two elements $x, y \in B_r^+$ we have

$$\|Fx - Fy\| \leqslant \frac{1}{1-b_1}\int_0^1 \left| \int_0^1 k(t,s) b_2 [x(\sigma(s)) - y(\sigma(s))]\, ds \right| dt$$

$$\leqslant \frac{1}{(1-b_1)M} b_2 \|K\| \|x-y\| \|x-y\|.$$

On applying contraction fixed point theorem we get a fixed point for F. This proves Theorem 1.

Theorem 2. *Assume that*

(i) $g:[0,1] \times R \to R$ *satisfies Caratheodory conditions and there exist* $a_1 \in L^1[0,1]$ *and* $b_1 > 0$ *such that*

$$|g(t, x(t))| \leqslant a_1(t) + b_1 |x(t)|$$

for a.e. $t \in [0,1]$ *and for* $x \in R$ *and*

$$|g(t, x(t) \cdot g(s, x(s))| \leqslant \omega(|t-s|)$$

where $\omega(|t-s|) \to 0$ *as* $t \to s$.

(ii) $f:[0,1] \times R \to R$ *satisfies Caratheodory condition and there exist* $a_2 \in L^1[0,1]$ *and* $b_2 > 0$ *such that*

$$|f(t, x(t))| \leqslant a_2(t) + b_2 |x(t)|$$

for a.e. $t \in [0,1]$ *and all* $x \in R$.

(iii) $k:[0,1] \times [0,1] \to R^+$ *satisfies Caratheodory condition and is measurable with respect to the second variable. Also the integral operator*

$$Kx(t) = \int_0^1 k(t,s) x(s)\, ds \text{ maps } L^1[0,1] \text{ into itself.}$$

(iv) $\sigma:[0,1] \to [0,1]$ *is absolutely continuous and there exists a constant* M *such that* $\sigma'(t) \geqslant M$ *for all* $t \in [0,1]$.

(v) $b_1 + \dfrac{b_2 \|K\|}{M} < 1.$

Then (2) has a solution in $L^1[0,1]$.

Proof. Since all the assumptions of Theorem 1 are satisfied, there exists a unique a.e. non-negative function φ such that

$$\varphi(t) = \frac{1}{1-b_1}\left\{ a_1(t) + \int_0^1 k(t,s)[a_2(s) + b_2 \varphi(\sigma(s))]\, ds \right\}.$$

First let us assume $\varphi = 0_{L^1[0,1]}$ in $L^1[0,1]$. In this case, if we take

$$y(t) = g(t, \varphi(t)) + \int_0^1 k(t,s) f(t, \varphi(\sigma(s)))\, ds$$

then

$$|y(t)| \leqslant a_1(t) + b_1\varphi(t) + \int_0^1 k(t,s)[a_2(s) + b_2\varphi(\sigma(s))]\,ds = \varphi(t)$$

and so $y(t) = 0$. Therefore $\varphi = 0_{L^1[0,1]}$ is the solution of (2). Now, assume that $\varphi \neq 0_{L^1[0,1]}$. Define a set Q in $L^1[0,1]$ by

$$Q = \{x \in L^1[0,1] : |x(t)| \leqslant \varphi(t) \text{ a.e.}\}.$$

Then clearly Q is nonempty, bounded, closed and convex set in $L^1[0,1]$. Define an operator $H : L^1[0,1] \to L^1[0,1]$ by

$$Hx(t) = g(t, x(t)) + \int_0^1 k(t,s) f(t, x(\sigma(s)))\,ds.$$

Then according to our assumptions H is continuous and for $x \in Q$, we have

$$\begin{aligned} |Hx(t)| &\leqslant a_1(t) + b_1|x(t)| + \int_0^1 k(t,s)[a_2(s) + b_2|x(\sigma(s))|]\,ds \\ &\leqslant a_1(t) + b_1\varphi(t) + \int_0^1 k(t,s)[a_2(s) + b_2\varphi(\sigma(s))]\,ds \\ &= \varphi(t). \end{aligned}$$

Therefore $H(Q) \subset Q$. Now we shall prove that $H(Q)$ is relatively compact. Using Lusin's and Scorza-Dragoni's theorems [see 5] for each positive integer n there exists a closed set $A_n \subset [0,1]$ such that $m(A_n^c) < (1/n)$ and $a_1|_{A_n}, \varphi|_{A_n}, k|_{A_n \times [0,1]}$ are uniformly continuous. Now let (y_k) be a sequence in Q. For $t', t'' \in A_n$ we have

$$\begin{aligned} |Hy_k(t') - Hy_k(t'')| &\leqslant |g(t', y_k(t')) - g(t'', y_k(t''))| \\ &\quad + \int_0^1 |k(t',s) - k(t'',s)|[a_2(s) + b_2\varphi(\sigma(s))]\,ds \\ &\leqslant \omega(|t' - t''|) + \int_0^1 |k(t',s) - k(t'',s)|[a_2(s) + b_2\varphi(\sigma(s))]\,ds. \end{aligned}$$

This proves that (Hy_k) is a sequence of equicontinuous functions on A_n. Also for every $t \in A_n$ we have

$$|Hy_k(t)| \leqslant a_1(t) + b_1\varphi(t) + \int_0^1 k(t,s)[a_2(s) + b_2\varphi(\sigma(s))]\,ds.$$

Because of the continuity of a_1 and φ on the compact set A_n and k on the compact set $A_n \times [0,1]$ the sequence (Hy_k) is equibounded on A_n. By applying the Ascoli–Arzela theorem we get for each n there exists a subsequence $(y_{k(h)})$ of (y_k) such that $(Hy_{k(h)})$ is a Cauchy sequence in the space $C^\circ(A_n)$ of all equicontinuous and equibounded functions on A_n. Now, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_A \varphi(s)\,ds < (\varepsilon/4)$ whenever $m(A) < \delta$. Choose a positive integer N such that $(1/N) < \delta$. Then $m(A_N^c) < \delta$. Therefore

$$\int_{A_N^c} |Hy_{k(h')}(t) - Hy_{k(h'')}(t)|\,dt \leqslant \int_{A_N^c} \varphi(t)\,dt + \int_{A_N^c} \varphi(t)\,dt < \frac{\varepsilon}{2}.$$

Also

$$\int_{A_N} |Hy_{k(h')}(t) - Hy_{k(h'')}(t)| \mathrm{d}t < \frac{\varepsilon}{2}$$

for sufficiently large h' and h'' since (Hy_k) is a Cauchy sequence in $C^\circ(A_N)$. Hence

$$\begin{aligned} \| Hy_{k(h')} - Hy_{k(h'')} \|_{L^1[0,1]} &= \int_{A_N^c} |Hy_{k(h')}(t) - Hy_{k(h'')}(t)| \mathrm{d}t \\ &\quad + \int_{A_N} |Hy_{k(h')}(t) - Hy_{k(h'')}(t)| \mathrm{d}t \\ &< \varepsilon \end{aligned}$$

for sufficiently large h' and h''. Therefore $(Hy_{k(h)})$ is a convergent subsequence of the sequence (Hy_k) in $L^1[0,1]$. This proves the relative compactness of $H(Q)$. Applying the Schauder fixed theorem we get a fixed point for H. This proves our theorem.

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On over-reflection of acoustic-gravity waves incident upon a magnetic shear layer in a compressible fluid

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Abstract. A study is made of over-reflection of acoustic-gravity waves incident upon a magnetic shear layer in an isothermal compressible electrically conducting fluid in the presence of an external magnetic field. The reflection and transmission coefficients of hydromagnetic acoustic-gravity waves incident upon magnetic shear layer are calculated. The invariance of wave-action flux is used to investigate the properties of reflection, transmission and absorption of the waves incident upon the shear layer, and then to discuss how these properties depend on the wavelength, length scale of the shear layers, and the ratio of the flow speed and phase speed of the waves. Special attention is given to the relationship between the wave-amplification and critical-level behaviour. It is shown that there exists a critical level within the shear layer and the wave incident upon the shear layer is over-reflected, that is, more energy is reflected back towards the source than was originally emitted. The mechanism of the over-reflection (or wave amplification) is due to the fact that the excess reflected energy is extracted by the wave from the external magnetic field. It is also found that the absence of critical level within the shear layer leads to non-amplification of waves. For the case of very large vertical wavelength of waves, the coefficients of incident, reflected and transmitted energy are calculated. In this limiting situation, the wave is neither amplified nor absorbed by the shear layer. Finally, it is shown that resonance occurs at a particular value of the phase velocity of the wave.

Keywords. Over-reflection; gravity waves; magnetic shear layer.

1. Introduction

During the last decade, considerable attention has been given to the phenomenon of over-reflection (wave amplification) of a hydrodynamic or hydromagnetic gravity wave incident upon a shear layer in an incompressible homogenous or stratified fluid. It has been known that the reflection coefficient for waves of one kind or another incident upon a shear layer can be greater than unity. This implies that more energy is reflected back towards the source than was originally emitted. This phenomenon known as over-reflection (wave amplification) occurs in various hydrodynamic and hydromagnetic fluid models under different conditions.

Several authors including Booker and Bretherton [3], Jones [7], Breeding [4], Jones and Houghton [8], Acheson [1, 2], McKenzie [9], Eltayeb and McKenzie [6] and Kandaswamy and Palaniswamy [10] have studied various aspects of the critical layer for internal gravity waves in a shear flow, critical layer for internal gravity waves in a shear flow, critical-level behaviour and over-reflection of a hydrodynamic

or hydromagnetic gravity wave incident upon a shear layer in an incompressible homogenous or stratified fluid. The over-reflection of internal gravity waves by a finite layer of constant shear separated by two uniform streams of incompressible fluid has been investigated analytically by Eltayeb and McKenzie [8] and numerically by Jones [7] and Breeding [4]. Mckenzie [8] has studied the reflection and refraction of a plane acoustic-gravity wave at an interface separating two fluids in relative motion. He predicted the phenomenon of over-reflection for pure acoustic waves provided the shear flow speed exceeds the horizontal phase speed of the incident gravity wave. A discussion of this result implies that the gravity waves can extract energy and momentum from the mean flow along with the idea of a critical layer at which the energy and momentum of gravity waves are absorbed into the mean flow. Acheson [2] has investigated the phenomenon of over-reflection for a variety of different systems involving waves propagating towards a shear layer. He studied the reflection of hydromagnetic internal gravity waves travelling in an incompressible fluid towards a vortex-current sheet with special attention to the relationship between over-reflection and critical layer absorption. Recently, the over-reflection of hydromagnetic gravity waves in a compressible stratified fluid was considered by Kandaswamy and Palaniswamy [10]. In spite of these works, attention is hardly given to the phenomenon of over-reflection of hydromagnetic waves in a compressible fluid.

The main objective of this paper is to study the phenomenon of over-reflection of acoustic-gravity waves incident upon a magnetic shear layer in an isothermal compressible electrically conducting fluid in the presence of an external magnetic fluid. The invariance of the wave-action flux is used to investigate the properties of reflection, transmission, and absorption of the acoustic-gravity waves incident upon the magnetic shear layer, and then to discuss how these properties depend on the wavelength, length scale of the shear layer, and the ratio of the flow speed and the phase speed of the waves. Special attention is given to the relationship between the wave amplification and critical-level behaviour. The over-reflection is due to the fact that the excess reflected energy is extracted by the wave from the external magnetic field. For the case of very large vertical wavelength, the coefficients of the incident, reflected and transmitted energy are calculated. In this limiting situation, the hydromagnetic acoustic-gravity wave is neither amplified nor absorbed by the magnetic shear layer. It is also shown that resonance occurs at a particular value of the phase velocity of the wave.

2. Basic equations

The basic hydromagnetic equations governing the unsteady motion of an isothermal compressible electrically conducting fluid in the presence of an external magnetic field **H** are in standard notation (Chandrasekhar [5]):

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} + [(\nabla \times \mathbf{H}) \times \mathbf{H}] \tag{2.1}$$

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) = 0 \tag{2.2}$$

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} \tag{2.3}$$

$$\frac{D\mathbf{H}}{Dt} = (\mathbf{H}\cdot\nabla)\mathbf{u} - \mathbf{H}(\nabla\cdot\mathbf{u}) \tag{2.4}$$

$$\nabla\cdot\mathbf{H} = 0 \tag{2.5}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla, \tag{2.6}$$

where $\mathbf{u}$ is the Eulerian velocity vector, ρ the fluid density, $\mathbf{g}$ the acceleration due to gravity, μ the magnetic permeability, p the hydrodynamic pressure, and c the constant speed of sound.

The equilibrium configuration is given by $\mathbf{u} = (0,0,0)$, $\mathbf{H} = (H_0(z),0,0)$, $\rho = \rho_0$, $p = p_0$ and $\mathbf{g} = (0,0,-g)$ where H_0, ρ_0, p_0 represent the basic magnetic field, density and the pressure respectively. In view of these results, the basic equations yield

$$-\frac{\partial p_0}{\partial x} = 0 = \frac{\partial p_0}{\partial y} \tag{2.7ab}$$

$$\frac{\partial p_0}{\partial z} = -\left(g\rho_0 + \mu H_0 \frac{\mathrm{d}H_0}{\mathrm{d}z}\right) \tag{2.8}$$

whence it follows that

$$p_0 = p(z), \quad \rho_0 = \rho_0(z) \quad \text{and} \quad -\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} = \beta \tag{2.9abc}$$

On the above equilibrium configuration, we superimpose a small disturbance of the form

$$\mathbf{u} = (u,v,w), \quad \mathbf{H} = (H_0(z) + h_x, h_y, h_z), \quad \rho = \rho_0 + \rho', \quad p = p_0 + p'. \tag{2.10abcd}$$

We assume that the disturbances are small enough compared to the initial state so that higher-order terms in perturbed quantities can be neglected. We then substitute (2.10abcd) in (2.1)–(2·5) and invoke linearization so that the resulting equations reduce to a set of linear partial differential equations. This system admits plane-wave solutions in which all perturbed quantities f may be written as

$$\tilde{f}(x,y,z,t) = f(z)\exp[i(kx + ly - \omega t)], \tag{2.11}$$

where (k,l) and ω are constants, and the former represents the wavenumber and the latter denotes the frequency of the wave.

Elimination of all perturbed variables but w leads to the equation

$$\frac{d^2w}{\mathrm{d}z^2} + \left[-\beta + \frac{Q}{(\omega^2 - A^2k^2)(Q-\omega^4)}\frac{\mathrm{d}}{\mathrm{d}z}\frac{(\omega^2 - A^2k^2)(Q-\omega^4)}{Q}\right]\frac{\mathrm{d}w}{\mathrm{d}z}$$
$$- \frac{\alpha^2}{(\omega^2 - A^2k^2)(Q-\omega^4)}\left[(\omega^2 - A^2k^2)\left\{Q + \alpha^2c^2\left(N^2 - \frac{g^2}{c^2}\right)\right\}\right.$$
$$\left. + l^2\omega^2A^2N^2 - g\omega^2Q\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\omega^2 - A^2k^2}{Q}\right)\right]w = 0, \tag{2.12}$$

where

$$Q=(\omega^2-A^2k^2)(\omega^2-\alpha^2c^2)-l^2\omega^2A^2, \tag{2.13}$$

with the Alfven velocity A, the wavenumber α and the Brunt-Väisälä frequency N being given by

$$A^2=\frac{\mu H_0^2}{\rho_0}, \quad \alpha^2=k^2+l^2, \quad N^2=g\beta. \tag{2.14abc}$$

Invoking the transformation $w=\phi\exp\left(\frac{\beta z}{2}\right)$, (2.12) assumes the form

$$\begin{aligned}\frac{d^2\phi}{dz^2}&+\left[\frac{Q}{(\omega^2-A^2k^2)(Q-\omega^4)}\frac{d}{dz}\frac{(\omega^2-A^2k^2)(Q-\omega^4)}{Q}\right]\frac{d\phi}{dz}\\&+\left[-\frac{\beta^2}{4}+\frac{\beta}{2}\frac{Q}{(\omega^2-A^2k^2)(Q-\omega^4)}\frac{d}{dz}\frac{(\omega^2-A^2k^2)(Q-\omega^4)}{Q}\right.\\&\quad-\frac{\alpha^2}{(\omega^2-A^2k^2)(Q-\omega^4)}(\omega^2-A^2k^2)\left\{Q+\alpha^2c^2\left(N^2-\frac{g^2}{c^2}\right)\right\}\\&\quad\left.+l^2\omega^2A^2N^2-gQ\omega^2\frac{d}{dz}\frac{\omega^2-A^2k^2}{Q}\right]\phi=0,\end{aligned} \tag{2.15}$$

In the next section we calculate the reflection and transmission coefficients for a hydromagnetic gravity wave incident upon a magnetic shear layer.

3. Reflection and transmission of hydromagnetic waves by a magnetic shear layer

We consider the problem of a hydromagnetic gravity wave incident upon a magnetic shear layer specified by

$$A^2=\begin{cases}A_1^2, & z\leqslant 0 \quad \text{(region I)}\\ A_2^2z, \quad A_2^2=\dfrac{1}{L}A_3^2, & 0\leqslant z\leqslant L \quad \text{(region II)}\\ A_3^2, & z\geqslant L \quad \text{(region III)}\end{cases} \tag{3.1abc}$$

A gravity wave from region I incident upon the magnetic shear layer (region II) gives rise to a reflected wave in region I, a transmitted wave in region III, and two waves, one moving upward and the other moving downward going in region II.

In region I, (2.15) reduces to the form

$$\begin{aligned}\frac{d^2\phi}{dz^2}&+\left[-\frac{1}{4}\beta^2-\frac{\alpha^2}{(\omega^2-A_1^2k^2)(Q_1-\omega^4)}(\omega^2-A_1^2k^2)\right.\\&\left.\left\{Q_1+\alpha^2c^2\left(N^2-\frac{g^2}{c^2}\right)\right\}+l^2\omega^2A_1^2N^2\right]\phi=0,\end{aligned} \tag{3.2}$$

and the corresponding solution has the form

$$\phi(z)=I\exp(i\alpha_{z_1})z+R\exp(-iz\alpha_{z_1}), \tag{3.3}$$

where I is the amplitude of the incident wave and R that of the reflected wave, and α_{z_1} is given by

$$\alpha_{z_1}^2 = -\frac{\beta^2}{4} - \frac{\alpha^2}{(\omega^2 - A_1^2k^2)(Q_1 - \omega^4)}(\omega^2 - A_1^2k^2)\left\{Q_1 + \alpha^2c^2\left(N^2 - \frac{g^2}{c^2}\right)\right\} + l^2\omega^2A_1^2N^2, \tag{3.4}$$

with

$$Q_1 = (\omega^2 - A_1^2k^2)(\omega^2 - \alpha^2c^2) - l^2\omega^2A_1^2. \tag{3.5}$$

If we take α_{z_1} as the positive root of (3.4), the choice of the signs in (3.3) ensures that the incident wave transports wave energy upwards (towards the magnetic shear layer) and the reflected wave carries wave energy downwards (away from the magnetic shear layer).

In region II, (2.15) takes the following form

$$\frac{d^2\phi}{dz^2} + \left[-\frac{A_2^2k^2}{\omega^2 - zA_2^2k^2} + \frac{Q_2'}{Q_2 - \omega^4} - \frac{Q_2'}{Q_2}\right]\frac{d\phi}{dz} + \left[-\frac{1}{4}\beta^2 + \frac{\beta}{2}\left\{-\frac{A_2^2k^2}{(\omega^2 - zA_2^2k^2)} + \frac{Q_2'}{Q_2 - \omega^4} - \frac{Q_2'}{Q_2}\right\}\right] - \frac{\alpha^2}{(\omega^2 - zA_2^2k^2)(Q_2 - \omega^4)}\left\{(\omega^2 - zA_2^2k^2)\left(Q_2 + \alpha^2c^2\left(N^2 - \frac{g^2}{c^2}\right)\right) + l^2\omega^2zA_2^2N^2 + g\omega^2A_2^2k^2\right\}\phi = 0, \tag{3.6}$$

where Q_2 represents the expression (2.13) with A^2 replaced with zA_2^2.

Making use of the transformation $\phi = \psi Q_2^{1/2}(Q_2 - \omega^4)^{-1/2}$, (3.6) becomes

$$\frac{d^2\psi}{dz^2} + \frac{1}{z - z_{c_1}}\frac{d\psi}{dz} + \left[\frac{j_1\gamma_1}{z - z_{c_2}} - \frac{\gamma_1^2}{4}\right]\psi = 0, \tag{3.7}$$

$$z_{c_1} = c_1^2/A_2^2, \quad c_1^2 = \omega^2/k^2, \quad j_1 = (R_1/\gamma_1) + \frac{1}{2}(\beta/\gamma_1) \tag{3.8abc}$$

$$R_1 = \frac{\alpha}{Q_2 - \omega^4}(l^2c_1^2zN^2 + g\omega^2) + \frac{Q_2'}{2}\left[\frac{1}{Q_2} - \frac{1}{Q_2 - \omega^4}\right], \tag{3.9}$$

$$\gamma_1^2 = \beta^2 + 2\beta Q_2'\left(\frac{1}{Q_2} - \frac{1}{Q_2 - \omega^4}\right) + Q_2'^2\left(\frac{1}{Q_2} - \frac{1}{Q_2 - \omega^4}\right) + \frac{4\alpha^2}{Q_2 - \omega^4}\{Q_2 + \alpha^2c^2(N^2 - g^2/c^2) + g\omega^2(Q_2'/Q_2)\}. \tag{3.10}$$

Using the transformations

$$\psi = \exp\left(-\frac{Z}{2}\right)\psi(Z), \quad Z = \gamma_1(z - z_{c_1}), \tag{3.11ab}$$

(3.7) can be transformed into the confluent hypergeometric equation

$$Z\frac{d^2\psi}{dZ^2}+(1-Z)\frac{d\psi}{dZ}-\left(\frac{1}{2}-j_1\right)\psi=0. \tag{3.12}$$

This admits two independent solutions in the form

$$\psi_1=1+\frac{\left(\frac{1}{2}-j_1\right)Z}{(1!)^2}+\frac{\left(\frac{1}{2}-j_1\right)\left(\frac{2}{3}-j_1\right)Z^2}{(2!)^2}+\cdots, \tag{3.13}$$

$$\psi_2=\psi_1\log Z+S_1, \tag{3.14}$$

where

$$S_1=\sum_{k=1}^{\infty}B_kZ^k,\quad B_k=\frac{\left(\frac{1}{2}-j_1+k\right)H_k}{(k!)^2\left(\frac{1}{2}-j_1\right)}, \tag{3.15ab}$$

$$H_k=\sum_{n=0}^{k-1}\left(\frac{1}{(1/2)-j_1+n}-\frac{2}{n+1}\right). \tag{3.16}$$

Thus, the solution of (3.7) can be written as

$$\phi=D_1W_1(Z)+D_2W_2(Z) \tag{3.17}$$

where

$$W_1(Z)=e^{-Z/2}Q_2^{1/2}(Q_2-\omega^4)^{-1/2}\left[1+\frac{\left(\frac{1}{2}-j_1\right)}{(1!)^2}Z+\frac{\left(\frac{1}{2}-j_1\right)\left(\frac{3}{2}-j_1\right)}{(2!)^2}Z^2+\cdots\right], \tag{3.18}$$

$$W_2(Z)=e^{-Z/2}Q_2^{1/2}(Q_2-\omega^4)^{-1/2}\times\left[1+\frac{\left(\frac{1}{2}-j_1\right)}{(1!)^2}Z+\frac{\left(\frac{1}{2}-j_1\right)\left(\frac{3}{2}-j_1\right)}{(2!)^2}Z^2+\cdots\right]\log Z+\left[2j_1Z+\left(3j_1^2+4j_1-\frac{1}{4}\right)Z^2+\cdots\right] \tag{3.19}$$

and D_1 and D_2 are amplitude constants.

Using the transformation $\phi=\psi Q_2^{1/2}(\omega^2-A_2^2zk^2)^{-1/2}$, (3.6) reduces to

$$\frac{d^2\psi}{dz^2}+\frac{1}{z-z_{c_2}}\frac{d\psi}{dz}+\left[\frac{j_2\gamma_2}{z-z_{c_2}}-\frac{\gamma_2^2}{4}\right]\psi=0, \tag{3.20}$$

where

$$z_{c_2} = \frac{c_2^2}{A_2^2}, \quad c_2^2 = \frac{\omega^2 c^2}{c^2 k^2 - \omega^2} \tag{3.21ab}$$

$$j_2 = \frac{1}{2}\left[\frac{Q_2'}{Q_2} - \frac{A_2^2 k^2}{\omega^2 - A_2^2 z k^2}\right] - \frac{Q_2 + \alpha^2 c^2\left(N^2 - \frac{g^2}{c^2}\right)}{(\omega^2 - k^2 c^2)A_2^2} - \frac{\alpha^2 g \omega^2}{Q_2} - \frac{(l\omega^2 z N^2 + g\omega^2 k^2)}{(\omega^2 - k^2 c^2)(\omega^2 - A_2^2 z k^2)} \tag{3.22}$$

$$\gamma_2^2 = -\beta^2 - 2\beta\left[\frac{A_2^2 k^2}{\omega^2 - A_2^2 z k^2} + \frac{Q_2'}{Q_2}\right] + \left[\frac{Q_2'}{Q_2} + \frac{A_2^2 k^2}{\omega^2 - A_2^2 k^2 z}\right]^2 + 2\left[\frac{Q_2'^2}{Q_2^2} - \frac{(A_2^2 k^2)^2}{(\omega^2 - A_2^2 k^2 z)^2}\right]. \tag{3.23}$$

Invoking the transformation

$$\psi = \exp\left(-\frac{Z}{2}\right)\psi(Z), \quad Z = \gamma_2(z - z_{c_2}), \tag{3.24ab}$$

(3.20) assumes the form

$$Z\frac{d^2\psi}{dZ^2} + (1 - Z)\frac{d\psi}{dZ} - \left(\frac{1}{2} - j_2\right)\psi = 0. \tag{3.25}$$

This gives two independent solutions in the form

$$\psi_3 = 1 + \frac{\left(\frac{1}{2} - j_2\right)}{(1!)^2} Z + \frac{\left(\frac{1}{2} - j_2\right)\left(\frac{3}{2} - j_2\right)}{(2!)^2} Z^2 + \cdots, \tag{3.26}$$

$$\psi_4 = \psi_3 \log Z + S_2, \tag{3.27}$$

where

$$S_2 = \sum_{k=1}^{\infty} B_k Z^k, \quad B_k = \frac{\left(\frac{1}{2} - j_2 + k\right)H_k}{(k!)^2\left(\frac{1}{2} - j_2\right)}, \tag{3.28ab}$$

$$H_k = \sum_{n=0}^{k-1}\left[\frac{1}{\frac{1}{2} - j_2 + n} - \frac{2}{n+1}\right]. \tag{3.29}$$

Therefore, the solution of (3.20) can be written as

$$\phi = D_3 W_3(Z) + D_4 W_4(Z) \tag{3.30}$$

where

$$W_3(Z) = e^{-Z/2} Q_2^{1/2} (\omega^2 - A_2^2 z k^2)^{-1/2} \left[1 + \frac{\left(\frac{1}{2} - j_2\right)}{(1!)^2} Z + \frac{\left(\frac{1}{2} - j_2\right)\left(\frac{3}{2} - j_2\right)}{(2!)^2} Z^2 + \cdots \right] \tag{3.31}$$

$$W_4(Z) = e^{-Z/2} Q_2^{1/2} (\omega^2 - A_2^2 z k^2)^{-1/2} \times \left[1 + \frac{\left(\frac{1}{2} - j_2\right)}{(1!)^2} Z + \frac{\left(\frac{1}{2} - j_2\right)\left(\frac{3}{2} - j_2\right)}{(2!)^2} Z^2 + \cdots \right] \log Z + \left[2 j_2 Z + \left(3 j_2^2 + 4 j_2 - \frac{1}{4} \right) Z^2 + \cdots \right], \tag{3.32}$$

and D_3 and D_4 are amplitude constants.

The transformation $\phi = \psi (Q_2 - \omega^4)^{-1/2} (\omega^2 - A_2^2 z k_2)^{-1/2}$ reduces (3.6) to the following form:

$$\frac{d^2\psi}{dz^2} + \frac{1}{z_{c_3} - z} \frac{d\psi}{dz} + \left(\frac{j_3 \gamma_3}{z_{c_3} - z} - \frac{\gamma_3^2}{4} \right) \psi = 0, \tag{3.33}$$

where

$$z_{c_3} = \frac{c_3^2}{A_2^2}, \quad c_3^2 = \frac{\omega^2 (\omega^2 - \alpha^2 c^2)}{\alpha^2 (\omega^2 - k^2 c^2)} \tag{3.34ab}$$

$$j_3 = \frac{R_3}{\gamma_3} + \frac{\beta}{2\gamma_3} \tag{3.35}$$

$$R_3 = \frac{1}{2} \left[\frac{A_2^2 k^2}{\omega^2 - A_2^2 z k^2} - \frac{Q_2'}{Q_2 - \omega^4} \right] + \frac{\alpha^2 g \omega^2}{Q_2 - \omega^4} \tag{3.36}$$

$$\gamma_3^2 = \beta^2 - 2\beta \left[\frac{Q_2'}{Q_2 - \omega^4} - \frac{A_2^2 k^2}{\omega^2 - A_2^2 z k^2} \right] - 2 \left[\frac{(A_2^2 k^2)^2}{(\omega^2 - A_2^2 z k^2)^2} + \frac{Q_2'^2}{(Q_2 - \omega^4)^2} \right] + \left[\frac{A_2^2 k^2}{\omega^2 - A_2^2 z k_2^2} - \frac{Q_2'}{Q_2 - \omega^4} \right]^2 + \frac{4\alpha^2}{Q_2 - \omega^4} \left[Q_2 + \alpha^2 c^2 \left(N^2 - \frac{g^2}{c^2} \right) \right] + \frac{4\alpha^2 [l^2 \omega^2 A_2^2 z N^2 + g \omega^2 A_2^2 k^2]}{(\omega^2 - A_2^2 z k^2)(Q_2 - \omega^4)}. \tag{3.37}$$

Equation (3.33) can be transformed to the following form

$$Z \frac{d^2\psi}{dZ^2} - (1 + Z) \frac{d\psi}{dZ} + \left(\frac{1}{2} + j_3 \right) \psi = 0 \tag{3.38}$$

by means of the transformation $\psi = e^{-(Z/2)} \psi(Z)$, $Z = \gamma_3 (z_{c_3} - z)$, which has two

independent solutions

$$\psi_5 = 1 - \frac{\left(\frac{1}{2}+j_2\right)}{(1!)^2}Z + \frac{\left(\frac{1}{2}+j_2\right)\left(\frac{3}{2}+j_2\right)}{(2!)^2}Z^2 - \cdots, \tag{3.39}$$

$$\psi_6 = \psi_5 \log Z + S_3, \tag{3.40}$$

where

$$S_3 = \sum_{k=1}^{\infty} B_k Z^k, \quad B_k = \frac{\left(\frac{1}{2}+j_3+k\right)H_k}{(k!)^2\left(\frac{1}{2}+j_3\right)}, \tag{3.41}$$

$$H_k = \sum_{n=0}^{k-1}\left[\frac{1}{(1/2)+j_3+n} - \frac{2}{n+1}\right]. \tag{3.42}$$

Therefore, the solution of (3.33) can be written as

$$\phi = D_5\, W_5(Z) + D_6\, W_6(Z) \tag{3.43}$$

where

$$W_5(Z) = e^{-Z/2}(Q_2-\omega^4)^{-1/2}(\omega^2 - A_2^2 z k^2)^{-1/2} \times\left[1 - \frac{\left(\frac{1}{2}+j_3\right)}{(1!)^2}Z + \frac{\left(\frac{1}{2}+j_3\right)\left(\frac{3}{2}+j_3\right)}{(2!)^2}Z^2 - \cdots\right], \tag{3.44}$$

$$W_6(Z) = e^{-Z/2}(Q_2-\omega^4)^{-1/2}(\omega^2 - A_2^2 z k^2)^{-1/2} \times\left[1 - \frac{\left(\frac{1}{2}+j_3\right)}{(1!)^2}Z + \frac{\left(\frac{1}{2}+j_3\right)\left(\frac{3}{2}+j_3\right)}{(2!)^2}Z^2 - \cdots\right]\log Z + \left[2j_3 Z - \left(3j_3^2 + 4j_3 - \frac{1}{4}\right)Z^2 + \cdots\right], \tag{3.45}$$

and D_5 and D_6 are amplitude constants.

For region III, (3.15) takes the following form

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + \left\{-\frac{\beta^2}{4} - \frac{\alpha^2}{(\omega^2 - A_3^2k^2)(Q_3-\omega^4)}\left[(\omega^2 - A_3^2k^2)\left(Q_3 + \alpha^2c^2\left(N^2 - \frac{g^2}{c^2}\right)\right) + l^2\omega^2 A_3^2 N^2\right]\right\}\phi = 0 \tag{3.46}$$

and has the solution of the form

$$\phi = T\exp(iz\alpha_{z_3}), \tag{3.47}$$

where T is the amplitude of the transmitted wave and α_{z_3} is given by

$$\alpha_{z_3}^2 = -\frac{\beta^2}{4} - \frac{\alpha^2}{(\omega^2 - A_3^2 k^2)(Q_2 - \omega^4)} \times \left\{(\omega^2 - A_2^2 k^2)\left(Q_3 + \alpha^2 c^2\left(N^2 - \frac{g^2}{c^2}\right)\right) + l^2 \omega^2 A_2^2 N^2\right\}. \tag{3.48}$$

Again the choice of the sign of α_{z_3} ensures that the transmitted wave transports wave energy upwards.

To simplify the calculations in the following discussion, we shall consider the low-frequency approximation, so that the dispersion equations (3.4) and (3.48) appropriate to region I and region III are approximated by

$$\alpha_{z_1}^2 = \frac{\alpha^2 N^2}{\omega^2 - A_1^2 k_2}, \quad \alpha_{z_3}^2 = \frac{\alpha^2 N^2}{\omega^2 - A_3^2 k^2}. \tag{3.49ab}$$

By using the boundary conditions at $z = 0$ and $z = L$, namely the continuity of the vertical component of velocity and continuity of pressure, we determine the amplitudes of the reflected and transmitted waves. Since the vertical component of velocity w is continuous, and from the relation $w = \phi e^{(\beta/2)z}$, ϕ is also continuous.

It follows from the equations of motion that the pressure p_t can be obtained in the form

$$p_t = \frac{i\rho_0(\omega^2 - A^2 k^2)}{Q}\frac{}{\omega^2}\left[(Q - \omega^4)\frac{dw}{dz} + (k^2 + l^2) g\omega^2 w\right]. \tag{3.50}$$

we put $w = \phi e^{(\beta/2)z}$ to transform (3.50) into the form

$$p_t = \frac{i\rho_0(\omega^2 - A^2 k^2)}{Q}\frac{}{\alpha^2}\left[(Q - \omega^4)\left(\frac{d\phi}{dz} + \frac{\beta}{2}\phi\right) + \alpha^2 g\omega^2 \phi\right] e^{(\beta/2)z}. \tag{3.51}$$

Therefore, the boundary conditions are equivalent to

$$[\phi] = \left[\frac{d\phi}{dz}\right] = 0 \quad \text{at } z = 0, L \tag{3.52}$$

where the square bracket denotes the jump in the quantity inside the square bracket.

Utilizing the boundary conditions (3.52), and when $A_3^2 < c_1^2$ yields $[\phi]_{0,L} = 0$ which implies

$$I + R = D_1 W_1(0) + D_2 W_2(0), \tag{3.53}$$

$$T\exp(iL\alpha_{z_3}) = D_1 W_1(L) + D_2 W_2(L), \tag{3.54}$$

and the condition $\left[\frac{d\phi}{dz}\right]_{0,L} = 0$ leads to

$$i\alpha_{z_1}(I - R) = \gamma_1 D_1 W_1'(0) + \gamma_1 D_2 W_2'(0). \tag{3.55}$$

$$i\alpha_{z_3} T\exp(iL\alpha_{z_3}) = \gamma_1 D_1 W_1'(L) + \gamma_1 D_2 W_2'(L). \tag{3.56}$$

m the above equations we obtain

$$\frac{R}{I}=\frac{D_1[i\alpha_{z_1}W_1(0)-\gamma_1 W_1'(0)]+D_2[i\alpha_{z_1}W_2(0)-\gamma_1 W_2'(0)]}{D_1[i\alpha_{z_1}W_1(0)+\gamma_1 W_1'(0)]+D_2[i\alpha_{z_1}W_2(0)+\gamma_1 W_2'(0)]}, \tag{3.57}$$

$$\frac{D_1}{D_2}=\frac{i\alpha_{z_3}W_2(L)-\gamma_1 W_2'(L)}{i\alpha_{z_3}W_1(L)-\gamma_1 W_1'(L)}, \tag{3.58}$$

$$\frac{T\exp(i\alpha_{z_3}L)}{I+R}=\frac{D_1W_1(L)+D_2W_2(L)}{D_1W_1(0)+D_2W_2(0)}. \tag{3.59}$$

the case $A_3^2>c_1^2$, the above expressions can be written as

$$\frac{D_1}{D_2}=\frac{i\alpha_{z_3}W_2(-L)-\gamma_1 W_2'(-L)}{i\alpha_{z_3}W_1(-L)-\gamma_1 W_1'(-L)}, \tag{3.60}$$

$$\frac{T\exp(i\alpha_{z_3}L)}{I+R}=\frac{D_1W_1(-L)+D_2W_2(-L)}{D_1W_1(0)+D_2W_2(0)}. \tag{3.61}$$

the above results can be written in the compact form

$$\frac{R}{I}=\frac{\delta_1 d_{i1}+d_{i2}}{\delta_1 d_{r1}+d_{r2}} \tag{3.62}$$

$$\frac{T\exp(i\alpha_{z_3}L)}{I+R}=\frac{\delta_1 W_1(L)+W_2(L)}{\delta_1 W_1(0)+W_2(0)} \tag{3.63}$$

$A_2^2<c_1^2$, and

$$\frac{R}{I}=\frac{\delta_2 d_{i1}+d_{i2}}{\delta_2 d_{r1}+d_{r2}} \tag{3.64}$$

$$\frac{T\exp(i\alpha_{z_3}L)}{I+R}=\frac{\delta_1 W_1(-L)+W_2(-L)}{\delta_2 W_1(0)+W_2(0)} \tag{3.65}$$

$A_3^2>c_1^2$, where

$$d_{in}=i\alpha_{z_1}W_n(0)-\gamma_1 W_n'(0),\ d_{rn}=i\alpha_{z_1}W_n(0)+\gamma_1 W_n'(0),\ n=1,2 \tag{3.66}$$

$$\delta_1=-c_{t_2}/c_{t_1},\ \delta_2=-b_{t_2}/b_{t_1} \tag{3.67ab}$$

$$c_{t_n}=\gamma_1 W_n'(L)-i\alpha_{z_3}W_n(L) \tag{3.68}$$

$$b_{t_n}=\gamma_1 W_n'(-L)-i\alpha_{z_3}W_n(-L) \tag{3.69}$$

d the following notations were used:

$$W_{1,2}(L)=W_{1,2}[\gamma_1(L-z_{c_1})] \tag{3.70}$$

$$W_{1,2}(0)=W_{1,2}[-\gamma_1 z_{c_1}] \tag{3.71}$$

$$W_{1,2}(-L)=W_{1,2}[-\gamma_1(z_{c_1}-L)]. \tag{3.72}$$

4. Wave amplification and critical-level behaviour

We investigate the properties of reflection, transmission and absorption of hydromagnetic acoustic-gravity waves incident upon a magnetic shear layer, and discuss how these properties depend on the wavelength, the length scale of the shear layer, and the ratio of the flow speed and the phase speed of the waves. We use the invariance of the wave action flux to prove some general properties.

The wave energy flux is

$$E = \overline{p_t q_z} = \frac{1}{4}(p_t^* q_z + p_t q_z^*) = \frac{1}{2}\mathrm{Re}(p_t q_z^*), \tag{4.1}$$

where the asterisk denotes the complex conjugate and Re stands for the real part.

Using (2.15), the expression for E becomes

$$E = \frac{1}{2}\mathrm{Re}\left[\frac{i\rho_0}{\omega Q}\frac{\omega^2 - A^2k^2}{\alpha^2}\left[(Q-\omega^4)\frac{\mathrm{d}q_z}{\mathrm{d}z}q_z^*\right]e^{\beta z}\right] \tag{4.2}$$

$$= \frac{1}{2}\mathrm{Re}\left[\frac{i\rho_c}{\omega\alpha^2}\frac{(\omega^2 - A^2k^2)(Q-\omega^4)}{Q}\phi'\phi^*\right] \tag{4.3}$$

where

$$q_z = \phi\exp\left(\frac{\beta z}{2}\right), \quad q_z^* = \phi^*\exp\left(\frac{\beta z}{2}\right), \quad \rho_c = \rho_0 e^{\beta z}. \tag{4.4abc}$$

We next define the wave action flux M as the ratio of the wave energy flux and the local relative frequency by

$$M = \frac{E}{-\omega} = \frac{1}{2}\mathrm{Re}\left[\frac{-i\rho_c(\omega^2 - A^2k^2)(Q-\omega^4)}{\omega^2\alpha^2}\frac{}{Q}\phi'\phi^*\right]. \tag{4.5}$$

It turns out that

$$\frac{\mathrm{d}M}{\mathrm{d}z} = 0 \quad \text{for } z \neq z_{c_r},\ r = 1,2,3. \tag{4.6}$$

This means that M is independent of z except at the critical levels where it is discontinuous. The invariance of M is closely linked to the invariance of both the vertical component of the total energy flux and of the horizontal component of momentum.

If

$$A_1^2 < c_2^2 \text{ and } A_3^2 < c_1^2, A_1^2 > c_1^2 \text{ and } A_3^2 > c_1^2, A_1^2 < c_2^2 \text{ and } A_3^2 < c_2^2 \tag{4.7abc}$$

and

$$A_1^2 > c_2^2 \text{ and } A_3^2 > c_2^2, A_1^2 > c_3^2 \text{ and } A_3^2 > c_3^2, A_1^2 < c_3^2 \text{ and } A_3^2 < c_3^2 \tag{4.8abc}$$

there is no critical level inside the magnetic shear layer. In view of the invariance of M over the whole domain, it turns out that

$$M = \frac{\rho_c \alpha_{z_1} k^2}{2\omega^2\alpha^2}\frac{(A_1^2 - c_1^2)(A_1^2 - c_2^2)}{(c_3^2 - A_1^2)}\ [|I|^2 - |R|^2] \text{ in region I} \tag{4.9}$$

$$M=\frac{\rho_c\alpha_{z_3}k^2}{2\omega^2k^2}\frac{(A_3^2-c_1^2)(A_3^2-c_2^2)}{(c_3^2-A_3^2)}|T|^2 \quad \text{in region III.} \tag{4.10}$$

These results combined with the invariance of M give

$$\alpha_{z_1}\left[\frac{(A_1^2-c_1^2)(A_1^2-c_2^2)}{(c_3^2-A_1^2)}\right][|I|^2-|R|^2]=\alpha_{z_3}\left[\frac{(A_3^2-c_1^2)(A_3^2-c_2^2)}{(c_3^2-A_3^2)}\right]|T|^2. \tag{4.11}$$

Since α_{z_1} and α_{z_3} are positive and terms within the square bracket are either positive or negative, the amplification of wave is impossible.

On the other hand, if

$$A_1^2>c_3^2 \text{ and } A_3^2<c_3^2, A_1^2<c_1^2 \text{ and } A_3^2>c_1^2, A_1^2<c_2^2 \text{ and } A_3^2>c_2^2 \tag{4.12abc}$$

there exists a critical level inside the magnetic shear layer. We use the following approximate solution near $z=z_{c_1}$:

$$\phi=D_1[1-j_1\gamma_1(z-z_{c_1})]+D_2\{[1-j_1\gamma_1(z-z_{c_1})]\log[\gamma_1(z-z_{c_1})]+2j_1\gamma_1(z-z_{c_1})\},\quad z<z_{c_1} \tag{4.13a}$$

$$\phi=D_1[1+j_1\gamma_1(z_{c_1}-z)]+D_2\{[1+j_1\gamma_1(z_{c_1}-z)]\log[\gamma_1(z_{c_1}-z)+i\pi]-2j_1\gamma_1(z_{c_1}-z)\},\quad z>z_{c_1}. \tag{4.13b}$$

We then calculate the values M_b of M below the critical level and the value M_a of M above the critical level. These values are

$$M_b=-\frac{\rho_c k^2}{2\omega^2\alpha^2}A_2^2\frac{(c_1^2-c_2^2)}{(c_3^2-c_1^2)}\mathrm{Im}(D_1^*D_2), \tag{4.14}$$

$$M_a=-\frac{\rho_c k^2}{2\omega^2\alpha^2}A_2^2\frac{(c_1^2-c_2^2)}{(c_3^2-c_1^2)}[\mathrm{Im}D_1^*D_2-|D_2|^2\pi]. \tag{4.15}$$

We obtain α_{z_1} from (4.7) and (4.11), and α_{z_3} from (4.8) and (4.12abc) so that they are given by

$$\alpha_{z_1}\frac{(c_1^2-A_1^2)(A_1^2-c_2^2)}{(c_3^2-A_1^2)}[|I|^2-|R|^2]=A_2^2\frac{(c_1^2-c_2^2)}{(c_3^2-c_1^2)}\mathrm{Im}(D_2^*D_2) \tag{4.16}$$

$$\alpha_{z_3}\frac{(c_1^2-A_3^2)(A_3^2-c_2^2)}{(c_3^2-A_3^2)}|T|^2=A_2^2\frac{(c_1^2-c_2^2)}{(c_3^2-c_1^2)}[\mathrm{Im}D_1^*D_2-|D_2|^2\pi]. \tag{4.17}$$

The results combined with (4.10) give the total energy flux in the shear layer:

$$\alpha_{z_1}|I|^2=\alpha_{z_1}|R|^2+\frac{(c_1^2-A_3^2)(A_3^2-c_2^2)(c_3^2-A_1^2)}{(c_1^2-A_1^2)(A_1^2-c_2^2)(c_3^2-A_3^2)}|T|^2\alpha_{z_3}+A_2^2\frac{(c_1^2-c_2^2)(c_3^2-A_1^2)}{(c_1^2-A_1^2)(A_1^2-c_2^2)(c_3^2-c_1^2)}\pi|D_2|^2. \tag{4.18}$$

The term on the left-side of (4.18) represents the total energy flux into the shear layer whereas the first two terms on the right-side denote the total energy flux out of the layers. The term on the right side of (4.18) is negative whenever the critical level exists. Thus, if the critical layer exists, the wave is amplified.

The solution near the critical level $z = z_{c_2}$ is given by

$$\phi = D_3[1 - j_2\gamma_2(z - z_{c_2})] + D_4\{[1 - j_2\gamma_2(z - z_{c_2})]\log[\gamma_2(z - z_{c_2})] + 2j_2\gamma_2(z - z_{c_2})\}, \quad z < z_{c_2}, \tag{4.19a}$$

$$\phi = D_3[1 + j_2\gamma_2(z - z_{c_2})] + D_4\{[1 + j_2\gamma_2(z - z_{c_2})][\log\gamma_2(z - z_{c_2}) + i\pi] - 2j_2\gamma_2(z - z_{c_2})\}, \quad z > z_{c_2}. \tag{4.19b}$$

The value M_b of M below the critical level and the value M_a of M above the critical level are

$$M_b = \frac{\rho_c}{2\omega^2\alpha^2}\frac{A_2^2k^2}{(z_{c_3} - z_{c_2})}(z_{c_1} - z_{c_2})\,\mathrm{Im}(D_3^*D_4), \tag{4.20}$$

$$M_a = \frac{\rho_c}{2\omega^2\alpha^2}\frac{A_2^2k^2}{(z_{c_3} - z_{c_2})}(z_{c_1} - z_{c_2})[\mathrm{Im}(D_3^*D_4) - |D_4|^2\pi]. \tag{4.21}$$

We obtain α_{z_1} from (4.9) and (4.20) and α_{z_3} from (4.10) and (4.21) in the form

$$\alpha_{z_1}\frac{(A_1^2 - c_1^2)(-c_2^2 + A_1^2)}{c_3^2 - A_1^2}(|I|^2 - |R|^2) = A_2^2\frac{(c_1^2 - c_2^2)}{(c_3^2 - c_2^2)}\mathrm{Im}(D_3^*D_4) \tag{4.22}$$

$$\alpha_{z_3}\frac{(A_3^2 - c_1^2)(-c_2^2 + A_3^2)}{(c_3^2 - A_3^2)}|T|^2 = A_2^2\frac{(c_1^2 - c_2^2)}{(c_3^2 - c_2^2)}[\mathrm{Im}(D_3^*D_4) - |D_4|^2\pi] \tag{4.23}$$

These relations combined with (4.19) yield the total energy flux

$$\alpha_{z_1}|I|^2 = \alpha_{z_1}|R|^2 + \alpha_{z_3}\frac{(A_3^2 - c_1^2)(-c_2^2 + A_3^2)(c_3^2 - A_1^2)}{(A_1^2 - c_1^2)(-c_2^2 + A_1^2)(c_3^2 - A_3^2)}|T|^2 + A_2^2\frac{(c_1^2 - c_2^2)(c_3^2 - A_1^2)}{(c_3^2 - c_2^2)(A_1^2 - c_1^2)(-c_2^2 + A_1^2)}|D_4|^2\pi. \tag{4.24}$$

The term on the left hand side of (4.24) represents the total energy flux into the shear layer whereas the first two terms in the right hand side denote the total energy flux out of the layer and the last term on the right hand side of (4.24) is negative whenever the critical layer exists. Thus the conclusion is that if the critical layer exists, the wave is amplified.

Finally, at the critical level $z = z_{c_3}$, the equation for the total energy flux is found to be

$$\alpha_{z_1}|I|^2 = \alpha_{z_1}|R|^2 + \alpha_{z_3}\frac{(A_3^2 - c_1^2)(A_3^2 - c_2^2)(c_3^2 - A_1^2)}{(c_3^2 - A_2^2)(A_1^2 - c_1^2)(A_1^2 - c_2^2)}|T|^2. \tag{4.25}$$

This implies that the total energy flux into the shear layer is equal to the total energy flux out of the shear layer. So, the wave is not amplified in this case.

5. Reflection coefficient for large vertical wavelength

A large vertical wavelength normalized by the thickness of the shear layer corresponds to $\gamma_1 L \ll 1$. The approximate solution can be obtained from (4.13). The following results can be found from (3.62)–(3.65):

$$\frac{R}{I} = \frac{(a_2 - ib_1)\log(1 - A_3^2/c_1^2) + (e_2 - if_1)}{(a_1 - ib_2)\log(1 - A_3^2/c_1^2) + (e_1 - if_2)} \quad \text{for } A_3^2 < c_1^2 \tag{5.1a}$$

$$\frac{R}{I} = \frac{(a_3 - ib_4)\log((A_3^2/c_1^2) - 1) + (e_3 - if_4)}{(a_4 - ib_3)\log((A_3^2/c_1^2) - 1) + (e_4 - if_3)} \quad \text{for } A_3^2 > c_1^2 \tag{5.1b}$$

where

$$a_1 = +\alpha_{z_1}\alpha_{z_3}[1 - j_1\gamma_1 z_{c_1} - \gamma j_1(L - z_{c_1}) + (j_1\gamma_1)^2 z_{c_1}(L - z_{c_1})] - (j_1\gamma_1)^2$$

$$a_2 = +\alpha_{z_1}\alpha_{z_3}[1 - j_1\gamma_1 z_{c_1} - j_1\gamma_1(L - z_{c_1}) + (j_1\gamma_1)^2 z_{c_1}(L - z_{c_1})] + (j_1\gamma_1)^2$$

$$a_3 = +\alpha_{z_1}\alpha_{z_3}[1 + j_1\gamma_1(z_{c_1} - L) + j_1\gamma_1 z_{c_1} + (j_1\gamma_1)^2 z_{c_1}(z_{c_1} - L)] - (j_1\gamma_1)^2$$

$$a_4 = +\alpha_{z_1}\alpha_{z_3}[1 + j_1\gamma_1(z_{c_1} - L) + j_1\gamma_1 z_{c_1} + (j_1\gamma_1)^2 z_{c_1}(z_{c_1} - L)] + (j_1\gamma_1)^2$$

$$b_1 = \alpha_{z_1} j_1\gamma_1(-1 - j_1\gamma_1 z_{c_1}) + \alpha_{z_3} j_1\gamma_1[j_1\gamma_1(L - z_{c_1}) - 1]$$

$$b_2 = \alpha_{z_1} j_1\gamma_1(-1 - j_1\gamma_1 z_{c_1}) - \alpha_{z_3} j_1\gamma_1[j_1\gamma_1(L - z_{c_1}) - 1]$$

$$b_3 = \alpha_{z_1} j_1\gamma_1(j_1\gamma_1 z_{c_1} + 1) + \alpha_{z_3} j_1\gamma_1[1 - j_1\gamma_1(z_{c_1} - L)]$$

$$b_4 = \alpha_{z_1} j_1\gamma_1(j_1\gamma_1 z_{c_1} + 1) - \alpha_{z_3} j_1\gamma_1[1 - j_1\gamma_1(z_{c_1} - L)]$$

$$e_1 = \alpha_{z_1}\alpha_{z_3}[2j_1\gamma_1(L - z_{c_1}) - 2j_1\gamma_1 z_{c_1}] + j_1\gamma_1\left(\frac{1}{L - z_{c_1}} + \frac{1}{z_{c_1}}\right)$$

$$e_2 = \alpha_{z_1}\alpha_{z_3}[2j_1\gamma_1(L - z_{c_1}) - 2j_1\gamma_1 z_{c_1}] - j_1\gamma_1\left(\frac{1}{L - z_{c_1}} + \frac{1}{z_{c_1}}\right)$$

$$e_3 = \alpha_{z_1}\alpha_{z_3}[i\pi j_1\gamma_1 z_{c_1} + (j_1\gamma_1)^2 z_{c_1}(L - z_{c_1}) - j_1\gamma_1(L - z_{c_1}) - 1$$
$$- 2j_1\gamma_1(z_{c_1} - L) + 2j_1\gamma_1 z_{c_1}] + j_1\gamma_1\left(\frac{1}{z_{c_1} - L} - \frac{1}{z_{c_1}} - i\pi j_1\gamma_1\right)$$

$$e_4 = \alpha_{z_1}\alpha_{z_3}[i\pi j_1\gamma_1 z_{c_1} + (j_1\gamma_1)^2 z_{c_1}(L - z_{c_1}) - j_1\gamma_1(1 - z_{c_1}) - 1$$
$$- 2j_1\gamma_1(z_{c_1} - L) + 2j_1\gamma_1 z_{c_1}] - j_1\gamma_1\left(\frac{1}{z_{c_1} - L} - \frac{1}{z_{c_1}} - i\pi j_1\gamma_1\right)$$

$$f_1 = \alpha_{z_1}\left[j_1\gamma_1 - \frac{1}{L - z_{c_1}} + j_1\gamma_1\left(1 - \frac{A_3^2}{c_1^2}\right)^{-1} + (j_1\gamma_1)^2 z_{c_1}\right]$$
$$+ \alpha_{z_3}\left[-3j_1\gamma_1 + \frac{1}{z_{c_1}} - j_1\gamma_1\left(1 - \frac{A_3^2}{c_1^2}\right) + (j_1\gamma_1)^2(L - z_{c_1})\right]$$

$$f_2 = \alpha_{z_1}\left[j_1\gamma_1 - \frac{1}{L - z_{c_1}} + j_1\gamma_1\left(1 - \frac{A_3^2}{c_1^2}\right)^{-1} + (j_1\gamma_1)^2 z_{c_1}\right]$$
$$- \alpha_{z_3}\left[-3j_1\gamma_1 + \frac{1}{z_{c_1}} - j_1\gamma_1\left(1 - \frac{A_3^2}{c_1^2}\right) + (j_1\gamma_1)^2(L - z_{c_1})\right]$$

$$f_3 = \alpha_{z_1}[i\pi j_1\gamma_1(j_1\gamma_1 z_{c_1} - 1) + j_1\gamma_1 + (j_1\gamma_1)^2 z_{c_1} + j_1\gamma_1((A_3^2/c_1^2) - 1)^{-1}]$$
$$+ \alpha_{z_3}\left[i\pi j_1\gamma_1(j_1\gamma_1(L - z_{c_1}) - 1) + j_1\gamma_1 + \frac{1}{z_{c_1}} + j_1\gamma_1((A_3^2/c_1^2) - 1) \right.$$
$$\left. + (j_1\gamma_1)^2(L - z_{c_1}) \right]$$
$$f_4 = \alpha_{z_1}[i\pi j_1\gamma_1(j_1\gamma_1 z_{c_1} - 1) + j_1\gamma_1 + (j_1\gamma_1)^2 z_{c_1} + j_1\gamma_1((A_3^2/c_1^2) - 1)^{-1}]$$
$$- \alpha_{z_3}\left[i\pi j_1\gamma_1(j_1\gamma_1(j_1\gamma_1(L - z_{c_1}) - 1) + j_1\gamma_1 + \frac{1}{z_{c_1}} + j_1\gamma_1((A_3^2/c_1^2) - 1) \right.$$
$$\left. + (j_1\gamma_1)^2(L - z_{c_1}) \right]$$

Also, we find

$$\frac{Te^{i\alpha_{z_3}L}}{I+R} = \frac{(j_1\gamma_1)^2(z_{c_1} - L) - \dfrac{1}{z_{c_1} - L}}{i\alpha_{z_3}\left[s_1 \log\left(1 - \dfrac{A_3^2}{c_1^2}\right) + 2j_1\gamma_1 z_{c_1} - 2j_1\gamma_1(z_c - L) \right] + s_2(1 - j_1\gamma_1 z_{c_1})},$$

for $A_3^2 < c_1^2$. (5.2a)

$$\frac{Te^{i\alpha_{z_3}L}}{I+R} = \frac{-[(j_1\gamma_1)^2(L - z_{c_1}) + (1/(L - z_{c_1}))]}{\Gamma}, \quad A_3^2 > c_1^2 \text{ for} \tag{5.2b}$$

$$\Gamma = i\alpha_{z_3}\left[t_1 \log\left(\frac{A_3^2}{c_1^2} - 1\right) + i\pi - 2j_1\gamma_1(L - z_{c_1}) - 2j_1\gamma_1 z_{c_1} \right]$$
$$+ t_2\left[\log\left(\frac{A_3^2}{c_1^2} - 1\right) + i\pi \right] - t_2. \tag{5.2c}$$

As $j_1\gamma_1 \to 0$ the results (5.1) and (5.2) reduces to the form

$$\frac{R}{I} = \frac{\alpha_{z_3}\alpha_{z_1}(L - z_{c_1})\log\left(1 - \dfrac{A_3^2}{c_1^2}\right) + i\alpha_{z_3}\left(1 - \dfrac{A_3^2}{c_1^2}\right) + i\alpha_{z_1}}{\alpha_{z_3}\alpha_{z_1}(L - z_{c_1})\log\left(1 - \dfrac{A_3^2}{c_1^2}\right) - i\alpha_{z_3}\left(1 - \dfrac{A_1^2}{c_1^2}\right) + i\alpha_{z_1}} \tag{5.3}$$

and

$$\frac{T}{I+R} = \frac{1}{1 + i\alpha_{z_3}(L - z_{c_1})\log(1 - (A_3^2/c_1^2))} \tag{5.4}$$

In the limit $L - z_{c_1} \to 0$, we obtain

$$\frac{R}{I} = \frac{\alpha_{z_1} + \alpha_{z_3}(1 - (A_3^2/c_1^2))}{\alpha_{z_1} - \alpha_{z_3}(1 - (A_3^2/c_1^2))}, \tag{5.5}$$

and

$$\frac{T}{I+R} = 1. \tag{5.6}$$

view of results (5.5) and (5.6), we conclude that the wave is neither amplified nor sorbed by the layer. Expressing the reflection and transmission co-efficients for the al energy flux as a function of A^2 and c_1^2 we obtain

$$\left|\frac{R}{I}\right|^2 = \frac{[c_1^2 + (c_1^2 - A_1^2)^{1/2}(c_1^2 - A_3^2)^{1/2}]^2}{[c_1^2 - (c_1^2 - A_1^2)^{1/2}(c_1^2 - A_3^2)^{1/2}]^2} \tag{5.7}$$

$$\left|\frac{T}{I}\right|^2 = \frac{4c^4}{[c_1^2 - (c_1^2 - A_1^2)^{1/2}(c_1^2 - A_3^2)^{1/2}]^2} \tag{5.8}$$

d

$$\tau^2 = \frac{(c_1^2 - A_3^2)(-c_2^2 + A_3^2)(c_3^2 - A_1^2)}{(c_1^2 - A_1^2)(-c_1^2 + A_1^2)(c_3^2 - A_3^2)}\left|\frac{T}{I}\right|^2 - \frac{z_3}{\alpha_{z_1}} \tag{5.9}$$

ere τ^2 is the ratio of the transmitted energy flux to the incident flux in the moving id. Also we choose the sign of α_{z_1} and α_{z_3} so that the above result is positive.
It follows from (5.7) that $|R/I| \to \infty$ provided

$$c_1^2 = \frac{A_1^2 A_3^2}{A_1^2 + A_3^2}. \tag{5.10}$$

is result reveals that resonance occurs (that is $|R/T| \to \infty$) only when the negative n of the square root of (5.10) is taken.

Discussion and conclusion

is clear from the above analysis that if (4.7abc) and (4.8abc) are satisfied, there is critical level within the magnetic shear layer. Consequently, the amplification of dromagnetic wave is impossible.
On the other hand, if the condition (4.12abc) is satisfied, there exists a critical level thin the magnetic shear layer. The wave action flux is found to be invariant erywhere in the fluid medium except at the critical level. In view of (4.12abc), the ave incident upon the shear layer is over-reflexed, that is, more energy is reflected ck towards the source than was originally emitted. In the present hydromagnetic alysis, the mechanism of the over-reflection is due to the fact that the excess reflected ergy is extracted by the wave from the external magnetic field.
When the vertical wavelength is very large, $\gamma_1 L \ll 1$ the incident energy I, the flected energy R and the transmitted energy T satisfy results (5.5) and (5.6). It is ident from these results that the wave is neither amplified nor absorbed by the agnetic shear layer.
Finally, result (5.7) reveals that $|R/T| \to \infty$ provided the phase velocity of the wave negative and given by

$$c_1 = -\left(\frac{A_1^2 A_3^2}{A_1^2 + A_3^2}\right)^{1/2}. \tag{6.1}$$

hus resonance occurs at this value of c_1. And this quantity c_1 can be expressed as

$$c_1 = \pm\left[\left\{-\frac{q_i}{2} + \left(\frac{q_i^2}{4} + \frac{r_i^3}{27}\right)^{1/2}\right\}^{1/3} + \left\{-\frac{q_i}{2} - \left(\frac{q_i^2}{4} + \frac{r_i^3}{27}\right)^{1/2}\right\}^{1/3} - \frac{a_i}{3}\right]^{1/2} \tag{6.2ab}$$

where

$$q_i = \frac{1}{27}a_i^3 - \frac{1}{3}a_i b_i + k_i, \quad r_i \equiv -\frac{1}{3}a_i^2 + b_i, \quad i = 1, 2 \tag{6.3ab}$$

$$a_1 = [c^4(1 - A_1^2/A_3^2) - 2c^4 - A_1^2 A_3^2 - 2c^2(A_1^2 + A_3^2)]/(A_3^2 + 2c^2 A_3^4), \tag{6.4}$$

$$b_1 = (2c^4 A_1^2 + A_3^2 c^4 + 2c^2 A_1^2 A_3^2)/(A_3^2 + 2c^2 A_3^4), \tag{6.5}$$

$$k_1 = -(A_1^2 A_3^2 c^4)/(A_3^2 + 2c^2 A_3^4), \tag{6.6}$$

$$a_2 = \frac{1}{X}[(\omega^4/\alpha^4 + c^4 - 2\omega^2 c^2/\alpha^2)A_3^2 - (A_1^2 + 2A_3^2)c^4 - A_1^2(\omega^4/\alpha^4 + A_3^4 - 2A_3^2\omega^2/\alpha^2) + 2c^2(A_1^2 + A_3^2)]. \tag{6.7}$$

$$b_2 = \frac{1}{X}[2c^4 A_1^2 A_3^2 + A_3^4 c^4 - 2c^2 A_1^2 A_3^2], \tag{6.8}$$

$$k_2 = -(A_1^2 A_3^2 c^4)/X, \tag{6.9}$$

$$X = 2\omega^2(c^2 - A_3^2)/\alpha^2 + A_3^4 - 2c^2 \tag{6.10}$$

and c is the constant speed of sound.

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Badly approximable p-adic integers

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Abstract. It is known that the p-adic integers that are badly approximable by rationals form a null set with respect to Haar measure. We define a $[0,1]$-valued dimension function on the p-adic integers analogous to Hausdorff dimension in $\mathbf{R}$ and show that with respect to this function the dimension of the set of badly approximable p-adic integers is 1.

Keywords. Diophantine approximation; p-adic numbers; Hausdorff dimension.

Introduction

A real number x is called *badly approximable* if, roughly speaking, there are no rationals p/q such that $x - p/q$ is small compared with q^{-2}. It is well known (see [5]) that the set of badly approximable real numbers has Lebesgue measure zero and Hausdorff dimension 1. As might be expected, we can in an analogous way define the set of badly approximable p-adic integers. It is known (see [6]) that this set is a null set with respect to Haar measure on the group $\mathbf{Z}_p$ of all p-adic integers. In this paper we describe a natural analog of Hausdorff dimension applicable to the space of p-adic integers and we show that with respect to this dimension the dimension of the set of badly approximable p-adic integers is 1.

The proof of this result makes use of an approximation scheme for p-adic numbers developed by Mahler in [7], the essential features of which are recalled in the course of §3 below. We also exploit a method initiated by Billingsley in [2], and further developed by the author in [1], for comparing Hausdorff-like dimension functions defined with respect to arbitrary non-atomic measures. The basic facts about this method are explained in §4. With the aid of Mahler's scheme we construct a measure with respect to which the set of badly approximable numbers has measure 1. We then apply Billingsley's method to complete the proof.

1. Notation and preliminary remarks

We denote by $\mathbf{N}$ the set of strictly positive integers and write $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. For a natural number N we denote by $[N]$ the set

$$\{h \in \mathbf{N} : h \leqslant N\}.$$

If z is a complex number we shall always write $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. For any real y_0 we denote by $\mathbf{U}_{y_0}$ the set

$$\{z \in \mathbf{C} : y > y_0\}.$$

We denote by Γ the modular group $\mathrm{SL}_2(\mathbf{Z})$, and by $\mathbf{I}$ the identity of Γ. As usual, we let Γ act on the upper half-plane $\mathbf{U}_0$ in the following way. For

$$\omega = \begin{pmatrix} \alpha & \alpha' \\ \beta & \beta' \end{pmatrix}$$

in Γ and z in $\mathbf{U}_0$ we put

$$\omega z = \frac{\alpha z + \alpha'}{\beta z + \beta'}.$$

We denote by R the standard fundamental region for this action of Γ given by $R = R_1 \cup R_2$ where

$$R_1 = \{z \in \mathbf{U}_0 : |z| > 1,\ -\tfrac{1}{2} \leqslant x < \tfrac{1}{2}\}$$

and

$$R_2 = \{z \in \mathbf{U}_0 : |z| = 1,\ -\tfrac{1}{2} \leqslant x \leqslant 0\}.$$

It is easy to check that for any ξ in R the expression

$$\frac{2(r - \xi s)(r - \bar{\xi} s)}{|\xi - \bar{\xi}|}$$

is a positive definite quadratic form in r and s. We may therefore define a positive-valued function Φ_ξ on $\mathbf{R} \times \mathbf{R}$ by setting

$$\Phi_\xi(r, s) = \left(\frac{2(r - \xi s)(r - \bar{\xi} s)}{|\xi - \bar{\xi}|} \right)^{1/2}.$$

For a fixed prime p, we denote by $\mathbf{Z}_p$ the ring of p-adic integers with the usual valuation $|\ |_p$. Thus a typical element ρ of $\mathbf{Z}_p$ is a sequence $(\rho_n)_{n \in \mathbf{N}_0}$, where each ρ_n is an element of the additive group $\mathbf{Z}/p^n\mathbf{Z}$, and for each n the natural homomorphism $\mathbf{Z}/p^{n+1}\mathbf{Z} \to \mathbf{Z}/p^n\mathbf{Z}$ sends ρ_{n+1} to ρ_n. Given $\rho = (\rho_n)_{n \in \mathbf{N}_0}$, $\rho' = (\rho'_n)_{n \in \mathbf{N}_0}$ in $\mathbf{Z}_p$ we define

$$\rho + \rho' = (\rho_n + \rho'_n)_{n \in \mathbf{N}_0}$$

and

$$\rho\rho' = (\rho_n \rho'_n)_{n \in \mathbf{N}_0}.$$

We define $|\rho|_p = p^{-v}$ where $v = v(\rho)$ is the least integer in $\mathbf{N}_0$ such that ρ_{v+1} is different from zero.

We say that ρ, ρ' are congruent modulo p^β, and write $\rho \equiv \rho' \pmod{p^\beta}$, if $|\rho - \rho'|_p \leqslant p^{-\beta}$.

We equip $\mathbf{Z}_p$ with the topology induced by the metric $d(\rho, \rho') = |\rho - \rho'|_p$. The space $\mathbf{Z}_p$ is homeomorphic to the topological product $[p]^\omega$, where $[p]$ is equipped with the discrete topology. Therefore $\mathbf{Z}_p$ is compact.

e put

$$B_h(\rho) = \{\rho' \in \mathbf{Z}_p : \|\rho - \rho'\|_p \leqslant p^{-h}\}$$

$$\mathfrak{B} = \{B_h(\rho) : \rho \in \mathbf{Z}_p,\ h \in \mathbf{N}_0\}.$$

set $\mathfrak{B}$ is a basis for $\mathbf{Z}_p$ consisting of closed open sets. An element of $\mathfrak{B}$ will be called *here*. The reader will observe that the sphere $B_h(\rho)$ is the set of all ρ' in $\mathbf{Z}_p$ with ρ_h. In sections 4–5 below we shall persistently abuse notation by writing ρ_h in e of $B_h(\rho)$.

et $(\mathbf{Z}_p, \mathscr{E}, \mu)$ be a probability space on $\mathbf{Z}_p$, where $\mathscr{E}$ is the σ-algebra generated by $\mathfrak{B}$. μ be any probability measure on $\mathbf{Z}_p$ that is non-atomic, i.e. $\mu(\{\rho\}) = 0$ for all $_p$. Suppose $\gamma > 0$. For $\theta > 0$ and $M \subset \mathbf{Z}_p$, write

$$\ell^{\gamma}_{\mu,\theta}(M) = \inf \sum (\mu(B_{h_i}(\rho^{(i)})))^{\gamma}.$$

e the infimum is taken over all coverings of M by subsets of $\mathfrak{B}$ of the form $\{\rho^{(i)}) : i \in \mathbf{N}\}$ such that $\mu(B_{h_i}(\rho^{(i)})) < \theta$ for all $i \in \mathbf{N}$. The (not necessarily finite) limit

$$\ell^{\gamma}_{\mu}(M) = \lim_{\theta \to 0} \ell^{\gamma}_{\mu,\theta}(M)$$

s for all M. For a simple proof the ℓ^{γ}_{μ} as thus defined is an outer measure see [2], 36, 141. It can be shown ([2], pp. 136–137, 141) that for each $M \subset \mathbf{Z}_p$ there exists a ue real number $\Delta = \Delta_{\mu}(M)$ such that $\ell^{\gamma}_{\mu} = \infty$ for all $\gamma < \Delta$ and $\ell^{\gamma}_{\mu} = 0$ for all $\gamma > \Delta$.

e define $\eta : \mathfrak{B}(\mathbf{Z}_p) \to \mathbf{R}$ by $\eta(B_h(a)) = p^{-h}$ for all $a \in \mathbf{Z}_p$, $h \in \mathbf{N}_0$. Then by the athéodory-Hopf extension theorem ([4], § 13, Theorem A), η can be extended to obability measure on $\mathbf{Z}_p$, also denoted by η. The measure η is clearly translation-riant and therefore by the Haar uniqueness theorem ([3], pp. 309–310) it coincides Haar measure on $\mathbf{Z}_p$. We call $\Delta_{\eta}(M)$ the *Hausdorff dimension* of M. This inology is appropriate because, as is proved in [2], p. 140, Hausdorff dimension can be defined by the same procedure with Lebesgue measure in place of η.

tatement of the result

each positive real number τ let us say that a p-adic integer ρ is *badly approximable* nd write $\rho \in J(\tau)$ if for all a, b in $\mathbf{Z}$ we have

$$|a + b\rho|_p \geqslant \tau(\max|a|, |b|)^{-2}.$$

us say that ρ is *badly approximable* if it is badly approximable (τ) for some $\tau > 0$. denote the set of badly approximable p-adic integers by J. Thus

$$J = \bigcup_{\tau > 0} J(\tau).$$

is well known (see for example [6], Th. 4.23) that $\eta(J) = 0$. Thus it is of interest to rmine $\Delta_{\eta}(J)$. Our purpose in this paper is to prove the following:

orem 2.1. *We have* $\Delta_{\eta}(J) = 1$.

We first recast this result in a more convenient form. For ξ in R and $\tau > 0$ let $J_\xi(\tau)$ denote the set of ρ such that

$$|a + b\rho|_p \geqslant \tau(\Phi_\xi(a, b))^{-2} \tag{2.1}$$

for all a, b *in* $\mathbf{Z}$, and write

$$J_\xi = \bigcup_{\tau > 0} J_\xi(\tau).$$

Since $(\Phi_\xi)^2$ is positive definite, a simple computation shows that J is identical with J_ξ for each ξ. Therefore Theorem 2.1 is a consequence of the following result, which, though more detailed than Theorem 2.1, appears to be no harder to prove.

Theorem 2.2. *There is a constant C depending only on p such that for any ξ in R and any K in* $\mathbf{N}$ *we have*

$$\Delta_\eta(J_\xi(p^{-K-C})) \geqslant 1 - \frac{1}{2K}.$$

Theorem 2.2 is analogous to the following result on Diophantine approximation in $\mathbf{R}$. Call r in $\mathbf{R}$ *badly approximable* if there is a constant τ such that $|a + br| \geqslant \tau b^{-1}$ for all a, b in $\mathbf{Z}$. Then we have:

Theorem 2.3. *The Hausdorff dimension of the set of badly approximable real numbers is* 1.

This was established by V Jarnik in [5], a pioneering paper in which dimension theory was applied for the first time in the study of Diophantine approximation. The proof of Theorem 2.3 depends on a special feature of $\mathbf{R}$, namely the availability of an appropriate continued fraction algorithm. It turns out that badly approximable real numbers are those whose simple continued fractions have bounded partial denominators.

To prove Theorem 2.2 we shall use an approximation scheme for p-adic integers developed by K Mahler in [7]. As Mahler points out, his scheme is a working substitute for a continued fraction algorithm in the sense that it yields all "good" approximations to a p-adic integer ρ, that is all potential counterexamples to (2.1). Lemma 3.2 below is a more precise statement of this fact. As we shall see, the badly approximable p-adic integers can be nicely characterized in the language of Mahler's scheme.

In the early days of research on Hausdorff dimension it was notoriously difficult to find sharp lower bounds for the dimension of sets like $J(\tau)$. It is now, in many cases, much easier, thanks to a method developed by P Billingsley which we review briefly in section 4 before applying it to the present problem.

3. Mahler's approximation scheme

Given a p-adic integer ρ we define, for each n in $\mathbf{N}_0$, an integer $E_n = E_n(\rho)$ by means of the relations

$$0 \leqslant E_n < p^n$$

$$|E_n - \rho|_p \leqslant p^{-n}.$$

s easy to check that exactly one integer E_n satisfies these two relations.
Fix ξ in R, and for each n in $\mathbf{N}_0$ define a complex number $Z_n = Z_n(\rho)$ by setting

$$Z_n = \frac{E_n + \xi}{p^n}.$$

rther for each n in $\mathbf{N}_0$ let

$$z_n = z_n(\rho) = x_n + iy_n = x_n(\rho) + iy_n(\rho)$$

the unique element of R that is equivalent to Z_n under the action of Γ on $\mathbf{U}_0$.
Suppose that

$$\omega_n = \begin{pmatrix} c_n & c'_n \\ b_n & b'_n \end{pmatrix}$$

he element of Γ satisfying

$$\omega_n z_n = Z_n,$$

l write

$$a_n = p^n c_n - E_n b_n, \quad a'_n = p^n c'_n - E_n b'_n.$$

o for each n in $\mathbf{N}_0$ write

$$T_n = \begin{pmatrix} a_n & a'_n \\ b_n & b'_n \end{pmatrix}$$

l for each n in $\mathbf{N}$ write

$$\Omega_n = T_{n-1}^{-1} T_n.$$

can now state the fundamental results due to Mahler on which our proof of eorem 2.2 will be based.

nma 3.1. ([7], p. 12). *For any ρ in $\mathbf{Z}_p$ and any n in $\mathbf{N}_0$ we have*

$$y_n(\rho) = \frac{p^n}{(\Phi_\xi(a_n, b_n))^2}.$$

nma 3.2. ([7], p. 51 (*Theorem* 18)). *Let a, b in $\mathbf{Z}$ satisfy*

$$|a + b\rho|_p \leqslant p^{-n},$$

$$\Phi_\xi(a, b) > 0.$$

en

$$\Phi_\xi(a, b) \geqslant \Phi_\xi(a_n, b_n).$$

Lemma 3.3. ([7], p. 15). *The subset* $M(p)$ *of* $\mathbf{GL}_2(\mathbf{Z}_p)$ *defined by*

$$M(p) = \{\Omega_n(\rho) : n \in \mathbf{N}, \rho \in \mathbf{Z}_p\}$$

is a finite set and the determinant of each element of $M(p)$ *is* p. *Moreover a matrix* Ω *is in* $M(p)$ *if and only if* $p\Omega^{-1}$ *is in* $M(p)$.

Note. It turns out that $M(p)$ is independent of the choice of ξ. However we do not require this fact.

Lemma 3.4. ([7], p. 14). *For each* n *in* $\mathbf{N}_0$ *we have*

$$z_{n+1} = \Omega_{n+1}^{-1} z_n.$$

Lemma 3.5. ([7], p. 14). *For each* n *in* $\mathbf{N}_0$ *the integers* a_n *and* b_n *are relatively prime.*

We now derive some consequences of the preceding lemmas.

Lemma 3.6. *The set of badly approximable p-adic integers coincides with the set of those* ρ *such that* $y_n(\rho)$ *remains bounded as* n *goes to infinity. More precisely, for each* $\tau > 0$ *we have*

$$J_\xi(\tau) = \{\rho \in \mathbf{Z}_p : y_n(\rho) \leqslant \tau^{-1} (\forall n \in \mathbf{N}_0)\}.$$

Proof. It suffices to prove the second statement. Suppose ρ is in $J_\xi(\tau)$. Then by the definition of $J_\xi(\tau)$ we have for each n in $\mathbf{N}_0$ that

$$|a_n + b_n\rho|_p \geqslant \tau(\Phi_\xi(a_n, b_n))^{-2}. \tag{3.1}$$

By the definition of a_n, b_n we have

$$p^{-n} \geqslant |a_n + b_n\rho|_p. \tag{3.2}$$

By Lemma 3.1 we have

$$(\Phi_\xi(a_n, b_n))^{-2} = y_n(\rho)p^{-n}. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) we have

$$1 \geqslant \tau y_n(\rho),$$

which proves that $J_\xi(\tau)$ is included in the set of those ρ such that $y_n(\rho)$ never exceeds τ^{-1}.

To prove the reverse inclusion, suppose that ρ satisfies $y_n(\rho) \leqslant \tau^{-1}$ for all n in $\mathbf{N}_0$. Let a, b be any integers and define $h = h(a, b)$ by the relation

$$|a + b\rho|_p = p^{-h}.$$

Then using Lemmas 3.1 and 3.2 we have

$$\begin{aligned} \tau|a + b\rho|_p^{-1} &= \tau p^h \\ &\leqslant (y_n(\rho))^{-1} p^h \\ &= (\Phi_\xi(a_n, b_n))^2 \\ &\leqslant (\Phi_\xi(a, b))^2 \end{aligned}$$

t ρ is in $J_\xi(\tau)$ as required.

a 3.7. There exists a constant C depending only on p such that for any ρ in $\mathbf{Z}_p$, ver $y_n(\rho) > p^C$ we have

$$y_{n+1} = p^{\pm 1} y_n \tag{3.4}$$

$n \geqslant 1$ we have

$$y_{n-1} = p^{-1} y_n. \tag{3.5}$$

By Lemmas 3.3 and 3.4 we have for any ρ in $\mathbf{Z}_p$ and any n in $\mathbf{N}_0$ that

$$z_{n+1} = \Omega z_n,$$

Ω is in $M(p)$. We write

$$\Omega = \begin{pmatrix} \alpha & \alpha' \\ \beta & \beta' \end{pmatrix}.$$

a fixed complex number z, the subset $\Upsilon(z)$ of $[0, 2\pi)$ defined by

$$\Upsilon(z) = \{|\arg \Omega z| : \Omega \in M(p)\}$$

e, by Lemma 3.3. Now suppose z is in R. We see easily that when y is large h we have $|\arg \Omega z| < \frac{1}{2}$ for all Ω satisfying $\alpha\beta \neq 0$. Moreover if $\alpha = 0$ we see that whenever y is sufficiently large. Thus there is a constant C such that for z in R, , and Ω in $M(p)$ we have Ωz in R only when $\beta = 0$.
then, since by Lemma 3.3 we have $\det \Omega = p$, we find that either $\alpha = p$ or $\beta' = p$. stablishes (3.4), and if $n \geqslant 1$ the same argument with z_{n-1} in place of z_n shes (3.5).

a 3.8. Suppose that for some n in $\mathbf{N}_0$ we have

$$y_n(\rho) > y_{n+1}(\rho) > p^C,$$

C is the same constant as in Lemma 3.7. Then we have

$$y_{n+2} = p^{-1} y_{n+1}.$$

In view of Lemma 3.7 we need only show that $y_{n+2} \neq p y_{n+1}$. We know, using a 3.7, that $y_n = p y_{n+1}$. Therefore by Lemma 3.4 we can write

$$\Omega_{n+1} = \begin{pmatrix} p & \alpha' \\ 0 & 1 \end{pmatrix},$$

iven z_{n+1} in R there is just one choice of α' such that $z_n = \Omega_{n+1} z_{n+1}$ is in R. fore the relation $y_{n+2} = p y_{n+1}$ would imply

$$\Omega_{n+2}^{-1} = p^{-1} \begin{pmatrix} p & \alpha' \\ 0 & 1 \end{pmatrix} = \Omega_{n+1},$$

so that

$$\Omega_{n+1}\Omega_{n+2} = pI,$$

and then each component of T_{n+2} would be divisible by p, which contradicts Lemma 3.5. We conclude that $y_{n+2} \neq py_{n+1}$ as claimed.

4. Billingsley's lower bound for the dimension of a set

A version of the following lemma, relating to subsets of $[0,1]$, is proved in [2], pp. 144–145, and the proof carries over to the present setting without significant alteration. Recall that we agreed to abuse notation by writing ρ_j in place of $B_j(\rho)$.

Lemma 4.1. *For any non-atomic Borel measures λ, μ on $\mathbf{Z}_p$ and for any $\delta \geqslant 0$, if*

$$M \subset \left\{\rho \in \mathbf{Z}_p : \liminf_{j\to\infty} \frac{\log \lambda(\rho_j)}{\log \mu(\rho_j)} \geqslant \delta\right\}$$

then $\Delta_\mu(M) \geqslant \delta \Delta_\nu(M)$.

Note: if either of the real numbers a, b is either 0 or 1, then $\log a/\log b$ is defined equal to 0, 1 or ∞ according as $a > b$, $a = b$ or $a < b$. The logarithms can be taken to any positive base except 1, *and in what follows we shall take all logarithms to the base p.*

In order to apply Lemma 4.1 to the problem at hand we need to construct a measure ν on $\mathbf{Z}_p$ such that

$$\Delta_\nu(J_\xi(p^{-K-C})) = 1$$

and such that

$$J_\xi(p^{-K-C}) \subset \left\{\rho \in \mathbf{Z}_p : \liminf_{j\to\infty} \frac{\log \nu(\rho_j)}{\log \eta(\rho_j)} \geqslant 1 - \frac{1}{2K}\right\}.$$

The construction of such a measure is made possible by the following result, which is a special case of Lemma 5.2 in [1]. If u is any sphere, we denote by $\sigma(u)$ the set of maximal proper subspheres of u.

Lemma 4.2. *Suppose that $\mu' : \mathfrak{B} \setminus \{\mathbf{Z}_p\} \to [0,1]$ satisfies*

$$\sum_{v \in \sigma(u)} \mu'(v) = 1$$

for all u in $\mathfrak{B}$. Then there is a unique Borel probability measure μ on $\mathbf{Z}_p$ satisfying

$$\mu(u)\mu'(v) = \mu(v)$$

for all u, v in $\mathfrak{B}$ with v in $\sigma(u)$.

5. Proof of Theorem 2.2

Let K be a fixed integer greater than 0, and let C be the constant whose existence is guaranteed by Lemma 3.7. For ρ in $\mathbf{Z}_p$ write

$$t_n(\rho) = \#\{\rho_n' \in \sigma(\rho_{n-1}) : \log y_n(\rho') \leqslant K + C\}.$$

One checks easily that $y_n(\rho')$ is actually determined by ρ_n'.

We show that if

$$C < \log y_{n-1}(\rho) \leqslant K + C$$

then

$$\#(t_n(\rho)) \geqslant 1. \tag{5.1}$$

Suppose the contrary. Then for every maximal subsphere ρ_n' of ρ_{n-1} we have

$$\log y_n(\rho') > K + C.$$

But there are at least two maximal subspheres ρ_n' contained in ρ_n (in fact there are p of them) and therefore there are at least two Ω in $M(p)$ satisfying

$$\Omega z_{n-1}(\rho) \in R \cap \mathbf{U}_{p^{K+C}}.$$

As in the proof of Lemma 3.8 any Ω satisfying

$$\Omega z_{n-1}(\rho) \in \mathbf{U}_{p^{K+C}}$$

must be of the form

$$\Omega = \begin{pmatrix} p & \alpha' \\ 0 & 1 \end{pmatrix},$$

and there is just one choice of $\alpha' \in \mathbf{Z}$ such that $\Omega z_{n-1}(\rho)$ is in R. Thus we have arrived at a contradiction and must conclude that (5.1) holds as claimed.

We may therefore define a function $v' = v_K'$ on $\mathfrak{B} \backslash \{\mathbf{Z}_p\}$ with values in $[0, 1]$ as follows.

Case (i). If

$$K - 1 + C < \log y_{n-1}(\rho) \leqslant K + C$$

and $\log y_n(\rho) > K + C$, we set

$$v'(\rho_n) = 0.$$

Case (ii). If

$$K - 1 + C < \log y_{n-1}(\rho) \leqslant K + C$$

and $\log y_n(\rho) \leqslant K + C$, we set

$$v'(\rho_n) = (t_n(\rho))^{-1}.$$

Case (iii). If $\log y_{n-1}(\rho)$ lies in the complement of the interval $(K-1+C, K+C]$, we set

$$v'(\rho_n) = p^{-1}.$$

One checks easily (using the definition of $t_n(\rho)$) that v' satisfies the hypotheses of Lemma 4.2, and so there is a probability measure v on $\mathbf{Z}_p$ satisfying (4.1).

To check that v is non-atomic, choose ρ in $\mathbf{Z}_p$, so that

$$\rho = \bigcap_{n \in \mathbf{N}_0} \rho_n.$$

We must show that $v(\rho_n)$ goes to 0 as n goes to ∞. By (4.1) and straightforward induction we have

$$v(\rho_n) = \prod_{1 \leqslant j \leqslant n} v'(\rho_n).$$

Now by Lemma 3.7 we cannot have both

$$K-1+C < \log y_{n-1}(\rho) \leqslant K+C$$

and

$$K-1+C < \log y_n(\rho) \leqslant K+C.$$

Hence for infinitely many n case (ii) of the definition of v' does not apply and for such n we have

$$v'(\rho_n) \leqslant p^{-1} < 1.$$

Therefore $v(\rho_n) \to 0$ as required, so v is non-atomic.

We now verify that $v(J(p^{-K-C})) = 1$. If ρ is in the complement of $J(p^{-K-C})$ then by Lemma 3.6 for some n in $\mathbf{N}_0$ we have $\log y_n(\rho) > K+C$. Choose N to be the least integer with this property. Then by Lemma 3.7 we have

$$K-1+C < \log y_{N-1}(\rho) \leqslant K+C.$$

Therefore case (i) of the definition of v' gives $v'(\rho_N) = 0$, so also $v(\rho_N) = 0$. Thus the complement of $J(p^{-K-C})$ is covered by elements of $\mathfrak{B}$ each of which has measure zero with respect to v. Since $\mathfrak{B}$ is countable we have $v(J(p^{-K-C})) = 1$ as claimed. Our next objective is to show that for all ρ in $J(p^{-K-C})$ we have

$$\liminf_{N \to \infty} \frac{\log v(\rho_N)}{\log \eta(\rho_n)} \geqslant 1 - \frac{1}{2K}. \tag{5.2}$$

For ρ in $J(p^{-K-C})$ let $H = H(\rho)$ be the subset of $\mathbf{N}_0$ consisting of those n for which

$$K-1+C < \log y_n(\rho).$$

Let ρ be in $J(p^{-K-C})$, and choose n in $H(\rho)$. By the choice of ρ we have $\log y_n \leqslant K+C$, and also by Lemma 3.6 we have $\log y_{n+1} \leqslant K+C$. Thus by Lemma 3.7 and the fact that $K-1+C < \log y_n$ we have

$$\log y_{n+1} = -1 + \log y_n > K-2+C.$$

2 we then have $\log y_{n+1} > C$, and since $\log y_n > \log y_{n+1}$ Lemma 3.8 implies that
'e

$$\log y_{n+2} = -2 + \log y_n > K - 3 + C.$$

uing in this way we find that

$$\log y_{n+h} = -h + \log y_n > K - h + C \geqslant C$$

h $h = 0, \ldots, K$. A further application of Lemma 3.7 shows that

$$\log y_{n+K+h} \leqslant K + C$$

h $h = 0, \ldots, K-1$.
s the difference between consecutive elements of $H(\rho)$ is at least $2K$, so we

$$\#([N] \setminus H(\rho)) > N - 1 - \frac{N}{2K} \tag{5.3}$$

N in $\mathbf{N}_0$.
if ρ is in $J(p^{-K-C})$, we have for each n in $\mathbf{N}$ either

$$v'(\rho_n) = (t_n(\rho))^{-1}$$

is in $H(\rho)$, or

$$v'(\rho_n) = p^{-1}$$

ise. Therefore for each N in $\mathbf{N}_0$ we have

$$\log v(\rho_N) = - \sum_{n \in [N] \cap H(\rho)} \log t_{n+1}(\rho) - \#([N] \setminus H(\rho)).$$

e clearly have

$$\log \eta(\rho_N) = -N,$$

using (5.3) and the fact that $t_n(\rho) \geqslant 1$ we have

$$\frac{\log v(\rho_n)}{\log \eta(\rho_n)} > 1 - \frac{1}{N} - \frac{1}{2K}.$$

ore letting N go to ∞ we have (5.2).
e $v(J(p^{-K-C})) = 1$, we certainly have

$$\Delta_v(J(p^{-K-C})) = 1,$$

w an appeal to Lemma 4.1 with $\lambda = v$, $\mu = \eta$, and $\delta = 1 - (1/2K)$ completes the

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ertainty principles on certain Lie groups

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Abstract. There are several ways of formulating the uncertainty principle for the Fourier transform on $\mathbb{R}^n$. Roughly speaking, the uncertainty principle says that if a function f is 'concentrated' then its Fourier transform $\tilde{f}$ cannot be 'concentrated' unless f is identically zero. Of course, in the above, we should be precise about what we mean by 'concentration'. There are several ways of measuring 'concentration' and depending on the definition we get a host of uncertainty principles. As several authors have shown, some of these uncertainty principles seem to be a general feature of harmonic analysis on connected locally compact groups. In this paper, we show how various uncertainty principles take form in the case of some locally compact groups including $\mathbb{R}^n$, the Heisenberg group, the reduced Heisenberg group and the Euclidean motion group of the plane.

Keywords. Fourier transform; Heisenberg group; motion group; uncertainty principle.

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are several ways of formulating the uncertainty principle for the Fourier form on $\mathbb{R}^n$. Roughly speaking, the uncertainty principle says that if a function 'concentrated' then its Fourier transform $\tilde{f}$ cannot be concentrated unless dentically zero. Of course, in the above, we should be precise about what we by 'concentration'. There are several ways of measuring 'concentration' and ding on the definition we get a host of uncertainty principles. As has been n in [1], [2], [4], [9], [12], [13], [17] etc, some of these uncertainty principles to be a general feature of harmonic analysis on connected locally compact os. We continue these investigations in this paper to see how various uncertainty ples take form in the case of some locally compact groups including $\mathbb{R}^n$, the nberg group, the reduced Heisenberg group and the Euclidean motion group plane. In a forthcoming paper [14] we consider semi-simple Lie groups and more general eigenfunction expansions on a manifold with respect to some c operator.

e way of measuring concentration is by considering the decay of the function at ty. In this context, a theorem of Hardy for the Fourier transform on $\mathbb{R}$ says the ving:

em 1. (*Hardy*) *Suppose f is a measurable function on $\mathbb{R}$ such that*

$$|f(x)| \leqslant Ce^{-\alpha x^2}, \quad |\tilde{f}(\xi)| \leqslant Ce^{-\beta \xi^2}, \quad x, \xi \in \mathbb{R} \tag{1.1}$$

where α, β are positive constants. If $\alpha\beta > \frac{1}{4}$ then $f = 0$ a.e. If $\alpha\beta < \frac{1}{4}$ there are infinitely many linearly independent functions satisfying (1.1) *and if $\alpha\beta = \frac{1}{4}$ then $f(x) = Ce^{-\alpha x^2}$.*

For a proof of the above theorem see [3]. A more general theorem due to Beurling, from which Hardy's theorem can be deduced, can be found in [10]. In this paper we establish an analogue of the above theorem for the Heisenberg group $\mathscr{H}_n$ (see § 2 for the precise formulation). We also prove Hardy's theorem in the case of $\mathbb{R}^n$, $n \geqslant 2$ and show that though the exact analogue for the reduced Heisenberg group fails, a slightly modified version continues to hold. In the final section we prove an analogue of Hardy's theorem for the Euclidean motion group of the plane.

Another natural way of measuring 'concentration' is in terms of the supports of the function f and its Fourier transform $\tilde{f}$. If f is non-trivial and compactly supported then $\tilde{f}$ extends to an entire function, and so $\tilde{f}$ cannot have compact support. A non-trivial extension of this result due to Benedicks [1] says: If $f \in L^1(\mathbb{R}^n)$ is such that $m\{x : f(x) \neq 0\} < \infty$ and $m\{\xi : f(\xi) \neq 0\} < \infty$ then $f = 0$ a.e. Here m stands for the Lebesgue measure on $\mathbb{R}^n$. This result of Benedicks has been extended in [2], [12], [4] etc. to a wide variety of locally compact groups. In particular, one has the following result for the Heisenberg group:

Theorem 2. (*Price–Sitaram*) *Let $f \in L^1 \cap L^2(\mathscr{H}_n)$. Suppose that $m\{t \in \mathbb{R} : f(z, t) \neq 0\} < \infty$ for a.e. $z \in \mathbb{C}^n$ and $m\{\lambda \in \mathbb{R}^* : \hat{f}(\lambda) \neq 0\} < \infty$. Then $f = 0$ a.e.*

In the above $\hat{f}(\lambda)$ stands for the group Fourier transform on $\mathscr{H}_n$ and $\mathbb{R}^*$ means $\mathbb{R} \setminus \{0\}$. Roughly speaking, the above theorem says that if $f \in L^2(\mathscr{H}_n)$ is concentrated in the t direction then $\hat{f}(\lambda)$ cannot be concentrated. It is the concentration in the t direction, not that in the z direction, which forces the spreading out of the Fourier transform. In fact, as was shown by Thangavelu in [17], we can have L^2 functions with compact support in the z variable for which $\hat{f}$ is also compactly supported. The special role played by the t variable in the above theorem (as well as in our Hardy's theorem in § 2) should not come as a surprise. The Fourier transform on $\mathscr{H}_n$ is more or less the Euclidean Fourier transform as far as the t variable is concerned. If one goes through the proof of the above theorem, one observes that it is a consequence of the corresponding theorem for the Euclidean Fourier transform in the t variable.

In view of the preceding remarks one would like to have an analogue of the above theorem which respects the z variable. We formulate and prove such a theorem in § 3. We will show that when f has compact support in the z variable then $\hat{f}(\lambda)$ (as an operator) cannot have 'compact support'. We will give a precise meaning to this statement in § 3.

We now turn our attention towards quantitative versions of the uncertainty principle, namely uncertainty inequalities. The classical Heisenberg–Pauli–Weyl uncertainty inequality for the Fourier transform on $\mathbb{R}^n$ says that

$$\|f\|_2^4 \leqslant C_n \left(\int |x|^2 |f(x)|^2 dx\right)\left(\int |\xi|^2 |\tilde{f}(\xi)|^2 d\xi\right). \tag{1.2}$$

For a proof of (1.2) with the precise value of C_n we refer to [6]. A version of the above inequality for the Heisenberg group was established by Thangavelu in [17]. Here we are concerned with local versions of the above inequality for the Heisenberg group.

For the Fourier transform on $\mathbb{R}^n$ one has the following local uncertainty inequality: For any measurable $E \subset \mathbb{R}^n$, and $0 < \theta < \frac{1}{2}$,

$$\int_E |\tilde{f}(\xi)|^2 d\xi \leqslant C_\theta m(E)^{2\theta} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{2n\theta} dx. \tag{1.3}$$

An analogue of the above inequality is known on the Heisenberg group. The following result is proved in [13].

Theorem 3. (*Price–Sitaram*) *Let* $\theta \in [0, \frac{1}{2})$. *Then, for each* $f \in L^1 \cap L^2(\mathscr{H}_n)$ *and measurable* $E \subset \mathbb{R}^*$, *one has*

$$\int_E \operatorname{tr}(\hat{f}(\lambda)^* \hat{f}(\lambda)) d\mu(\lambda) \leqslant C_\theta m(E)^{2\theta} \int_{\mathscr{H}_n} |f(z,t)|^2 |t|^{2\theta} dz\, dt. \tag{1.4}$$

(In the above tr stands for the canonical semifinite trace and $d\mu$ is the Plancherel measure on $\mathscr{H}_n$—see § 2.) Again we observe that the t variable plays a special role. As in the case of the Euclidean Fourier transform one would like to have an inequality which is more symmetric in all the variables. In § 4 we formulate and prove a local uncertainty inequality with the right hand side being

$$\int_{\mathscr{H}_n} |f(w)|^2 |w|^{2\theta Q} dw \tag{1.5}$$

where $|w|^4 = |z|^4 + t^2$ and $Q = (2n+2)$ is the homogeneous dimension of $\mathscr{H}_n$. From the local uncertainty inequality we will also deduce a global inequality similar to the classical Heisenberg–Pauli–Weyl uncertainty inequality.

Finally, for various facts about the Heisenberg group we refer to the monographs of Folland [6] and Thangavelu [19]. We closely follow the notations of the latter which differ from the former by a factor of 2π.

2. Analogues of Hardy's theorem for $\mathbb{R}^n$ and $\mathscr{H}_n$

Before we prove Hardy's theorem for the Heisenberg group, consider the case of $\mathbb{R}^n$, $n \geqslant 2$. The proof of Hardy's theorem (for $n = 1$) depends heavily on complex analysis. As we have not found a reference in the literature for the higher dimensional case of Hardy's theorem we take this opportunity to present a proof which follows easily from the one-dimensional case via the Radon transform.

Theorem 4. *Let* f *be a measurable function on* $\mathbb{R}^n$ *and* α, β *two positive constants. Further assume that*

$$|f(x)| \leqslant Ce^{-\alpha|x|^2}, |\tilde{f}(\xi)| \leqslant Ce^{-\beta|\xi|^2}, \quad x, \xi \in \mathbb{R}^n. \tag{2.1}$$

If $\alpha\beta > \frac{1}{4}$, *then* $f = 0$ *a.e. If* $\alpha\beta < \frac{1}{4}$, *there are infinitely many linearly independent solutions for* (2.1) *and if* $\alpha\beta = \frac{1}{4}$, f *is a constant multiple of* $e^{-\alpha|x|^2}$.

Proof. As mentioned above, we will use theorem 1. So, assume that $n \geqslant 2$. We use the Radon transform to reduce the problem to the one-dimensional case. Recall that the

Radon transform Rg of an integrable function g on $\mathbb{R}^n$ is a function of two variables (ω, s) where $\omega \in S^{n-1}$ and $s \in \mathbb{R}$ and is given by

$$\mathrm{Rg}(\omega, s) = \int_{x.\omega = s} g(x)\,\mathrm{d}x. \tag{2.2}$$

where dx is the Euclidean measure on the hyperplane $x.\omega = s$. Actually, for each fixed ω, the above makes sense for almost all $s \in \mathbb{R}$ which may depend on ω. However for functions with sufficient rapid decay at infinity it makes sense for all s. For various properties of the Radon transform we refer to [5] and [8].

Our definition of the Fourier transform of a function f on $\mathbb{R}^n$ is

$$\tilde{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix.\xi}\,\mathrm{d}x. \tag{2.3}$$

Then it can be easily seen that

$$\tilde{f}(s\omega) = (Rf)^{\sim}(\omega, s) \tag{2.4}$$

where $s \in \mathbb{R}$, $\omega \in S^{n-1}$ and $(Rf)^{\sim}$ stands for the Fourier transform of Rf in the s-variable alone. From the definition of the Radon transform Rf and the relation (2.4), the conditions on f and $\tilde{f}$ translate into conditions on Rf and $(Rf)^{\sim}$. For each fixed ω, we therefore get

$$|Rf(\omega, r)| \leqslant Ce^{-\alpha r^2}, \quad r \in \mathbb{R} \tag{2.5}$$

$$|(Rf)^{\sim}(\omega, s)| \leqslant Ce^{-\beta s^2}, \quad s \in \mathbb{R}. \tag{2.6}$$

By appealing to Hardy's theorem for $\mathbb{R}$ we conclude that for $\alpha\beta > \frac{1}{4}$, $Rf(\omega, .) = 0$, for almost all ω. In view of the inversion theorem for the Radon transform this implies $f = 0$ a.e. When $\alpha\beta = \frac{1}{4}$, $(Rf)^{\sim}(\omega, s) = \tilde{f}(s\omega) = A(\omega)e^{-\alpha s^2}$ where A is a measurable function on the unit sphere S^{n-1}. Because $f \in L^1(\mathbb{R}^n)$, $\tilde{f}$ is continuous at zero and by taking $s \to 0$ we obtain $A(\omega) = \tilde{f}(0)$. Hence $\tilde{f}(\xi) = \tilde{f}(0)e^{-\beta|\xi|^2}$ so that $f(x) = Ce^{-\alpha|x|^2}$ for some constant C. If $\alpha\beta < \frac{1}{4}$, the n-dimensional suitably scaled Hermite functions Φ_μ satisfy (2.1).

We now consider the case of the Heisenberg group $\mathscr{H}_n = \mathbb{C}^n \times \mathbb{R}$. The multiplication law of the group $\mathscr{H}_n$ is given by

$$(z, t)(w, s) = (z + w, t + s + \tfrac{1}{2}\mathrm{Im}(z.\bar{w})), \tag{2.7}$$

where $z, w \in \mathbb{C}^n$, $t, s \in \mathbb{R}$. Then $\mathscr{H}_n$ becomes a step-two nilpotent Lie group with Haar measure $\mathrm{d}z\mathrm{d}t$. In order to define the group Fourier transform we need to recall some facts about the representations of the Heisenberg group. For each $\lambda \in \mathbb{R}^*$, there is an irreducible unitary representation π_λ of $\mathscr{H}_n$ realised on $L^2(\mathbb{R}^n)$ and is given by

$$(\pi_\lambda(z, t)\phi)(\xi) = e^{i\lambda t} e^{i\lambda(x.\xi + \frac{1}{2}x.y)} \phi(\xi + y), \tag{2.8}$$

where $z = x + iy$ and $\phi \in L^2(\mathbb{R}^n)$. A theorem of Stone–von Neumann says that all the infinite dimensional irreducible unitary representations of $\mathscr{H}_n$ are given by π_λ, $\lambda \in \mathbb{R}^*$, (up to unitary equivalence). The Plancherel measure $\mathrm{d}\mu = |\lambda|^n \mathrm{d}\lambda$ is supported on $\mathbb{R}^*$. (There is another family of one-dimensional representations of $\mathscr{H}_n$ which do not play a role in the Plancherel theorem.)

Given a function f, say in $L^1(\mathscr{H}_n)$, its group Fourier transform $\hat{f}$ is defined to be the operator valued function

$$\hat{f}(\lambda) = \int_{\mathscr{H}_n} f(z,t)\pi_\lambda(z,t)\mathrm{d}z\,\mathrm{d}t. \tag{2.9}$$

(The above integral being interpreted suitably). For each $\lambda \in \mathbb{R}^*$, $\hat{f}(\lambda)$ is a bounded operator on $L^2(\mathbb{R}^n)$. A simple calculation shows that $\hat{f}(\lambda)$ is an integral operator with kernel $K_f^\lambda(\xi,\eta)$ given by

$$K_f^\lambda(\xi,\eta) = \mathscr{F}_{13} f\left(\frac{\lambda(\xi+\eta)}{2},\ \xi-\eta,\ \lambda\right), \tag{2.10}$$

where we have written $f(z,t) = f(x,y,t)$ and $\mathscr{F}_{13} f$ stands for the Fourier transform of f in the first and the third set of variables. For f in $L^1 \cap L^2(\mathscr{H}_n)$ a simple calculation shows that

$$\|\hat{f}(\lambda)\|_{HS}^2 = C|\lambda|^{-n} \int_{\mathbb{C}^n} |\mathscr{F}_3 f(z,\lambda)|^2 \mathrm{d}z, \tag{2.11}$$

(for a suitable constant C) where $\|\cdot\|_{HS}$ is the Hilbert–Schmidt norm. From this and the Euclidean–Plancherel theorem, the Plancherel theorem for the Heisenberg group follows:

$$\|f\|_2^2 = C_n \int_{\mathbb{R}^*} \|\hat{f}(\lambda)\|_{HS}^2 \mathrm{d}\mu(\lambda), \tag{2.12}$$

where $\mathrm{d}\mu(\lambda) = |\lambda|^n \mathrm{d}\lambda$ and C_n is a constant depending only on the dimension.

We now state and prove the following analogue of Hardy's theorem for $\mathscr{H}_n$.

Theorem 5. *Suppose f is a measurable function on $\mathscr{H}_n$ satisfying the estimates*

$$|f(z,t)| \leqslant g(z)e^{-\alpha t^2}, \quad z \in \mathbb{C}^n, \quad t \in \mathbb{R}, \tag{2.13}$$

$$\|\hat{f}(\lambda)\|_{HS} \leqslant Ce^{-\beta|\lambda|^2}, \quad \lambda \in \mathbb{R}^*, \tag{2.14}$$

where $g \in L^1 \cap L^2(\mathbb{C}^n)$ and α, β are positive constants. Then, if $\alpha\beta > \frac{1}{4}$, $f = 0$ a.e.; if $\alpha\beta < \frac{1}{4}$ there are infinitely many linearly independent functions satisfying the above estimates.

Proof. For a function f on $\mathscr{H}_n$ define f^* to be the function $f^*(z,t) = \overline{f(z,-t)}$ and let $f *_3 f^*$ stand for the convolution of f and f^* in the t-variable. Then, a simple calculation shows that

$$\int_{\mathscr{H}_n} (f *_3 f^*)(z,t)e^{i\lambda t}\mathrm{d}z\,\mathrm{d}t = \int_{\mathbb{C}^n} \mathscr{F}_3(f *_3 f^*)(z,\lambda)\mathrm{d}z$$

$$= \int_{\mathbb{C}^n} |\mathscr{F}_3 f(z,\lambda)|^2 \mathrm{d}z \tag{2.15}$$

which, in view of (2.11), equals $C^{-1}|\lambda|^n \|\hat{f}(\lambda)\|_{HS}^2$. Define a function h on $\mathbb{R}$ by

$$h(t) = \int_{\mathbb{C}^n} (f *_3 f^*)(z,t)\mathrm{d}z. \tag{2.16}$$

Then one has

$$\tilde{h}(\lambda) = C^{-1}|\lambda|^n \|\hat{f}(\lambda)\|_{HS}^2. \tag{2.17}$$

Now the conditions (2.13) and (2.14) on f and $\hat{f}$ translate into the conditions

$$|h(t)| \leqslant Ce^{-(\alpha/2)t^2}, \quad |\tilde{h}(\lambda)| \leqslant Ce^{-2\beta'|\lambda|^2}, \quad t\in\mathbb{R}, \quad \lambda\in\mathbb{R}^*, \tag{2.18}$$

where β' can be chosen so that $\alpha\beta' > \frac{1}{4}$ or $< \frac{1}{4}$ according as $\alpha\beta > \frac{1}{4}$ or $< \frac{1}{4}$. If $\alpha\beta > \frac{1}{4}$, then $\alpha\beta' > \frac{1}{4}$, so that Hardy's theorem for $\mathbb{R}$ implies that $h = 0$ a.e. This means $\|\hat{f}(\lambda)\|_{HS}^2 = 0$ for all $\lambda\in\mathbb{R}^*$ and consequently $f = 0$ a.e. by the Plancherel theorem for $\mathscr{H}_n$. If $\alpha\beta < \frac{1}{4}$, then any function of the form $g(z)h_k(t)$ where h_k is a suitably scaled Hermite function satisfies the hypothesis of the theorem.

The following is the exact analogue of Hardy's theorem for $\mathscr{H}_n$.

COROLLARY 6

Suppose f is a measurable L^1-function on $\mathscr{H}_n$ and

$$|f(z,t)| \leqslant Ce^{-\alpha(|z|^2+|t|^2)}, \quad z\in\mathbb{C}^n, \quad t\in\mathbb{R} \tag{2.19}$$

$$\|\hat{f}(\lambda)\|_{HS} \leqslant Ce^{-\beta|\lambda|^2}, \quad \lambda\in\mathbb{R}^* \tag{2.20}$$

for some positive constants α and β. If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e. If $\alpha\beta < \frac{1}{4}$, then there are infinitely many such linearly independent functions.

We shall now consider the case of the reduced Heisenberg group $\mathscr{H}_n^{\mathrm{red}} = \mathbb{C}^n \times S^1$. The multiplication law is as in (2.7) except for the understanding that t is a real number modulo 1. The reduced Heisenberg group $\mathscr{H}_n^{\mathrm{red}}$ is also a step two nilpotent Lie group with Haar measure $\mathrm{d}z\mathrm{d}t$ where $\mathrm{d}t$ denotes the normalized Lebesgue measure on S^1. For each $m\in Z^* = Z\backslash\{0\}$, there is an irreducible unitary representation π_m of $\mathscr{H}_n^{\mathrm{red}}$ realized on $L^2(\mathbb{R}^n)$ and is defined exactly as in (2.8). As in the case of $\mathscr{H}_n$, we get (up to unitary equivalence) that all the infinite dimensional irreducible unitary representations of $\mathscr{H}_n^{\mathrm{red}}$ are given by π_m, $m\in Z^*$. Apart from this there is a class of one dimensional representations, $\pi_{a,b}$, $a, b\in\mathbb{R}^n$ given by

$$\pi_{a,b}(z,t) = e^{2\pi i(ax+by)} \quad \text{for} \quad (z,t)\in\mathscr{H}_n^{\mathrm{red}}. \tag{2.21}$$

The dual $\hat{\mathscr{H}}_n^{\mathrm{red}}$ can be thought of as the disjoint union of Z^* and $\mathbb{R}^{2n}$. The Plancherel measure is the counting measure on Z^* with a weight function $C|m|^n$ (for a suitable constant C) and the Lebesgue measure on $\mathbb{R}^{2n}$. (This is in sharp contrast to the case of Heisenberg group.)

Given f in $L^1(\mathscr{H}_n^{\mathrm{red}})$, we can write

$$f(z,t) = \sum_{k=-\infty}^{\infty} \Psi_k(z)e^{ikt} \tag{2.22}$$

as a Fourier series in the central variable t. (Here f can be thought of as the L^1-limit of the Cesàro means of the right hand side of (2.22).) Hence, as in the case of $\mathscr{H}_n$, if we compute the group Fourier transform $\hat{f}(m)$, $m \in Z^*$ we see that it is an integral operator with kernel $K_f^m(\xi, \eta)$ given by

$$K_f^m(\xi, \eta) = \mathscr{F}_1 \Psi_{-m}\left(\frac{m(\xi+\eta)}{2}, \xi - \eta\right) \tag{2.23}$$

where $\mathscr{F}_1 \Psi_{-m}$ stands for the Fourier transform of Ψ_{-m} in the first set of variables. Therefore, for $f \in L^1 \cap L^2(\mathscr{H}_n^{\text{red}})$, a simple calculation shows that

$$\|\hat{f}(m)\|_{HS}^2 = |m|^{-n} \|\mathscr{F}_1 \Psi_{-m}\|_{L^2(\mathbb{C}^n)}^2, \quad m \in Z^*. \tag{2.24}$$

Remark 7. We will now show by an example that the exact analogue of Hardy's theorem on $\mathscr{H}_n^{\text{red}}$ is not valid. Since t varies over a compact set in this case, one might be tempted to consider the following analogue of Hardy's theorem:

Suppose f is a measurable L^1-function on $\mathscr{H}_n^{\text{red}}$ and f satisfies the following estimates:

$$|f(z,t)| \leqslant Ce^{-\alpha|z|^2}, \quad \|\hat{f}(m)\|_{HS} \leqslant Ce^{-\beta|m|^2}, \quad z \in \mathbb{C}^n, m \in Z^*, \tag{2.25}$$

for positive constants α, β. Then if $\alpha\beta > \frac{1}{4}$, $f = 0$ a.e.

However, the following demonstrates that this is not the case.

Observe that as f satisfies (2.25), f belongs to $L^1 \cap L^2(\mathscr{H}_n^{\text{red}})$ and the series in (2.22) converges to f in L^2-sense. Now take $f(z,t) = e^{-\alpha|z|^2} e^{ikt}$, for some $k \in Z^*$. Using (2.24) one can see that f is a non-trivial function satisfying the conditions (2.25).

However the following, which can be viewed as a "sort of" uncertainty principle still holds:

Suppose f is a measurable L^1-function on $\mathscr{H}_n^{\text{red}}$ satisfying

$$|f(z,t)| \leqslant \alpha(z)\beta(t), \quad z \in \mathbb{C}^n, t \in S^1 \tag{2.26}$$

$$\|\hat{f}(m)\|_{HS} \leqslant Ce^{-\gamma|m|}, \quad m \in Z^*, \tag{2.27}$$

where α is any function with reasonably rapid decay at infinity, β is any function that vanishes to infinite order at some point $t_0 \in S^1$ and γ is a positive constant. Then $f = 0$ a.e.

Remark 8. Since S^1 is compact the point t_0 can be "viewed" as the point at infinity and therefore condition (2.26) can be thought of as the analogue of the decay of the function at infinity.

3. An uncertainty principle for the Heisenberg group

In this section we formulate and prove an uncertainty principle for the Fourier transform on the Heisenberg group. In the uncertainty principle stated in theorem 2 as well as in the analogue of Hardy's theorem the Fourier transform has been considered as a function of the continuous parameter λ. The properties of the given function f as a function of the t variable are reflected in $\hat{f}(\lambda)$ as a function of λ. But if we want to

investigate how the properties of f as a function of z are affecting $\hat{f}(\lambda)$ one has to view the Fourier transform as a function of two parameters, one continuous and the other discrete.

To justify the above claim let us write down the formula for $\hat{f}(\lambda)$ when f is a radial function. In what follows, by a radial function we mean a function which is radial in the z variable. In order to state the formula we need to introduce some more notation. For each multi index $\alpha \in \mathbb{N}^n$ let $\Phi_\alpha(x)$ stand for the normalized Hermite functions on $\mathbb{R}^n$. For $\lambda \in \mathbb{R}^*$ we let $\Phi_\alpha^\lambda(x) = |\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x)$ and define $P_k(\lambda)$ to be the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha^\lambda : |\alpha| = k\}$. By $\varphi_k^\lambda(r)$ we denote the scaled Laguerre function

$$\varphi_k^\lambda(r) = L_k^{n-1}(\tfrac{1}{2}|\lambda|r^2)e^{-(1/4)|\lambda|r^2}, \tag{3.1}$$

$L_k^{n-1}(t)$ being the kth Laguerre polynomial of type $(n-1)$.

Now let $f(z,t)$ be a radial function and write $f(r,t)$ in place of $f(z,t)$ when $|z| = r$. Then we have the following formula for the Fourier transform of f:

$$\hat{f}(\lambda) = \sum_{k=0}^{\infty} R_k(\lambda, f)P_k(\lambda) \tag{3.2}$$

where the coefficients $R_k(\lambda, f)$ are given by

$$R_k(\lambda, f) = C_n \frac{k!}{(k+n-1)!} \int_0^\infty \tilde{f}(r,\lambda)\varphi_k^\lambda(r)r^{2n-1}\,dr. \tag{3.3}$$

In the above $\tilde{f}(r,\lambda)$ stands for the Fourier transform of $f(r,t)$ in the t-variable and C_n is a constant. From the above formula it follows that we can identify $\hat{f}(\lambda)$ with the sequence of functions $\{R_k(\lambda, f)\}$. The support properties of f as a function of t are reflected on the properties of $R_k(\lambda, f)$ as a function of λ. Likewise, one expects that the z support of f will influence the properties of $R_k(\lambda, f)$ as a function of k. We will show that this is indeed the case.

More generally we consider the Fourier transform $\hat{f}(\lambda)$ as a family of linear functionals $F(\lambda, \alpha)$ on $L^2(\mathbb{R}^n)$ indexed by $(\lambda, \alpha) \in \mathbb{R}^* \times \mathbb{N}^n$. For each (λ, α) the linear functional $F(\lambda, \alpha)$ is given by

$$F(\lambda, \alpha)\varphi = (\varphi, \hat{f}(\lambda)\Phi_\alpha^\lambda), \quad \varphi \in L^2(\mathbb{R}^n). \tag{3.4}$$

With the above notations the uncertainty principle stated in theorem 2 can be restated as follows. If $m\{t : f(z,t) \neq 0\} < \infty$ for a.e. z and $m\{\lambda : F(\lambda, \alpha) \neq 0\} < \infty$ then $f = 0$. Now to state our uncertainty principle let

$$A(\lambda) = \{z : \tilde{f}(z, \lambda) \neq 0\} \tag{3.5}$$

and

$$B(\lambda) = \{\alpha : F(\lambda, \alpha) \neq 0\}. \tag{3.6}$$

Then we have the following result.

Theorem 9. *Suppose $f \in L^1 \cap L^2(\mathscr{H}_n)$ is such that $m(A(\lambda)) < \infty$ and $B(\lambda)$ is finite for a.e. $\lambda \in \mathbb{R}^*$. Then $f = 0$.*

Before going into the proof of the theorem we make the following remarks concerning the statement of the theorem. If there exists a compact set $K \subset \mathbb{C}^n$ such that $f(z,t)=0$ whenever $z \notin K$ and $t \in \mathbb{R}$ then it follows that $A(\lambda)$ is compact for each λ and hence $m(A(\lambda)) < \infty$ is satisfied. The condition $B(\lambda)$ is finite simply means that $\hat{f}(\lambda)\Phi_\alpha^\lambda \neq 0$ only for finitely many α and consequently there is a $k=k(\lambda)$ such that $\hat{f}(\lambda)P_j(\lambda)=0$ for all $j>k$. Let S_k^λ be the span of $\{\Phi_\alpha^\lambda : |\alpha|=k\}$. Then it has been observed by Geller in [7] that S_k^λ are the analogues of the spheres $|x|=r$ in $\mathbb{R}^n$. In other words we can think of S_k^λ as a sphere in $L^2(\mathbb{R}^n)$ of radius $(2k+n)|\lambda|$. This view has turned out to be fruitful in other problems also as can be seen from [18].

Thus we can let B_k^λ to be the span of $\{\Phi_\alpha^\lambda : |\alpha| \leqslant k\}$ which is the analogue of a ball in $\mathbb{R}^n$ and the condition $\hat{f}(\lambda)P_j(\lambda)=0$ for $j>k$ simply means that $\hat{f}(\lambda)=0$ in the orthogonal complement of B_k^λ in $L^2(\mathbb{R}^n)$. Let us say that $\hat{f}(\lambda)$ has compact support in B_k^λ when the above holds. With this definition we can restate the above theorem in the following form.

Theorem 10. *Let $f \in L^1 \cap L^2(\mathscr{H}_n)$. Suppose for each λ the Fourier transform $\hat{f}(\lambda)$ is compactly supported. Then $\tilde{f}(.,\lambda)$ cannot have compact support for each λ unless $f=0$.*

We now come to the proof of theorem 9. We need to use some facts about the special Hermite expansions for which we refer the reader to [19]. If $f \in L^2(\mathbb{C}^n)$ then we have the expansion

$$f = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k. \tag{3.7}$$

In the above $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-(1/4)|z|^2}$ and $f \times \varphi_k$ stands for the twisted convolution

$$(f \times \varphi_k)(z) = \int_{\mathbb{C}^n} f(z-w)\varphi_k(w)e^{(i/2)im(z,\bar{w})}\mathrm{d}w. \tag{3.8}$$

The functions φ_k are eigenfunctions of the operator

$$L = -\Delta + \frac{1}{4}|z|^2 - i\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right) \tag{3.9}$$

with eigenvalues $(2k+n)$ and $f \to f \times \varphi_k$ is the projection of $L^2(\mathbb{C}^n)$ onto the k-th eigenspace of the operator L. We also have for any m

$$L^m(f \times \varphi_k) = (2k+n)^m f \times \varphi_k \tag{3.10}$$

and in view of the orthogonality the relation

$$\|L^m f\|_2^2 = (2\pi)^{-2n}\sum_{k=0}^{\infty}(2k+n)^{2m}\|f \times \varphi_k\|_2^2. \tag{3.11}$$

We need the following proposition in order to prove theorems 9 and 10.

PROPOSITION 11

Suppose $f \in L^2(\mathbb{C}^n)$ *is such that* $\|f \times \varphi_k\|_2 \leqslant Ce^{-\alpha(2k+n)}$ *for some* $\alpha > 0$. *Then* f *is real analytic.*

Proof. By the Sobolev's embedding theorem it is easy to see that f is in $C^\infty(\mathbb{C}^n)$. We want to apply an elliptic regularity theorem of Kotakè–Narasimhan to prove the proposition (see [11], theorem 3.8.9). In view of their theorem it suffices to show that for any positive integer m

$$\|L^m f\|_2 \leqslant M^{m+1}(2m)! \tag{3.12}$$

holds with some constant M. Under the assumption on f, the relation (3.11) gives

$$\|L^m f\|_2^2 \leqslant (2\pi)^{-2n} \sum_{k=0}^{\infty} (2k+n)^{2m} e^{-2\alpha(2k+n)}. \tag{3.13}$$

The series can be estimated by

$$\int_0^\infty t^{2m} e^{-2\alpha t} \mathrm{d}t \tag{3.14}$$

which gives the estimate

$$\|L^m f\|_2^2 \leqslant C^{2m+1}(2m)! \tag{3.15}$$

which is more than what we need.

Now we can give proofs of theorems 9 and 10. Define a radial function $G_j(z, t)$ by

$$G_j(z, t) = \int_{\mathbb{R}} e^{-i\lambda t} e^{-(1/2)\lambda^2} \varphi_k^\lambda(z) |\lambda|^n \mathrm{d}\lambda. \tag{3.16}$$

:n it follows from (3.2) that

$$\hat{G}_j(\lambda) = C_n e^{-(1/2)\lambda^2} P_j(\lambda), \tag{3.17}$$

vhere C_n is some constant which we do not bother to calculate. Setting $g_j = f * G_j$ and taking the (group) Fourier transform we get

$$\hat{g}_j(\lambda) = \hat{f}(\lambda)\hat{G}_j(\lambda) = C_n e^{-(1/2)\lambda^2} \hat{f}(\lambda) P_j(\lambda). \tag{3.18}$$

Now fix λ. Then under the hypothesis of the theorem we have $\hat{g}_j(\lambda) = 0$ for $j > k$ which in view of (2.11) means that for a.e. z in $\mathbb{C}^n g_j^\lambda(z) = 0$ for $j > k$ where we have set $g_j^\lambda(z)$ to stand for $\tilde{g}_j(z, \lambda)$ the Fourier transform of g_j in the t-variable.

Recalling the definition of the convolution $g_j = f * G_j$ on $\mathscr{H}_n$ and taking the Fourier transform in the t-variable we get with the same notation as above

$$g_j^\lambda(z) = f^\lambda *_\lambda G_j^\lambda(z), \tag{3.19}$$

where the λ-twisted convolution is given by

$$f^\lambda *_\lambda G_j^\lambda(z) = \int_{\mathbb{C}^n} f^\lambda(z - w) G_j^\lambda(w) e^{i(\lambda/2)\mathrm{im}(z.\bar{w})} \mathrm{d}w. \tag{3.20}$$

Let $f_\lambda^\lambda(z) = f^\lambda(2^{-1}|\lambda|^{-(1/2)}z)$. Then it follows from the definition of G_j that

$$(f^\lambda *_\lambda G_j^\lambda)(2^{-1}|\lambda|^{-(1/2)}z) = C_n e^{-(1/2)\lambda^2}(f_\lambda^\lambda \times \varphi_j)(z). \tag{3.21}$$

Under the hypothesis of either of the theorems we have $(f_\lambda^\lambda \times \varphi_j)(z) = 0$ for $j > k$. This means that f_λ^λ satisfies the conditions of proposition 11 and consequently $\tilde{f}(z, \lambda)$ is real analytic for a.e. λ as a function of (x, y). But then the set $\{z : \tilde{f}(z, \lambda) \neq 0\}$ cannot have finite measure unless $\tilde{f}(z, \lambda) = 0$ for a.e. z. This implies $f = 0$ and hence theorem 9 follows. It is clear that the hypothesis of theorem 10 implies that of theorem 9. Hence both theorems are proved.

4. Some uncertainty inequalities for the Heisenberg group

In this section we establish a local uncertainty inequality for the Fourier transform on $\mathscr{H}_n$ and deduce a global inequality too. As we have remarked in the previoussection we consider the Fourier transform $\hat{f}(\lambda)$ as a family of linear functionals $F(\lambda, \alpha)$ indexed by $(\lambda, \alpha) \in \mathbb{R}^* \times \mathbb{N}^n$. From the definition of $F(\lambda, \alpha)$ it follows that

$$\operatorname{tr}(\hat{f}(\lambda)^* \hat{f}(\lambda)) = \sum_\alpha \|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2 = \sum_\alpha \|F(\lambda, \alpha)\|^2, \tag{4.1}$$

where $\|F(\lambda, \alpha)\|$ is the norm of the linear functional $F(\lambda, \alpha)$. In this notation the uncertainty inequality of theorem 3 can be written as

$$\sum_\alpha \int_A \|F(\lambda, \alpha)\|^2 \mathrm{d}\mu(\lambda) \leqslant C_\theta m(A)^{2\theta} \int_{\mathscr{H}_n} |f(z, t)|^2 |t|^{2\theta} \mathrm{d}z\,\mathrm{d}t. \tag{4.2}$$

In the next theorem we will prove an inequality which is more symmetric in both variables.

Let ν be the counting measure on $\mathbb{N}^n$ and let $\sigma = \mu \times \nu$ on $\mathbb{R}^* \times \mathbb{N}^n$. We now prove the following inequality. We let $Q = (2n + 2)$ and $|w|^4 = |z|^4 + t^2$ for $w = (z, t) \in \mathscr{H}_n$.

Theorem 12. *Given $\theta \in [0, \frac{1}{2})$, for each $f \in L^1 \cap L^2(\mathscr{H}_n)$ and $E \subset \mathbb{R}^* \times \mathbb{N}^n$ with $\sigma(E) < \infty$ one has*

$$\int_E \|F(\lambda, \alpha)\|^2 \mathrm{d}\sigma \leqslant C_\theta^2 \sigma(E)^{2\theta} \int_{\mathscr{H}_n} |f(w)|^2 |w|^{2\theta Q} \mathrm{d}w, \tag{4.3}$$

where C_θ depends only on θ and Q.

Proof. Let $r > 0$ be a positive number to be chosen later. We write $f = g + h$ where $g(w) = f(w)$ when $|w| \leqslant r$ and $g(w) = 0$ otherwise. We then have

$$\int_E \|F(\lambda, \alpha)\|^2 \mathrm{d}\sigma \leqslant 2 \left\{ \int_E \|\hat{g}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma + \int_E \|\hat{h}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma \right\}. \tag{4.4}$$

Since

$$\|\hat{g}(\lambda)\Phi_\alpha^\lambda\|_2 \leqslant \|\hat{g}(\lambda)\|\,\|\Phi_\alpha^\lambda\|_2 = \|\hat{g}(\lambda)\|, \tag{4.5}$$

where $\|\hat{g}(\lambda)\|$ is the operator norm of $\hat{g}(\lambda)$ on $L^2(\mathbb{R}^n)$ and as $\|\hat{g}(\lambda)\| \leqslant \|g\|_1$ we obtain

$$\begin{aligned}\int_E \|\hat{g}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma &\leqslant \sigma(E)\left(\int_{\mathscr{H}_n} |g(w)|\mathrm{d}w\right)^2 \\ &\leqslant \sigma(E)\left(\int_{\mathscr{H}_n} |f(w)|^2|w|^{2\theta Q}\mathrm{d}w\right)\left(\int_{|w|\leqslant r} |w|^{-2\theta Q}\mathrm{d}w\right) \\ &\leqslant C\sigma(E)r^{-(2\theta-1)Q}\left(\int_{\mathscr{H}_n} |f(w)|^2|w|^{2\theta Q}\mathrm{d}w\right)\end{aligned} \tag{4.6}$$

where we have applied Cauchy–Schwarz to get the second inequality.

On the other hand by the Plancherel theorem

$$\begin{aligned}\int_E \|\hat{h}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma &\leqslant \int_{\mathbb{R}^*\times\mathbb{N}^n} \|\hat{h}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma \\ &= \int_{\mathbb{R}^*} \|\hat{h}(\lambda)\|_{HS}^2 \mathrm{d}\mu(\lambda) \\ &= C_n \int_{\mathscr{H}_n} |h(w)|^2 \mathrm{d}w \\ &= C_n \int_{\mathscr{H}_n} |h(w)|^2|w|^{-2\theta Q}|w|^{2\theta Q}\mathrm{d}w \\ &\leqslant C_n r^{-2\theta Q}\left(\int_{\mathscr{H}_n} |f(w)|^2|w|^{2\theta Q}\mathrm{d}w\right).\end{aligned} \tag{4.7}$$

Therefore, we have proved the inequality

$$\int_E \|F(\lambda,\alpha)\|^2 \mathrm{d}\sigma \leqslant (2C\sigma(E)r^{(1-2\theta)Q} + 2C_n r^{-2\theta Q})\left(\int_{\mathscr{H}_n} |f(w)|^2|w|^{2\theta Q}\mathrm{d}w\right). \tag{4.8}$$

Minimizing the right hand side by a judicious choice of r we get the inequality

$$\int_E \|F(\lambda,\alpha)\|^2 \mathrm{d}\sigma \leqslant C_\theta \sigma(E)^{2\theta}\left(\int_{\mathscr{H}_n} |f(w)|^2|w|^{2\theta Q}\mathrm{d}w\right). \tag{4.9}$$

This completes the proof of the theorem.

As in the case of $\mathbb{R}^n$ we can now deduce a global uncertainty inequality from the above local inequality. To state the inequality we need some more notation. Let $\mathscr{L}$ be the sublaplacian on the Heisenberg group and let $H(\lambda)$ be the Hermite operator whose spectral decomposition is given by

$$H(\lambda) = \sum_{k=0}^{\infty} (2k+n)|\lambda| P_k(\lambda). \tag{4.10}$$

For the definition of $\mathscr{L}$ we refer to [16] and we remark that when $\lambda = 1$, $H(\lambda) = -\Delta + |x|^2$ on $\mathbb{R}^n$. The relation between $\mathscr{L}$ and $H(\lambda)$ is given by

$$(\mathscr{L}f)\hat{}(\lambda) = \hat{f}(\lambda)H(\lambda), \tag{4.11}$$

for any reasonable function f on $\mathscr{H}_n$. We can define any fractional power $\mathscr{L}^\gamma$ by the equation

$$(\mathscr{L}^\gamma f)\hat{}(\lambda) = \hat{f}(\lambda)(H(\lambda))^\gamma, \tag{4.12}$$

where $(H(\lambda))^\gamma$ is given by the decomposition

$$(H(\lambda))^\gamma = \sum_{k=0}^{\infty} ((2k+n)|\lambda|)^\gamma P_k(\lambda). \tag{4.13}$$

We can now prove the following global uncertainty inequality for $\mathscr{H}_n$.

Theorem 13. *For f in $L^2(\mathscr{H}_n)$, $0 \leqslant \gamma < Q/2$ one has*

$$\|f\|_2^4 \leqslant K\left(\int_{\mathscr{H}_n} |f(w)|^2 |w|^{2\gamma} \mathrm{d}w\right)\left(\int_{\mathscr{H}_n} |\mathscr{L}^{\gamma/2} f(w)|^2 \mathrm{d}w\right) \tag{4.14}$$

where K is a constant.

Before going into the proof of the above inequality the following remarks are in order. When $\gamma = 1$ the above inequality reduces to

$$\|f\|_2^4 \leqslant K\left(\int_{\mathscr{H}_n} |f(w)|^2 |w|^2 \mathrm{d}w\right)\left(\int_{\mathscr{H}_n} |\mathscr{L}^{1/2} f(w)|^2 \mathrm{d}w\right) \tag{4.15}$$

and this is the analogue of the classical uncertainty inequality for the Fourier transform on $\mathbb{R}^n$. The analogy can be seen clearly if we write the inequality (1.2) in the form

$$\|f\|_2^4 \leqslant K\left(\int |f(x)|^2 |x|^2 \mathrm{d}x\right)\left(\int |(-\Delta)^{1/2} f(x)|^2 \mathrm{d}x\right). \tag{4.16}$$

The inequality (4.15) is valid even if we replace $|w|$ by $|z|$ as was shown in [17] and then a precise value for K can also be obtained.

Now we prove theorem 13. As in the case of the previous theorem the proof is modelled after the proof in the Euclidean case. Let E_r denote the set

$$E_r = \{(\lambda, \alpha) : (2|\alpha| + n)|\lambda| \leqslant r^2\}. \tag{4.17}$$

We claim that $\sigma(E_r) \leqslant Cr^Q$. To see this we first note that

$$E_r = \bigcup_{k=0}^{\infty} \bigcup_{|\alpha|=k} \{\lambda : (2|\alpha| + n)|\lambda| \leqslant r^2\} \times \{\alpha\} \tag{4.18}$$

and therefore

$$\sigma(E_r) \leqslant \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \mu\{\lambda : (2k+n)|\lambda| \leqslant r^2\}. \tag{4.19}$$

Since $\mu\{\lambda:(2k+n)|\lambda| \leqslant r^2\} \leqslant Cr^Q(2k+n)^{-n-1}$ and $\Sigma_{|\alpha|=k} 1 \leqslant C(2k+n)^{n-1}$ we get

$$\sigma(E_r) \leqslant Cr^Q \sum_{k=0}^{\infty} (2k+n)^{-2} \leqslant Cr^Q \tag{4.20}$$

and this proves the claim.

Let E_r' stand for the complement of E_r and write

$$\begin{aligned} \|f\|_2^2 &= C_n \int_{\mathbb{R}} \|\hat{f}(\lambda)\|_{HS}^2 \mathrm{d}\mu(\lambda) \qquad (4.21)\\ &= C_n \int \|F(\lambda,\alpha)\|^2 \mathrm{d}\sigma \\ &= C_n \left(\int_{E_r} \|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma + \int_{E_r'} \|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma \right). \end{aligned}$$

Applying the local uncertainty inequality to the first integral with $\theta = \gamma/Q < \frac{1}{2}$ and making use of the claim we obtain

$$\int_{E_r} \|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma \leqslant Cr^{2\gamma} \int_{\mathscr{H}_n} |f(w)|^2 |w|^{2\gamma} \mathrm{d}w. \tag{4.22}$$

For the second integral one has the following chain of inequalities:

$$\begin{aligned} \int_{E_r'} \|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma &\leqslant r^{-2\gamma} \int_{E_r'} ((2|\alpha|+n)|\lambda|)^\gamma \|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma \qquad (4.23)\\ &= r^{-2\gamma} \int_{E_r'} \|\hat{f}(\lambda)(H(\lambda))^{\gamma/2}\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma \\ &\leqslant r^{-2\gamma} \int \|(\mathscr{L}^{\gamma/2} f)\hat{}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\sigma \\ &= Cr^{-2\gamma} \int_{\mathscr{H}_n} |\mathscr{L}^{\gamma/2} f(w)|^2 \mathrm{d}w. \end{aligned}$$

Thus we have obtained the inequality

$$\|f\|_2^2 \leqslant C\left\{ r^{2\gamma} \int |f(w)|^2 |w|^{2\gamma} \mathrm{d}w + r^{-2\gamma} \int |\mathscr{L}^{\gamma/2} f(w)|^2 \mathrm{d}w \right\}. \tag{4.24}$$

Minimizing the right hand side we obtain

$$\|f\|_2^4 \leqslant K\left(\int |f|^2 |w|^{2\gamma} \mathrm{d}w \right)\left(\int |\mathscr{L}^{\gamma/2} f(w)|^2 \mathrm{d}w \right). \tag{4.25}$$

which proves the theorem.

5. The Euclidean motion group

In this section we shall state and prove an analogue of Hardy's theorem for the Euclidean motion group, $M(2)$. The group $G = M(2)$ is the semidirect product of

$SO(2)(\simeq S^1)$ and $\mathbb{R}^2(\simeq \mathbb{C})$. A typical element of G is denoted by (z, α) and this element acts on $\mathbb{R}^2$ as $t(z)r(\alpha)$ where $t(z)$ is the translation by $z \in \mathbb{C}(\simeq \mathbb{R}^2)$ and $r(\alpha)$ is the rotation by an angle α, $0 \leqslant \alpha \leqslant 2\pi$. The multiplication law is given by the composition of such maps. Haar measure on G is $dz\,d\alpha$ where dz is Lebesgue measure on $\mathbb{C}(\simeq \mathbb{R}^2)$ and $d\alpha$ is the normalized Haar measure on $SO(2)(\simeq S^1)$. For any unexplained terminology and notation in this section see [15].

For $a \in \mathbb{R}^+ = (0, \infty)$, we have the unitary irreducible representation U^a of G as operators in $\mathcal{U}(L^2(S^1))$ defined by

$$(U^a(z, \alpha)\phi)(\theta) = e^{i\langle z, r(\theta)a\rangle}\phi(\theta - \alpha), \tag{5.1}$$

where $\phi \in L^2(S^1)$, $0 \leqslant \theta \leqslant 2\pi$ and $\langle .,. \rangle$ is the inner product on $\mathbb{R}^2$. Here one is identifying $a \in \mathbb{R}^+$ with $(0, a) \in \mathbb{C}$. The Plancherel measure μ on $\hat{G}$ is supported on this family of representations parametrized by $\mathbb{R}^+$, and is given by $a\,da$, where da is Lebesgue measure on $\mathbb{R}^+$.

The Fourier transform $\hat{f}$ of $f \in L^1(G)$ is a function on $\mathbb{R}^+$ taking values in $\mathfrak{B}(L^2(S^1))$, and is defined by

$$\hat{f}(a) = U^a(f) = \int_{M(2)} U^a(z, \alpha) f(z, \alpha)\,dz\,d\alpha \tag{5.2}$$

(the integral interpreted suitably) and therefore we have

$$(\hat{f}(a)\phi)(\theta) = \int_{\mathbb{C}} \int_{SO(2)} f(z, \alpha) e^{i\langle z, r(\theta)a\rangle}\phi(\theta - \alpha)\,dz\,d\alpha \tag{5.3}$$

for $\phi \in L^2(S^1)$ and $\theta \in [0, 2\pi)$.

The following is an analogue of Hardy's theorem for the Euclidean motion group $M(2)$:

Theorem 14. *Suppose f is a measurable function on G satisfying the following conditions for some positive constants α, β and C:*

$$|f(z, \theta)| \leqslant Ce^{-\alpha|z|^2}, \ . \ (z, \theta) \in G, \tag{5.4}$$

$$\|\hat{f}(a)\|_{HS} \leqslant Ce^{-\beta|a|^2}, \quad a \in \mathbb{R}^+. \tag{5.5}$$

If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e.

Remark 15. Since functions on $\mathbb{R}^2$ can be thought of as functions on G invariant under right action by $SO(2)$, Hardy's theorem for $\mathbb{R}^2$ shows that $\frac{1}{4}$ is the best possible constant.

Proof. For $n \in Z$, define χ_n on $SO(2)$ as $\chi_n(\theta) = e^{in\theta}$. It is enough to show that if $\alpha\beta > \frac{1}{4}$, $\chi_n * f * \chi_m = 0$ for all n, m. This is because if f is a L^1-function (or more generally a distribution) and $\chi_n * f * \chi_m$ is zero for all n, $m \in Z$, then f is itself zero. A simple calculation shows that if f satisfies (5.4) and (5.5) then for all n, m, $\chi_n * f * \chi_m$ also satisfy (5.4) and (5.5). For $n, m \in Z$, define

$$L^1_{n,m}(G) = \{g \in L^1(G) : g(r(\theta)xr(\gamma)) = \chi_n(\theta)g(x)\chi_m(\gamma)$$
$$\text{a.e. } x \in G, \quad \text{a.e. } r(\theta), r(\gamma) \in SO(2)\}.$$

Observe that if $h = \chi_n * f * \chi_m$ then h belongs to $L^1_{n,m}(G)$. Therefore it is enough to prove the theorem for a function h in $L^1_{n,m}(G)$. It is easy to check that if $h \in L^1_{n,m}(G)$ then $\hat{h}(a)$ maps $\chi_m \in L^2(S^1)$ to a multiple of χ_n and is zero on the orthogonal complement of χ_m. In fact,

$$\hat{h}(a)\chi_m = \langle \hat{h}(a)\chi_m, \chi_n \rangle_{L^2(S^1)} \chi_n,$$

$$\hat{h}(a)\chi_l = 0 \quad \text{for } l \neq m.$$

Therefore

$$\|\hat{h}(a)\|_{HS} = |\langle \hat{h}(a)\chi_m, \chi_n \rangle_{L^2(S^1)}|.$$

Using the transformation property of h, it can be shown that

$$|\langle \hat{h}(a)\chi_m, \chi_n \rangle_{L^2(S^1)}| = |\mathcal{F}_1 h(r(\theta)a, \gamma)| \tag{5.6}$$

for a.e. θ and γ in $[0, 2\pi)$ where $\mathcal{F}_1 h$ denotes the Euclidean Fourier transform of h in the $\mathbb{C}(\simeq \mathbb{R}^2)$-variable z. Thus from (5.5) and (5.6) it will follow that:

$$|\mathcal{F}_1 h(\xi, \gamma)| \leqslant C e^{-\beta|\xi|^2} \tag{5.7}$$

for $\xi \in \mathbb{C}(\simeq \mathbb{R}^2)$ and a.e. γ in $[0, 2\pi)$. But h also satisfies (5.4). Using the analogue of Hardy's theorem for $\mathbb{R}^2(\simeq \mathbb{C})$ we conclude that $h(., \gamma) = 0$ for a.e. γ in $[0, 2\pi)$. This implies that $h = 0$ a.e.

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On subsemigroups of semisimple Lie groups

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Abstract. In this paper we classify the subsemigroups of any connected semisimple Lie group G which are K-bi-invariant, where $G = KAN$ is an Iwasawa decomposition of G.

Keywords. Lie group; semisimple; subsemigroup.

In a recent investigation of the support behaviour of certain Gauss measures on a connected semisimple Lie group (see [KM]), we encountered the following question.

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$ having a Cartan decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ (in the usual notation of Helgason [He]), and let K be the analytic subgroup corresponding to $\mathfrak{t}$. Can one classify the subsemigroups S of G such that $K \subseteq S$? Here "subsemigroup" means only a subset of G which is closed under the group multiplication. In this note we show that this problem has a very simple answer.

To describe this, we let

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$$

be the decomposition of $\mathfrak{g}$ into its simple ideals $\mathfrak{g}_j$, $1 \leqslant j \leqslant n$, and we recall that $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for all $1 \leqslant i < j \leqslant n$, and $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal w.r.t. the Killing form on $\mathfrak{g}$. If $\mathfrak{g}_j = \mathfrak{t}_j + \mathfrak{p}_j$ is a Cartan decomposition for $\mathfrak{g}_j$, then $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$, where $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2 + \cdots + \mathfrak{t}_n$ and $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2 + \cdots + \mathfrak{p}_n$.

Let N be a subset of $\{1, 2, \ldots, n\}$ and form $\mathfrak{g}_N := \mathfrak{t} + \sum_{j \in N} \mathfrak{p}_j$. It is easy to see that $\mathfrak{g}_N$ is a reductive subalgebra containing $\mathfrak{t}$, and if G_N is the corresponding analytic subgroup, then $G_N = (\prod_{j \in N} G_j)(\prod_{j \notin N} K_j)$, where G_j, K_j are the analytic subgroups determined by $\mathfrak{g}_j$ and $\mathfrak{t}_j$, respectively.

Our question raised above is now answered by the following result.

Theorem. *Let G be a connected semisimple Lie group and let S be any subsemigroup of G bi-invariant under K. Then $S = G_N$ for some subset N of $\{1, 2, \ldots, n\}$.*

In the special case when G is simple (and noncompact) this theorem tells us that K is a maximal proper subsemigroup of G. This special case therefore implies the observation of Hilgert and Hofmann that $SO(2)$ is a maximal proper subsemigroup of $SL(2, \mathbb{R})$ ([Hi H], Corollary 4.20, p. 49) and extends the theorem of Brun (see [B] or [He], Exercise A.3, p. 275) that K is a maximal proper subgroup of G, in the simple case.

PROPOSITION 1

Any subgroup of a connected semisimple Lie group G which contains K is of the form G_N, for some $N \subseteq \{1, 2, \ldots, n\}$.

Proof. (i) Let $x \in G \backslash K$ and let H_x denote the subgroup of G generated by K and x. We may write $x = x_1 x_2 \ldots x_n$, where $x_j \in G_j$ for $1 \leqslant j \leqslant n$, and set $N_x = \{1 \leqslant j \leqslant n : x_j \notin K_j\}$. Since x determines x_j up to translation by a central element, and the centre of G lies inside K, x determines N_x uniquely.

For each $j \in N_x$, $H_x \cap G_j$ contains K_j and $x_j K_j x_j^{-1}$. As G_j is simple, Brun's theorem implies that the normaliser of K_j in G_j is K_j, hence $H_x \cap G_j \neq K_j$, so by Brun's theorem again, $H_x \cap G_j = G_j$. It follows that H_x contains G_{N_x}. As G_{N_x} clearly contains K and x, we conclude that $H_x = G_{N_x}$.

(ii) Now let H be an arbitrary subgroup of G containing K and with $H \neq K$. Then

$$H = \bigcup_{x \in H} G_{N_x} = G_N,$$

where $N = \bigcup_{x \in H} N_x$. □

Given semisimple $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$, we choose a maximal abelian subspace $\mathfrak{a}_\mathfrak{p}$ of $\mathfrak{p}$ and denote by Σ the set of all roots of $\mathfrak{g}$ relative to $\mathfrak{a}_\mathfrak{p}$ (see [He], p. 263, and note that we follow the notation there except that $\mathfrak{a}_\mathfrak{p}$ replaces $\mathfrak{h}_{\mathfrak{p}_0}$ and the subscript 0 on $\mathfrak{g}$ and the subspaces of $\mathfrak{g}$ is dropped). We write $N_K(\mathfrak{a}_\mathfrak{p})$ for the normaliser of $\mathfrak{a}_\mathfrak{p}$ in K.

PROPOSITION 2

There exists $k \in N_K(\mathfrak{a}_\mathfrak{p})$, and some $m \geqslant 1$, such that for all $X \in \mathfrak{a}_\mathfrak{p}$,

$$-X = \sum_{j=1}^{m-1} \mathrm{Ad}(k^j)(X).$$

Proof. For each $\alpha \in \Sigma$, let $r_\alpha : \mathfrak{a}_\mathfrak{p} \to \mathfrak{a}_\mathfrak{p}$ denote the reflection in the hyperplane $\{Y \in \mathfrak{a}_\mathfrak{p} : \alpha(Y) = 0\}$ w.r.t. the restriction to $\mathfrak{a}_\mathfrak{p}$ of the Killing form on $\mathfrak{g}$. We choose a basis of simple roots $\{\alpha_1, \ldots, \alpha_l\}$ from Σ, and set

$$s = r_{\alpha_1} \circ r_{\alpha_2} \circ \cdots \circ r_{\alpha_l},$$

which is a Coxeter element of the Weyl group W of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$. Since $\alpha_1, \ldots, \alpha_l$ are linearly independent in $\mathfrak{a}_\mathfrak{p}^*$, we have $s(Y) = Y$ for $Y \in \mathfrak{a}_\mathfrak{p}$ if and only if $r_{\alpha_j}(Y) = Y$ for all $1 \leqslant j \leqslant l$ (c.f. [Ca], Proposition 10.5.6, p. 165). Hence $s(Y) = Y$ if and only if $\alpha_j(Y) = 0$ for all $1 \leqslant j \leqslant l$, which is equivalent to $Y = 0$ since $\alpha_1, \ldots, \alpha_l$ span $\mathfrak{a}_\mathfrak{p}^*$. Hence the linear map $I - s : \mathfrak{a}_\mathfrak{p} \to \mathfrak{a}_\mathfrak{p}$ is invertible.

Let the order of s be m, then from the identity

$$(I + s + \cdots + s^{m-1})(I - s) = I - s^m = 0$$

and the invertibility of $I - s$, it follows that on $\mathfrak{a}_\mathfrak{p}$,

$$I + s + \cdots + s^{m-1} = 0. \tag{1}$$

Because W can also be realised as $N_K(\mathfrak{a}_\mathfrak{p})/C_K(\mathfrak{a}_\mathfrak{p})$, where $C_K(\mathfrak{a}_\mathfrak{p})$ is the centraliser of $\mathfrak{a}_\mathfrak{p}$ in K, we can find $k \in N_K(\mathfrak{a}_\mathfrak{p})$ such that $s = \mathrm{Ad}\, k|_{\mathfrak{a}^\mathfrak{p}}$. Then (1) gives that for all $X \in \mathfrak{a}_\mathfrak{p}$,

$$X + \mathrm{Ad}(k)(X) + \mathrm{Ad}(k^2)(X) + \cdots + \mathrm{Ad}(k^{m-1})(X) = 0,$$

gives the result. □

OLLARY 1

exists $k \in N_K(\mathfrak{a}_p)$ *and* $m \geqslant 1$ *such that for each* $a \in A = \exp \mathfrak{a}_p$,

$$a^{-1} = (ka)^{m-1} k^{-m+1}.$$ □

OLLARY 2

y connected semisimple Lie group G, any K-bi-invariant subsemigroup of G is a oup containing K.

Let S be a K-bi-invariant subsemigroup and let $x \in S$, then $x = k_1 a k_2$ for some and $k_1, k_2 \in K$. Hence $x^{-1} = k_2^{-1} a^{-1} k_1^{-1} \in S$ by Corollary 1. Also $1 \in S$ and so . □

roof of the theorem stated earlier is now immediate by Propositions 1 and 2, lary 2.

rk. We note the following consequence of the theorem. If G is a connected mple Lie group and C is any K-bi-invariant subset of G, then there is some such that C^r is a neighbourhood of the identity in $G(C)$, the subgroup of G ated by C.

, by the theorem above,

$$G(C) = \bigcup_{s=1}^{\infty} C^s,$$

is Haar measure on $G(C)$, there exists $n \in \mathbb{N}$ such that $\lambda(C^n) > 0$. But we may $C = KDK$ for $D \subseteq A$, and by Proposition 2, Corollary 2,

$$D^{-1} \subseteq (KDK)^{m-1}$$

o

$$C^n C^{-n} \subseteq C^{nm}.$$

e result now follows because $C^n C^{-n}$ is a neighbourhood of the identity in $G(C)$, V], bottom of page 50.

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Induced representation and Frobenius reciprocity for compact quantum groups

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Abstract. Unitary representations of compact quantum groups have been described as isometric comodules. The notion of an induced representation for compact quantum groups has been introduced and an analogue of the Frobenius reciprocity theorem is established.

Keywords. Induced representation; compact quantum group; Hilbert C^*-module.

Quantum groups, like their classical counterparts, have a very rich representation theory. In the representation theory of classical groups, induced representation plays a very important role. Among other things, for example, one can obtain families of irreducible unitary representations of many locally compact groups as representations induced by one-dimensional representations of appropriate subgroups. Therefore, it is natural to try and see how far this notion can be developed and exploited in the case of quantum groups. As a first step, we do it here for compact quantum groups. First we give an alternative description of a unitary representation as an isometric comodule map. This is trivial in the finite-dimensional case, but requires a little bit of work if the comodule is infinite-dimensional. Using the comodule description, the notion of an induced representation is defined. We then go on to prove that an exact analogue of the Frobenius reciprocity theorem holds for compact quantum groups. As an application of this theorem, an alternative way of decomposing the action of $SU_q(2)$ on the Podleś sphere S^2_{q0} is given.

Notations. $\mathscr{H}, \mathscr{K}$ etc, with or without subscripts, will denote complex separable Hilbert spaces. $\mathscr{B}(\mathscr{H})$ and $\mathscr{B}_0(\mathscr{H})$ denote respectively the space of bounded operators and the space of compact operators on $\mathscr{H}$. $\mathscr{A}, \mathscr{B}, \mathscr{C}$ etc denote C^*-algebras. All the C^*-algebras used in this article have been assumed to act nondegenerately on Hilbert spaces. More specifically, given any C^*-algebra $\mathscr{A}$, it is assumed that there is a Hilbert space $\mathscr{K}$ such that $\mathscr{A} \subseteq \mathscr{B}(\mathscr{K})$ and for $u \in \mathscr{K}$, $a(u) = 0$ for all $a \in \mathscr{A}$ implies $u = 0$. Tensor product of C^*-algebras will always mean their spatial tensor product. The identity operator on Hilbert spaces is denoted by I, and on C^*-algebras by *id*. For two vector spaces X and Y, $X \otimes_{\text{alg}} Y$ denote their algebraic tensor product.

Let $\mathscr{A}$ be a C^*-algebra acting on $\mathscr{K}$. The subalgebras $\{a \in \mathscr{B}(\mathscr{K}) : ab \in \mathscr{A} \; \forall b \in \mathscr{A}\}$ and $\{a \in \mathscr{B}(\mathscr{K}) : ab, ba \in \mathscr{A} \; \forall b \in \mathscr{A}\}$ of $\mathscr{B}(\mathscr{K})$ are called respectively the *left multiplier algebra* and the *multiplier algebra* of $\mathscr{A}$. We denote them by $LM(\mathscr{A})$ and $M(\mathscr{A})$ respectively. A good reference for multiplier algebras and other topics in C^*-algebra theory is [4]. See [9] for another equivalent description of multiplier algebras that is often very useful.

1. Preliminaries

1.1 Let $\mathcal{A}$ be a unital C^*-algebra. A vector space X having a right $\mathcal{A}$-module structure is called a *Hilbert $\mathcal{A}$-module* if it is equipped with an $\mathcal{A}$-valued inner product that satisfies

(i) $\langle x, y\rangle^* = \langle y, x\rangle$,
(ii) $\langle x, x\rangle \geqslant 0$,
(iii) $\langle x, x\rangle = 0 \Rightarrow x = 0$,
(iv) $\langle x, yb\rangle = \langle x, y\rangle b$ for $x, y \in X$, $b \in \mathcal{A}$,

and if $\|x\| := \|\langle x, x\rangle\|^{1/2}$ makes X a Banach space.

Details on Hilbert C^*-modules can be found in [1], [2] and [3]. We shall need a few specific examples that are listed below.

Examples. (a) Any Hilbert space $\mathcal{H}$ with its usual inner product is a Hilbert $\mathbb{C}$-module.
(b) Any unital C^*-algebra $\mathcal{A}$ with $\langle a, b\rangle = a^*b$ is a Hilbert $\mathcal{A}$-module.
(c) $\mathcal{H} \otimes \mathcal{A}$, the 'external tensor product' of $\mathcal{H}$ and $\mathcal{A}$, is a Hilbert $\mathcal{A}$-module.
(d) $\mathcal{B}(\mathcal{H}, \mathcal{K})$, with $\langle S, T\rangle = S^*T$ is a Hilbert $\mathcal{B}(\mathcal{H})$-module.

1.2 We have seen above that $\mathcal{H} \otimes \mathcal{B}(\mathcal{K})$ and $\mathcal{B}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})$ both are Hilbert $\mathcal{B}(\mathcal{K})$-modules. It is easy to see that the map $\vartheta: \Sigma u_i \otimes a_i \mapsto \Sigma u_i \otimes a_i(\cdot)$ from $\mathcal{H} \otimes_{alg} \mathcal{B}(\mathcal{K})$ to $\mathcal{B}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})$ extends to an isometric module map from $\mathcal{H} \otimes \mathcal{B}(\mathcal{K})$ to $\mathcal{B}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})$, i.e. ϑ obeys

$$\langle \vartheta(x), \vartheta(y)\rangle = \langle x, y\rangle, \quad \forall x, y \in \mathcal{H} \otimes \mathcal{B}(\mathcal{K}),$$
$$\vartheta(xb) = \vartheta(x)b, \quad \forall x \in \mathcal{H} \otimes \mathcal{B}(\mathcal{K}), b \in \mathcal{B}(\mathcal{K}).$$

Thus ϑ embeds $\mathcal{H} \otimes \mathcal{B}(\mathcal{K})$ in $\mathcal{B}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})$. Observe two things here: first, if $\mathcal{H} = \mathbb{C}$, ϑ is just the identity map. And, ϑ is *onto* if and only if $\mathcal{H}$ is finite-dimensional. The following lemma, the proof of which is fairly straightforward, gives a very useful property of ϑ.

Lemma. Let ϑ_i be the map ϑ constructed above with $\mathcal{H}_i$ replacing $\mathcal{H}$, $i = 1, 2$. Let $S \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $x \in \mathcal{H}_1 \otimes \mathcal{B}(\mathcal{K})$. Then $\vartheta_2((S \otimes id)x) = (S \otimes I)\vartheta_1(x)$.

1.3 For an operator $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, and a vector $u \in \mathcal{H}$, let T_u denote the operator $v \mapsto T(u \otimes v)$ from $\mathcal{K}$ to $\mathcal{H} \otimes \mathcal{K}$. It is not too difficult to show that $T_u \in \vartheta(\mathcal{H} \otimes \mathcal{B}(\mathcal{K}))$ if $T \in LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))$. Define a map $\Psi(T)$ from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{B}(\mathcal{K})$ by: $\Psi(T)(u) = \vartheta^{-1}(T_u)$. Then Ψ is the unique linear injective contraction from $LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))$ to $\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathcal{B}(\mathcal{K}))$ for which $\vartheta(\Psi(T)(u))(v) = T(u \otimes v)$ $\forall u \in \mathcal{H}, v \in \mathcal{K}$, $T \in LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))$. Here are a few interesting properties of this map Ψ.

PROPOSITION

Let $\Psi: LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})) \to \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathcal{B}(\mathcal{K}))$ be the map described above. Then we have the following:

(i) *Ψ maps isometries in $LM(\mathcal{B}_0(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))$ onto the isometries in $\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathcal{B}(\mathcal{K}))$.*

(ii) *For any* $T \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{B}(\mathscr{K}))$ *and* $S \in \mathscr{B}_0(\mathscr{H})$,

$$\Psi(T(S \otimes I)) = \Psi(T) \circ S,\ \Psi((S \otimes I)T) = (S \otimes id) \circ \Psi(T).$$

(iii) *If* $\mathscr{A}$ *is any C*-subalgebra of* $\mathscr{B}(\mathscr{K})$ *containing its identity, then* $T \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A})$ *if and only if range* $\Psi(T) \subseteq \mathscr{H} \otimes \mathscr{A}$.

Proof. (i) Suppose $T \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{B}(\mathscr{K}))$ is an isometry. By 1.2, $\langle \Psi(T)u, \Psi(T)v \rangle = \langle \vartheta^{-1}(T_u), \vartheta^{-1}(T_v) \rangle = \langle T_u, T_v \rangle = \langle u, v \rangle I$ for $u, v \in \mathscr{H}$. Thus $\Psi(T)$ is an isometry.

Conversely, take an isometry $\pi: \mathscr{H} \to \mathscr{H} \otimes \mathscr{B}(\mathscr{K})$ and define an operator T on the product vectors in $\mathscr{H} \otimes \mathscr{K}$ by $T(u \otimes v) = \vartheta(\pi(u))(v)$, ϑ being the map constructed in 1.2. It is clear that T is an isometry. It is enough, therefore, to show that $T(|u\rangle\langle v| \otimes S) \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{B}(\mathscr{K})$ whenever $S \in \mathscr{B}(\mathscr{K})$ and u, v are unit vectors in $\mathscr{H}$ such that $\langle u, v \rangle = 0$ or 1.

Choose an orthonormal basis $\{e_i\}$ for $\mathscr{H}$ such that $e_1 = u, e_r = v$ where

$$r = \begin{cases} 0 & \text{if } \langle u, v \rangle = 0, \\ 1 & \text{if } \langle u, v \rangle = 1. \end{cases}$$

Let $\pi_{ij} = (\langle e_i| \otimes id)\pi(e_j)$. Then $T(|u\rangle\langle v| \otimes S) = \Sigma |e_i\rangle\langle e_r| \otimes \pi_{i1} S$ where the right-hand side converges strongly. Since $\pi(e_1) \in \mathscr{H} \otimes \mathscr{B}(\mathscr{K})$, it follows that $\Sigma_i \pi_{i1}{}^* \pi_{i1}$ converges in norm. Consequently the right-hand side above converges in norm, which means $T(|u\rangle\langle v| \otimes S) \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{B}(\mathscr{K})$.

(ii) Straightforward.

(iii) Take $T = |u\rangle\langle v| \otimes a, u, v \in \mathscr{H}, a \in \mathscr{A}$. For any $w \in \mathscr{H}$, $\Psi(T)(w) = \langle v, w \rangle u \otimes a \in \mathscr{H} \otimes \mathscr{A}$. Since Ψ is a contraction, and the norm closure of all linear combinations of such T's is $\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}$, we have range $\Psi(T) \subseteq \mathscr{H} \otimes \mathscr{A}$ for all $T \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}$.

Assume next that $T \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A})$. Then $T(|u\rangle\langle u| \otimes I) \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}$ for all $u \in \mathscr{H}$. Hence $\Psi(T(|u\rangle\langle u| \otimes I))(u) \in \mathscr{H} \otimes \mathscr{A}$, which means, by part (ii), that $\Psi(T)(u) \in \mathscr{H} \otimes \mathscr{A}$ for all $u \in \mathscr{H}$. Thus range $\Psi(T) \subseteq \mathscr{H} \otimes \mathscr{A}$.

To prove the converse, it is enough to show that $T(|u\rangle\langle v| \otimes a) \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}$ whenever $a \in \mathscr{A}$ and $u, v \in \mathscr{H}$ are such that $\langle u, v \rangle = 0$ or 1. Rest of the proof goes along the same lines as the proof of the last part of (i). ∎

1.4 Let $\mathscr{K}_1, \mathscr{K}_2$ be two Hilbert spaces, $\mathscr{A}_i$ being a C^*-subalgebra of $\mathscr{B}(\mathscr{K}_i)$ containing its identity. Suppose ϕ is a unital $*$-homomorphism from $\mathscr{A}_1$ to $\mathscr{A}_2$. Then $id \otimes \phi$: $S \otimes a \mapsto S \otimes \phi(a)$ extends to a $*$-homomorphism from $\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_1$ to $\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_2$. Moreover $\{((id \otimes \phi)(a))b : a \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_1, b \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_2\}$ is total in $\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_2$. Therefore $id \otimes \phi$ extends to an algebra homomorphism by the following prescription: for all $a \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_1)$, $b \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_1$, $c \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_2$,

$$((id \otimes \phi)a)(((id \otimes \phi)b)c) := ((id \otimes \phi)(ab))c.$$

PROPOSITION

Let ϕ *be as above, and* Ψ_i *be the map* Ψ *constructed earlier with* $\mathscr{K}_i$ *replacing* $\mathscr{K}$. *Then for* $T \in LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A}_1)$,

$$(I \otimes \phi)\Psi_1(T) = \Psi_2((id \otimes \phi)T).$$

Proof. It is enough to prove that

$$(\langle u|\otimes id)((I\otimes\phi)\Psi_1(T)(v))=(\langle u|\otimes id)\Psi_2((id\otimes\phi)T)(v),\forall u,v\in\mathscr{H}.$$

Rest now is a careful application of 1.2. ∎

1.5 Consider the homomorphic embeddings $\phi_{12}:\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1\to\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1\otimes\mathscr{A}_2$ and $\phi_{13}:\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_2\to\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1\otimes\mathscr{A}_2$ given on the product elements by

$$\phi_{12}(a\otimes b)=a\otimes b\otimes I,\ \phi_{13}(a\otimes c)=a\otimes I\otimes c,$$

respectively. Each of their ranges contains an approximate identity for $\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1\otimes\mathscr{A}_2$, so that their extensions respectively to $LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1)$ and $LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_2)$ are also homomorphic embeddings.

PROPOSITION

Let Ψ_1,Ψ_2 be as in the previous proposition, and let Ψ_0 be the map Ψ with $\mathscr{A}_1\otimes\mathscr{A}_2$ replacing $\mathscr{A}$. Let $S\in LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_1)$, $T\in LM(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}_2)$. Then

$$\Psi_0(\phi_{12}(S)\phi_{13}(T))=(\Psi_1(S)\otimes id)\Psi_2(T).$$

Proof. Observe that for $u_1,\ldots,u_n\in\mathscr{H}$, $(\langle\Psi_1(S)(u_i),\Psi_1(S)(u_j)\rangle)\leqslant\|S\|^2(\langle u_i,u_j\rangle I)$. Therefore $\Psi_1(S)\otimes id$ is a well-defined bounded operator from $\mathscr{H}\otimes\mathscr{A}_2$ to $\mathscr{H}\otimes\mathscr{A}_1\otimes\mathscr{A}_2$. Take an orthonormal basis $\{e_i\}$ for $\mathscr{H}$. Define S_{ij}'s and T_{ij}'s as follows:

$$S_{ij}:v\mapsto(\langle e_i|\otimes I)S(e_j\otimes v),\ T_{ij}:v\mapsto(\langle e_i|\otimes I)\,T(e_j\otimes v).$$

Let $P_n:=\Sigma_{i=1}^n|e_i\rangle\langle e_i|$. Then $(\Psi_1(S)\otimes id)(P_n\otimes id)\Psi_2(T)(e_i)=(\Psi_1(S)\otimes id)(\Sigma_{j\leqslant n}e_j\otimes T_{ij})=\Sigma_{j\leqslant n}(\Sigma_k e_k\otimes S_{kj})\otimes T_{ij}$. Hence for $v\in\mathscr{K}_1,w\in\mathscr{K}_2$,

$$\begin{aligned}&\vartheta((\Psi_1(S)\otimes id)(P_n\otimes id)\Psi_2(T)(e_i))(v\otimes w)\\&=\sum_{j\leqslant n}\sum_k e_k\otimes S_{kj}(v)\otimes T_{ji}(w)\\&=\Big(\sum_{j\leqslant n}\sum_{k,r}|e_k\rangle\langle e_r|\otimes S_{kj}\otimes T_{ji}\Big)(e_i\otimes v\otimes w)\\&=\phi_{12}(S)(P_n\otimes I\otimes I)\phi_{13}(T)(e_i\otimes v\otimes w).\end{aligned}$$

This converges to $\phi_{12}(S)\phi_{13}(T)(e_i\otimes v\otimes w)$ as $n\to\infty$. On the other hand,

$$\lim_{n\to\infty}(\Psi_1(S)\otimes id)(P_n\otimes id)\Psi_2(T)(e_i)=(\Psi_1(S)\otimes id)\Psi_2(T)(e_i),$$

which implies $\lim_{n\to\infty}\vartheta((\Psi_1(S)\otimes id)(P_n\otimes id)\Psi_2(T)(e_i))=\vartheta((\Psi_1(S)\otimes id)\Psi_2(T)(e_i))$. Therefore $\vartheta((\Psi_1(S)\otimes id)\Psi_2(T)(e_i))(v\otimes w)=\phi_{12}(S)\phi_{13}(T)(e_i\otimes v\otimes w)=\vartheta(\Psi_0(\phi_{12}(S)\phi_{13}(T))(e_i))(v\otimes w)$. Thus $(\Psi_1(S)\otimes id)\Psi_2(T)=\Psi_0(\phi_{12}(S)\phi_{13}(T))$. ∎

2. Representations of compact quantum groups

2.1 We start by recalling a few facts from [6] on compact quantum groups.

DEFINITION

Let $\mathscr{A}$ be a separable unital C^*-algebra, and $\mu:\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$ be a unital $*$-homomorphism. We call $G=(\mathscr{A},\mu)$ a compact quantum group if the following two conditions are satisfied:

(i) $(id\otimes\mu)\mu=(\mu\otimes id)\mu$, and
(ii) $\{(a\otimes I)\mu(b):a,b\in\mathscr{A}\}$ and $\{(I\otimes a)\mu(b):a,b\in\mathscr{A}\}$ both are total in $\mathscr{A}\otimes\mathscr{A}$.

μ is called the *comultiplication map* associated with G. We shall very often denote the underlying C^*-algebra $\mathscr{A}$ by $C(G)$ and the map μ by μ_G.

A representation of a compact quantum group G acting on a Hilbert space $\mathscr{H}$ is an element π of the multiplier algebra $M(\mathscr{B}_0(\mathscr{H})\otimes C(G))$ that obeys $\pi_{12}\pi_{13}=(id\otimes\mu)\pi$, where π_{12} and π_{13} are the images of π in the space $M(\mathscr{B}_0(\mathscr{H})\otimes C(G)\otimes C(G))$ under the homomorphisms ϕ_{12} and ϕ_{13} which are given on the product elements by:

$$\phi_{12}(a\otimes b)=a\otimes b\otimes I,\ \phi_{13}(a\otimes b)=a\otimes I\otimes b.$$

A representation π is called a *unitary representation* if $\pi\pi^*=I=\pi^*\pi$. One also has the notions of irreducibility, direct sum and tensor product of representations. As in the case of classical groups, any unitary representation decomposes into a direct sum of finite-dimensional irreducible unitary representations. Let $A(G)$ be the unital $*$-subalgebra of $C(G)$ generated by the matrix entries of finite-dimensional unitary representations of G. Then one has the following result (see [8]).

Theorem. ([8]) *Suppose G is a compact quantum group. Let $A(G)$ be as above. Then we have the following:*

(a) *$A(G)$ is a dense unital $*$-subalgebra of $C(G)$ and $\mu(A(G))\subseteq A(G)\otimes_{alg}A(G)$.*
(b) *There is a complex homomorphism $\varepsilon:A(G)\to\mathbb{C}$ such that*

$$(\varepsilon\otimes id)\mu=id=(id\otimes\varepsilon)\mu.$$

(c) *There exists a linear antimultiplicative map $\kappa:A(G)\to A(G)$ obeying*

$$m(id\otimes\kappa)\mu(a)=\varepsilon(a)I=m(\kappa\otimes id)\mu(a),\ \text{and}\ \kappa(\kappa(a^*)^*)=a$$

for all $a\in A(G)$, where m is the operator that sends $a\otimes b$ to ab.

The maps ε and κ in the above theorem are called the *counit* and *coinverse* respectively of the quantum group G.

2.2 Let $G=(C(G),\mu_G)$ and $H=(C(H),\mu_H)$ be two compact quantum groups. A C^*-homomorphism ϕ from $C(G)$ to $C(H)$ is called a quantum group homomorphism from G to H if it obeys $(\phi\otimes\phi)\mu_G=\mu_H\phi$.

One can show that if G, H are compact quantum groups, then H is a subgroup of G if and only if there is a homomorphism from G to H that maps $C(G)$ *onto* $C(H)$.

2.3 Let $G = (\mathscr{A}, \mu)$ be a compact quantum group. From now onward we shall assume that $\mathscr{A}$ acts nondegenerately on a Hilbert space $\mathscr{K}$, i.e. $\mathscr{A}$ is a C^*-subalgebra of $\mathscr{B}(\mathscr{K})$ containing its identity. We call a map π from $\mathscr{H}$ to $\mathscr{H} \otimes \mathscr{A}$ an *isometry* if $\langle \pi(u), \pi(v) \rangle = \langle u, v \rangle I$ for all $u, v \in \mathscr{H}$. If $\pi: \mathscr{H} \to \mathscr{H} \otimes \mathscr{A}$ is an isometry, then $\pi \otimes id$: $u \otimes a \mapsto \pi(u) \otimes a$ extends to a bounded map from $\mathscr{H} \otimes \mathscr{A}$ to $\mathscr{H} \otimes \mathscr{A} \otimes \mathscr{A}$. π is called an *isometric comodule map* if it is an isometry, and satisfies $(\pi \otimes id)\pi = (I \otimes \mu)\pi$. The pair $(\mathscr{H}, \pi)$ is called an *isometric comodule*. We shall often just say π is a comodule, omitting the $\mathscr{H}$.

The following theorem says that for a compact quantum group isometric comodules are nothing but the unitary representations.

Theorem. *Let π be an isometric comodule map acting on $\mathscr{H}$. Then $\Psi^{-1}(\pi)$ is a unitary representation acting on $\mathscr{H}$. Conversely, if $\hat{\pi}$ is a unitary representation of G on $\mathscr{H}$, then $(\mathscr{H}, \Psi(\hat{\pi}))$ is an isometric comodule.*

We need the following lemma for proving the theorem.

Lemma. Let $(\mathscr{H}, \pi)$ be an isometric comodule. Then $\mathscr{H}$ decomposes into a direct sum of finite dimensional subspaces $\mathscr{H} = \oplus \mathscr{H}_\alpha$ such that each $\mathscr{H}_\alpha$ is π-invariant and $\pi|_{\mathscr{H}_\alpha}$ is an irreducible isometric comodule.

Proof. By 1.3, there is an isometry $\hat{\pi}$ in $LM(\mathscr{B}_0(\mathscr{H}) \otimes \mathscr{A})$ such that $\Psi(\hat{\pi}) = \pi$. Using 1.4 and 1.5, we get $\hat{\pi}_{12}\hat{\pi}_{13} = (id \otimes \mu)\hat{\pi}$ where $\hat{\pi}_{12} = \phi_{12}(\hat{\pi})$, $\hat{\pi}_{13} = \phi_{13}(\hat{\pi})$, ϕ_{12} and ϕ_{13} being as in 1.5 with $\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}$.

Let $\mathscr{I} = \{a \in \mathscr{A} : h(a^*a) = 0\}$. From the properties of the haar state, $\mathscr{I}$ is an ideal in $\mathscr{A}$. For any unit vector u in $\mathscr{H}$, let $Q(u) = (id \otimes h)(\hat{\pi}(|u\rangle\langle u| \otimes I)\hat{\pi}^*)$. Then $Q(u)^* = Q(u) \in \mathscr{B}_0(\mathscr{H})$. If $Q(u) = 0$, then $|\hat{\pi}(|u\rangle\langle u| \otimes I)\hat{\pi}^*|^{1/2} \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{I}$. Therefore $\hat{\pi}(|u\rangle\langle u| \otimes I)\hat{\pi}^* \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{I}$. It follows then that $|u\rangle\langle u| \otimes I \in \mathscr{B}_0(\mathscr{H}) \otimes \mathscr{I}$. This forces u to be zero. Thus for a nonzero u, $Q(u) \neq 0$. Choose and fix any nonzero u. Then

$$
\begin{aligned}
&\hat{\pi}(Q(u) \otimes I)\hat{\pi}^* \\
&= (id \otimes id \otimes h)(\hat{\pi}_{12}\hat{\pi}_{13}(|u\rangle\langle u| \otimes I \otimes I)\hat{\pi}_{13}^*\hat{\pi}_{12}^*) \\
&= (id \otimes id \otimes h)(\hat{\pi}_{12}\hat{\pi}_{13}(|u\rangle\langle u| \otimes I \otimes I)(\hat{\pi}_{12}\hat{\pi}_{13}(|u\rangle\langle u| \otimes I \otimes I))^*) \\
&= (id \otimes id \otimes h)((id \otimes \mu)(\hat{\pi})(id \otimes \mu)(|u\rangle\langle u| \otimes I) \\
&\quad \times ((id \otimes \mu)(\hat{\pi})(id \otimes \mu)(|u\rangle\langle u| \otimes I))^*) \\
&= (id \otimes id \otimes h)((id \otimes \mu)(\hat{\pi}(|u\rangle\langle u| \otimes I))((id \otimes \mu)(\hat{\pi}(|u\rangle\langle u| \otimes I)))^*) \\
&= (id \otimes id \otimes h)((id \otimes \mu)(\hat{\pi}(|u\rangle\langle u| \otimes I))(id \otimes \mu)((|u\rangle\langle u| \otimes I)\hat{\pi}^*)) \\
&= (id \otimes id \otimes h)(id \otimes \mu)(\hat{\pi}(|u\rangle\langle u| \otimes I)\hat{\pi}^*) \\
&= (id \otimes (id \otimes h)\mu)(\hat{\pi}(|u\rangle\langle u| \otimes I)\hat{\pi}^*) \\
&= Q(u) \otimes I.
\end{aligned}
$$

Thus $\hat{\pi}(Q(u) \otimes I) = (Q(u) \otimes I)\hat{\pi}$. If P is any finite-dimensional spectral projection of $Q(u)$, then $\hat{\pi}(P \otimes I) = (P \otimes I)\hat{\pi}$, which means, by an application of part (ii) of 1.3, that $\pi P = (P \otimes id)\pi$. Standard arguments now tell us that π can be decomposed into

ect sum of finite-dimensional isometric comodules. Finite-dimensional comodules, rn, can easily be shown to decompose into a direct sum of irreducible isometric odules. The proof is thus complete. ■

oof of the theorem: Let $\hat{\pi}$ be a unitary representation. By 1.3, $\Psi(\hat{\pi})$ is an isometry $\mathscr{H}$ to $\mathscr{H}\otimes C(G)$. Using 1.4 and 1.5, we conclude that $\Psi(\hat{\pi})$ is an isometric comodule. or the converse, take an isometric comodule π. If π is finite-dimensional, it is easy ee that $\Psi^{-1}(\pi)$ is a unitary representation. So, assume that π is infinite-ensional. By the lemma above, there is a family $\{P_\alpha\}$ of finite-dimensional ections in $\mathscr{B}(\mathscr{H})$ satisfying

$$P_\alpha P_\beta = \delta_{\alpha\beta}P_\alpha, \sum P_\alpha = I, \pi P_\alpha = (P_\alpha\otimes id)\pi \ \forall\alpha \tag{2.1}$$

that $\pi|_{P_\alpha\mathscr{H}} = \pi P_\alpha$ is an irreducible isometric comodule. $\pi|_{P_\alpha\mathscr{H}}$ is finite-dimensional, fore $\Psi^{-1}(\pi|_{P_\alpha\mathscr{H}})$ is a unitary element of $LM(\mathscr{B}_0(P_\alpha\mathscr{H})\otimes\mathscr{A}) = \mathscr{B}(P_\alpha\mathscr{H})\otimes\mathscr{A}$. Let us te $\Psi^{-1}(\pi)$ by $\hat{\pi}$. Then the above implies that in the bigger space $\mathscr{B}(\mathscr{H}\otimes\mathscr{K})$,

$$(\hat{\pi}(P_\alpha\otimes I))^*(\hat{\pi}(P_\alpha\otimes I)) = P_\alpha\otimes I = (\hat{\pi}(P_\alpha\otimes I))(\hat{\pi}(P_\alpha\otimes I))^*.$$

second equality implies that $\hat{\pi}(P_\alpha\otimes I)\hat{\pi}^* = P_\alpha\otimes I$ for all α, so that $\hat{\pi}\hat{\pi}^* = I$. We dy know by 1.3 that $\hat{\pi}^*\hat{\pi} = I$ and by 1.4 and 1.5 that $\hat{\pi}_{12}\hat{\pi}_{13} = (id\otimes\mu)\hat{\pi}$. Thus it ins only to show that $\hat{\pi}\in M(\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A})$. It is enough to show that for any $_0(\mathscr{H})$ and $a\in\mathscr{A}$, $(S\otimes a)\hat{\pi}\in\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}$. Now from (2.1) and 1.3, $\hat{\pi}(P_\alpha\otimes I) =$ $\otimes I)\hat{\pi}$ for all α. Therefore $(S\otimes a)(P_\alpha\otimes I)\hat{\pi} = (S\otimes a)\hat{\pi}(P_\alpha\otimes I)\in\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}$. Since $a)\hat{\pi}$ is the norm limit of finite sums of such terms, $(S\otimes a)\hat{\pi}\in\mathscr{B}_0(\mathscr{H})\otimes\mathscr{A}$. Thus $\hat{\pi}$ is itary representation acting on $\mathscr{H}$. ■

Next we introduce the right regular comodule. Denote by $L_2(G)$ the GNS space ciated with the haar state h on G. Then $\mathscr{A}$ is a dense subspace of $L_2(G)$. One can see that $\mathscr{A}\otimes\mathscr{A}$ can be regarded as a subspace of $L_2(G)\otimes\mathscr{A}$. Consider the $\mu:\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$.

$$\langle\mu(a),\mu(b)\rangle = (h\otimes id)(\mu(a)^*\mu(b)) = (h\otimes id)\mu(a^*b) = h(a^*b)I = \langle a,b\rangle I$$

ll $a,b\in\mathscr{A}$. Therefore μ extends to an isometry from $L_2(G)$ into $L_2(G)\otimes\mathscr{A}$. Denote y $\mathfrak{R}$. The maps $(I\otimes\mu)\mathfrak{R}$ and $(\mathfrak{R}\otimes id)\mathfrak{R}$ both are isometries from $L_2(G)$ to $G)\otimes\mathscr{A}\otimes\mathscr{A}$ and they coincide on $\mathscr{A}$. Hence $(I\otimes\mu)\mathfrak{R} = (\mathfrak{R}\otimes id)\mathfrak{R}$. Thus $\mathfrak{R}$ is an ietric comodule map. We call it the *right-regular comodule* of G. By theorem 2.3, $(\mathfrak{R})$ is a unitary representation acting on $L_2(G)$. This is the right-regular represen-on introduced by Woronowicz ([8]).

inally let us state here a small lemma which is a direct consequence of the r-Weyl theorem for compact quantum groups.

Lemma. $\{u\in L_2(G):\mathfrak{R}(u)\in L_2(G)\otimes_{alg}C(G)\} = A(G)$.

nduced representations

his section we shall introduce the concept of an induced representation and show Frobenius reciprocity theorem holds for compact quantum groups. Throughout

this section $G = (C(G), \mu_G)$ will denote a compact quantum group and $H = (C(H), \mu_H)$, a subgroup of G. We start with a lemma concerning the boundedness of the left convolution operator.

3.1 *Lemma. Let $G = (\mathscr{A}, \mu)$ be a compact quantum group. Then the map $L_\rho : \mathscr{A} \to \mathscr{A}$ given by $L_\rho(a) = (\rho \otimes id)\mu(a)$ extends to a bounded operator from $L_2(G)$ into itself.*

Proof. The proof follows from the following inequality: for any two states ρ_1 and ρ_2 on $\mathscr{A}$, we have

$$\rho_1((\rho_2 * a)^*(\rho_2 * a)) \leqslant \rho_2 * \rho_1(a^* a)\ \forall a \in \mathscr{A},$$

where $\rho_i * a := (\rho_i \otimes id)\mu(a)$. ∎

3.2 Let $\hat{\pi}$ be a unitary representation of H acting on the space $\mathscr{H}_0$. $\pi := \Psi(\hat{\pi})$ is then an isometric comodule map from $\mathscr{H}_0$ to $\mathscr{H}_0 \otimes C(H)$. Consider the following map from $\mathscr{H}_0 \otimes L_2(G)$ to $\mathscr{H}_0 \otimes L_2(G) \otimes C(G)$:

$$I \otimes \mathfrak{R}^G : u \otimes v \mapsto u \otimes \mathfrak{R}^G(v)$$

where $\mathfrak{R}^G$ is the right-regular comodule of G. It is easy to see that this is an isometric comodule map acting on $\mathscr{H}_0 \otimes L_2(G)$.

Let p be the homomorphism from G to H (cf. 2.2). Let $\mathscr{H} = \{u \in \mathscr{H}_0 \otimes L_2(G) : (I \otimes L_{\rho \cdot p})u = (id \otimes \rho)\pi \otimes I)u$ for all continuous linear functionals ρ on $C(H)\}$. Then $I \otimes \mathfrak{R}^G$ keeps $\mathscr{H}$ invariant; the restriction of $I \otimes \mathfrak{R}^G$ to $\mathscr{H}$ is therefore an isometric comodule, so that $\Psi^{-1}((I \otimes \mathfrak{R}^G)|_{\mathscr{H}})$ is a unitary representation of G acting on $\mathscr{H}$. We call this *the representation induced by* $\hat{\pi}$, and denote it by $\mathrm{ind}_H^G \hat{\pi}$ or simply by $\mathrm{ind}\,\hat{\pi}$ when there is no ambiguity about G and H.

Let $\hat{\pi}_1$ and $\hat{\pi}_2$ be two unitary representations of H. Then clearly we have

(i) $\mathrm{ind}\,\hat{\pi}_1$ and $\mathrm{ind}\,\hat{\pi}_2$ are equivalent whenever $\hat{\pi}_1$ and $\hat{\pi}_2$ are equivalent, and
(ii) $\mathrm{ind}(\hat{\pi}_1 \oplus \hat{\pi}_2)$ and $\mathrm{ind}\,\hat{\pi}_1 \oplus \mathrm{ind}\,\hat{\pi}_2$ are equivalent.

Before going to the Frobenius reciprocity theorem, let us briefly describe what we mean by restriction of a representation to a subgroup. Let $\hat{\pi}^G$ be a unitary representation of G acting on a Hilbert space $\mathscr{H}_0$. We call $(id \otimes p)\hat{\pi}^G$ the restriction of $\hat{\pi}^G$ to H and denote it by $\hat{\pi}^{G|H}$. To see that it is indeed a unitary representation, observe that $\Psi((id \otimes p)\hat{\pi}^G) = (I \otimes p)\Psi(\hat{\pi}^G)$ which is clearly an isometric comodule. Therefore by 2.3, $\hat{\pi}^{G|H}$ is a unitary representation of H acting on $\mathscr{H}_0$. Denote $\Psi(\hat{\pi}^G)$ by π^G and $\Psi(\hat{\pi}^{G|H})$ by $\pi^{G|H}$.

3.3 **Theorem.** *Let $\hat{\pi}^G$ and $\hat{\pi}^H$ be irreducible unitary representations of G and H respectively. Then the multiplicity of $\hat{\pi}^G$ in $\mathrm{ind}_H^G \hat{\pi}^H$ is the same as that of $\hat{\pi}^H$ in $\hat{\pi}^{G|H}$.*

Proof. Let $\mathscr{I}(\hat{\pi}^{G|H}, \hat{\pi}^H)$ (respectively $\mathscr{I}(\hat{\pi}^G, \mathrm{ind}\,\hat{\pi}^H)$) denote the space of intertwiners between $\hat{\pi}^{G|H}$ and $\hat{\pi}^H$ (respectively $\hat{\pi}^G$ and $\mathrm{ind}\,\hat{\pi}^H$). Assume that $\hat{\pi}^G$ and $\hat{\pi}^H$ act on $\mathscr{K}_0$ and $\mathscr{H}_0$ respectively. $\mathscr{K}_0 \otimes C(G)$ can be regarded as a subspace of $\mathscr{K}_0 \otimes L_2(G)$ and hence π^G, as a map from $\mathscr{K}_0$ into $\mathscr{K}_0 \otimes L_2(G)$. Since $\pi^G = \Psi(\hat{\pi}^G)$ is unitary, we have for $u, v \in \mathscr{K}_0$,

$$\langle \pi^G(u), \pi^G(v) \rangle_{\mathscr{K}_0 \otimes L_2(G)} = h(\langle \pi^G(u), \pi^G(v) \rangle_{\mathscr{K}_0 \otimes C(G)}) = h(\langle u, v \rangle I) = \langle u, v \rangle.$$

Thus $\pi^G: \mathscr{H}_0 \to \mathscr{H}_0 \otimes L_2(G)$ is an isometry. Let $S: \mathscr{H}_0 \to \mathscr{H}_0$ be an element of $\mathscr{I}(\hat{\pi}^{G|H}, \hat{\pi}^H)$. $(S \otimes I)\pi^G$ is then a bounded map from $\mathscr{H}_0$ into $\mathscr{H}_0 \otimes L_2(G)$. Denote it by $f(S)$. It is not too difficult to see that $f(S)$ actually maps $\mathscr{H}_0$ into $\mathscr{H}$, and intertwines $\hat{\pi}^G$ and $\operatorname{ind} \hat{\pi}^H$. $f: S \mapsto f(S)$ is thus a linear map from $\mathscr{I}(\hat{\pi}^{G|H}, \hat{\pi}^H)$ to $\mathscr{I}(\hat{\pi}^G, \operatorname{ind} \hat{\pi}^H)$.

We shall now show that f is invertible by exhibiting the inverse of f. Take a $T: \mathscr{H}_0 \to \mathscr{H}$ that intertwines $\hat{\pi}^G$ and $\operatorname{ind} \hat{\pi}^H$. For any $u \in \mathscr{H}_0$, $T^u := (\langle u| \otimes I)T$ is a map from $\mathscr{H}_0$ to $L_2(G)$ intertwining $\hat{\pi}^G$ and the right regular representation $\mathfrak{R}^G$ of G, i.e. $\mathfrak{R}^G T^u = (T^u \otimes id)\pi^G$. Now, π^G is finite-dimensional, so that $\pi^G(\mathscr{H}_0) \subseteq \mathscr{H}_0 \otimes_{alg} A(G)$. Hence $\mathfrak{R}^G T^u(\mathscr{H}_0) \subseteq L_2(G) \otimes_{alg} A(G)$. By 2.5, $T^u(\mathscr{H}_0) \subseteq A(G)$. Since this is true for all $u \in \mathscr{H}_0$, $T(\mathscr{H}_0) \subseteq \mathscr{H}_0 \otimes_{alg} A(G)$. Therefore $(I \otimes \varepsilon_G)T$ is a bounded operator from $\mathscr{H}_0$ to $\mathscr{H}_0$. Denote it by $g(T)$.

For a comodule π and a linear functional ρ, denote $(id \otimes \rho)\pi$ by π_ρ. Let ρ be a linear functional on $C(H)$. Then $\pi^H_\rho g(T) = \pi^H_\rho (I \otimes \varepsilon_G)T = (I \otimes \varepsilon_G)(\pi^H_\rho \otimes id)T = (I \otimes \varepsilon_G)(I \otimes L_{\rho \cdot p})T = (I \otimes \rho \circ p)T$. On the other hand, since T intertwines $\hat{\pi}^G$ and $\operatorname{ind} \hat{\pi}^H$, we have $g(T)(\pi^{G|H})_\rho = g(T)(I \otimes \rho)\pi^{G|H} = g(T)(I \otimes \rho)(I \otimes p)\pi^G = (I \otimes \varepsilon_G)\ T\pi^G_{\rho \cdot p} = (I \otimes \varepsilon_G)(I \otimes \mathfrak{R}^G_{\rho \cdot p})T = (I \otimes \rho \circ p)T$. Thus $\pi^H_\rho g(T) = g(T)(\pi^{G|H})_\rho$ for all continuous linear functionals ρ on $C(H)$, which implies $g(T) \in \mathscr{I}(\hat{\pi}^{G|H}, \hat{\pi}^H)$. The map $T \mapsto g(T)$ is the inverse of f. Therefore $\mathscr{I}(\hat{\pi}^{G|H}, \hat{\pi}^H) \cong \mathscr{I}(\hat{\pi}^G, \operatorname{ind} \hat{\pi}^H)$, which proves the theorem. ■

COROLLARY 1.

For any unitary representation $\hat{\pi}^G$ of G and $\hat{\pi}^H$ of H, the spaces $\mathscr{I}(\hat{\pi}^{G|H}, \hat{\pi}^H)$ and $\mathscr{I}(\hat{\pi}^G, \operatorname{ind} \hat{\pi}^H)$ are isomorphic.

COROLLARY 2.

Let H be a subgroup of G and K be a subgroup of H. Suppose $\hat{\pi}$ is a unitary representation of K. Then $\operatorname{ind}_K^G \hat{\pi}$ and $\operatorname{ind}_H^G(\operatorname{ind}_K^H \hat{\pi})$ are equivalent.

3.4 Action of $SU_q(2)$ on the sphere S^2_{q0} has been decomposed by Podleś (see [5]). Here we give an alternative way of doing it using the Frobenius reciprocity theorem.

Let us start with a few observations. Let u be the function $z \mapsto z$, $z \in S^1$, where S^1 is the unit circle in the complex plane. Then u is unitary, and generates the C^*-algebra $C(S^1)$ of continuous functions on S^1. Let α and β be the two elements that generate the algebra $C(SU_q(2))$ and obey the following relations:

$$\alpha^*\alpha + \beta^*\beta = I = \alpha\alpha^* + q^2\beta\beta^*,$$
$$\alpha\beta - q\beta\alpha = 0 = \alpha\beta^* - q\beta^*\alpha, \quad \beta^*\beta = \beta\beta^*.$$

The map $p: \alpha \mapsto u$, $\beta \mapsto 0$ extends to a C^*-homomorphism from $C(SU_q(2))$ onto $C(S^1)$. It is in fact a quantum group homomorphism. By 2.2, S^1 is a subgroup of $SU_q(2)$.

For any $n \in \{0, 1/2, 1, 3/2, \ldots\}$, if we restrict the right-regular comodule $\mathfrak{R}$ of $SU_q(2)$ to the subspace $\mathscr{H}_n$ of $L_2(SU_q(2))$ spanned by

$$\{\alpha^{*i}\beta^{2n-i} : i = 0, 1, \ldots, 2n\}, \tag{3.1}$$

then we get an irreducible isometric comodule. Denote it by $u^{(n)}$. It is a well-known fact ([6], [7]) that these constitute *all* the irreducible comodules of $SU_q(2)$. If we take

the basis of $\mathscr{H}_n$ to be (3.1) with proper normalization, the matrix entries of $u^{(n)}$ turn out to be

$$u_{ij}^{(n)} = (d_i^{(n)}/d_j^{(n)})^{1/2} \sum_{r=(i-j)\vee 0}^{(2n-j)\wedge i} \binom{i}{r}_{q^{-2}} \binom{2n-i}{r+j-i}_{q^{-2}} (-1)^r q^{r(2i-r+1)+(j-i)(2n-j)} \times \alpha^{*i-r}\alpha^{2n-j-r}\beta^{r+j-i}\beta^{*r},$$

where

$$d_k^{(n)} = \sum_{r=0}^{k} \binom{k}{r}_{q^{-2}} (-1)^r q^{r(2k-r+1)} \frac{1-q^2}{1-q^{4n+2r-2k+2}};$$

$$\binom{r}{s}_{q^{-2}} := \frac{(r)_{q^{-2}}(r-1)_{q^{-2}}\ldots(1)_{q^{-2}}}{(s)_{q^{-2}}(s-1)_{q^{-2}}\ldots(1)_{q^{-2}}(r-s)_{q^{-2}}(r-s-1)_{q^{-2}}\ldots(1)_{q^{-2}}};$$

$$(k)_{q^{-2}} := 1 + q^{-2} + q^{-4} + \cdots + q^{-2k+2}.$$

Since $u^{(n)}|^{S^1} = (I \otimes p)u^{(n)}$, matrix entries of $u^{(n)}|^{S^1}$ are given by

$$(u^{(n)}|^{S^1})_{ij} = \begin{cases} u^{2(n-i)} & \text{if } i=j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.2}$$

Therefore if n is an integer then the trivial representation occurs in $u^{(n)}|^{S^1}$ with multiplicity 1, and does not occur otherwise.

Consider now the action of $SU_q(2)$ on S^2_{q0}. Recall ([5]) that $C(S^2_{q0}) = \{a \in C(SU_q(2)): (p \otimes id)\mu(a) = I \otimes a\}$ and the action is the restriction of μ to $C(S^2_{q0})$. From the above description, $C(S^2_{q0})$ can easily be shown to be equal to $\{a \in C(S^2_{q0}): L_{\rho \cdot p}(a) = \rho(I)a$ for all continuous linear functionals ρ on $C(S^1)\}$. Therefore when we take the closure of $C(S^2_{q0})$ with respect to the invariant inner product that it carries and extend the action there as an isometry, what we get is the restriction of the right-regular comodule $\mathfrak{R}$ of $SU_q(2)$ to the subspace $\mathscr{H} = \{u \in L_2(SU_q(2)): L_{\rho \cdot p}(u) = \rho(I)u$ for all continuous linear functionals ρ on $C(S^1)\}$, which is nothing but the representation $\hat{\pi}$ of $SU_q(2)$ induced by the trivial representation of S^1 on $\mathbb{C}$. Hence the multiplicity of $u^{(n)}$ in $\hat{\pi}$ is same as that of the trivial representation of S^1 in $u^{(n)}|^{S^1}$ which is, from (3.2), 1 if n is an integer and 0 if n is not. Thus the action splits into a direct sum of all the integer-spin representations.

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Differential subordination and Bazilevič functions

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Abstract. Let $M(z)=z^n+\cdots, N(z)=z^n+\cdots$ be analytic in the unit disc Δ and let $\lambda(z)=N(z)/zN'(z)$. The classical result of Sakaguchi–Libera shows that $\mathrm{Re}(M'(z)/N'(z))>0$ implies $\mathrm{Re}(M(z)/N(z))>0$ in Δ whenever $\mathrm{Re}(\lambda(z))>0$ in Δ. This can be expressed in terms of differential subordination as follows: for any p analytic in Δ, with $p(0)=1$,

$$p(z)+\lambda(z)zp'(z)\prec\frac{1+z}{1-z} \quad \text{implies } p(z)\prec\frac{1+z}{1-z}, \quad \text{for } \mathrm{Re}\,\lambda(z)>0, \quad z\in\Delta.$$

In this paper we determine different type of general conditions on $\lambda(z)$, $h(z)$ and $\phi(z)$ for which one has

$$p(z)+\lambda(z)zp'(z)\prec h(z) \quad \text{implies } p(z)\prec\phi(z)\prec h(z), \quad z\in\Delta.$$

Then we apply the above implication to obtain new theorems for some classes of normalized analytic funotions. In particular we give a sufficient condition for an analytic function to be starlike in Δ.

Keywords. Differential subordination; univalent; starlike and convex functions.

1. Introduction

Let f and g be analytic in the unit disc Δ. The function f is *subordinate* to g, written $f\prec g$, or $f(z)\prec g(z)$, if g is univalent, $f(0)=g(0)$ and $f(\Delta)\subset g(\Delta)$. Define $\mathscr{A}=\{f: f(0)=f'(0)-1=0\}$, $\mathscr{A}_k=\{f: f(z)=z+a_{k+1}z^{k+1}+\cdots\}$, and $\mathscr{A}'=\{f: f(0)=1\}$. Let $\lambda(z)$ be a function defined on Δ with $\mathrm{Re}\,\lambda(z)>\eta>0$, $z\in\Delta$ and let $p\in\mathscr{A}'$. Then a recent paper [8, Theorem 1] establishes the following:

$$\mathrm{Re}[p(z)+\lambda(z)zp'(z)]>\beta \text{ implies } \mathrm{Re}\,p(z)>\frac{2\beta+\eta}{2+\eta}, \quad \text{for } z\in\Delta. \tag{1}$$

Let μ and λ satisfy $|\mathrm{Im}\,\mu(z)|\leqslant\mathrm{Re}\,\lambda(z)$, $z\in\Delta$ and let $p\in\mathscr{A}'$. Then a result of Miller and Mocanu [5, Theorem 8] shows that

$$\mathrm{Re}[\mu(z)p(z)+\lambda(z)zp'(z)]>0 \text{ implies } \mathrm{Re}\,p(z)>0, \quad \text{for } z\in\Delta. \tag{2}$$

(2) is equivalent to (1) if we take $\mu(z)=1$ in (2) and $\beta=\eta=0$ in (1).

Let M and N be analytic in Δ, with $M'(0)/N'(0) = 1$ and let β be real. If N maps Δ onto a multisheeted starlike domain with respect to the origin, then from [4, Theorem 10] we get

$$\operatorname{Re}\frac{M'(z)}{N'(z)} < \beta \text{ (or } > \beta \text{ resp.) implies } \operatorname{Re}\frac{M(z)}{N(z)} < \beta \text{ (or } > \beta \text{ resp.)}, \quad \text{for } z \in \Delta. \tag{3}$$

A well-known condition for a function $p \in \mathcal{A}$ subordinate to q is that [6]

$$p(z) + \frac{zp'(z)}{p(z)} \prec q(z) + \frac{zq'(z)}{q(z)},$$

under some conditions on $q(z)$. Suppose we let $p(z) = zf'(z)/f(z)$ and $q(z) = 2(1+z)/(2-z)$, then we get

$$\operatorname{Re}\left[\frac{(zf'(z))'}{f'(z)}\right] < \frac{3}{2} \text{ implies } \left|\frac{zf'(z)}{f(z)} - \frac{2}{3}\right| < \frac{2}{3}, \quad \text{for } z \in \Delta.$$

Similarly it follows from a result of Mocanu *et al* [7] that for $p \in \mathcal{A}'$,

$$\operatorname{Re}[p(z) + zp'(z)] > 0 \text{ implies } |\arg p(z)| < 0 < \pi/3, \quad \text{for } z \in \Delta,$$

where θ lies between 0.911621904 and 0.911621907. This improves the relation (2) whenever $\mu(z) = \lambda(z) = 1$ for $z \in \Delta$.

However the example $M(z) = zf'(z)$, $N(z) = f(z)$ and $\beta = 3/2$ [or $M(z) = zf'(z)$, $N(z) = z$ and $\beta = 0$ resp.] in (3) suggests that there may exist some conditions on M and N so that

$$\operatorname{Re}\left[(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}\right] \begin{cases} < \beta_1 \\ > \beta_2 \end{cases} \text{ implies } \frac{M(z)}{N(z)} \prec \begin{cases} h_1(z) \\ h_2(z) \end{cases}, \quad \text{for } z \in \Delta \tag{4}$$

for some h_i $(i = 1, 2)$ to be specified.

Thus it is interesting to ask whether there exist such conditions for our implication. By writing (4) in terms of differential subordination, in this article we determine some new sufficient conditions on $\lambda(z)$, β_i and $h_i(z)$, $(i = 1, 2)$ for $\operatorname{Re}[p(z) + \lambda(z)zp'(z)] > \beta_i$ to imply $p(z)$ is subordinate to $h_i(z)$. Some interesting applications of this are given. In particular they improve the previous works of different authors [1, 8, 9, 12].

All of the inequalities in this article involving functions of z, such as (2), hold uniformly in the unit disc Δ. So the condition 'for $z \in \Delta$' will be omitted in the remaining part of the paper.

2. Preliminaries

Let $f \in \mathcal{A}$ and $S^* = \{f \in \mathcal{A}: f(\Delta) \text{ is starlike}\}$. Then for $\gamma > 0$ and $\beta < 1$, we say $f \in B(\gamma, \beta)$ if, and only if, there exists $g \in S^*$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)^{1-\gamma}g(z)^{\gamma}}\right) > \beta,$$

where all powers are chosen as principal ones.

Denote by $B_1(\gamma, \beta)$, the subclass consisting of those functions in $B(\gamma, \beta)$ for which $g \in S^*$ can be taken as the identity map on Δ. As usual we let $B_1(1, \beta) = R(\beta)$ and $B_1(0, \beta) = S^*(\beta)$. From (1), for $0 \leqslant \beta < 1$, $\gamma > 0$ and for $f \in B_1(\gamma, \beta)$, we easily have

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\gamma} > \frac{2\beta\gamma + 1}{2\gamma + 1}. \tag{5}$$

In Lemma 1 of section 3 below, we obtain a more general result which improves the above inequality. Lemma 1 has been used in [9] to obtain new sufficient conditions for starlikeness.

We use the following two lemmas in our proofs.

Lemma A. [5] *Let F be analytic in Δ and let G be analytic and univalent on $\bar{\Delta}$, with $F(0) = G(0)$. If F is not subordinate to G, then there exist points $z_0 \in \Delta$ and $\zeta_0 \in \partial\Delta$, and $m \geqslant 1$ for which $F(|z| < |z_0|) \subset G(|z| < |z_0|)$, $F(z_0) = G(\zeta_0)$, and $z_0 F'(z_0) = m\zeta_0 G'(\zeta_0)$.*

Lemma B. [5, 6] *Let $\Omega \subset \mathbb{C}$ and let q be analytic and univalent on $\bar{\Delta}$ except for those $\zeta \in \partial\Delta$ for which $\operatorname{Lt}_{z \to \zeta} q(z) = \infty$. Suppose that $\psi: \mathbb{C}^2 \times \Delta \to \mathbb{C}$ satisfies the condition*

$$\psi(q(\zeta), m\zeta q'(\zeta); z) \notin \Omega \tag{6}$$

when $q(z)$ is finite, $m \geqslant k \geqslant 1$ and $|\zeta| = 1$. If p and q are analytic in Δ, $p(z) = p(0) + p_k z^k + \cdots$, $p(0) = q(0)$, and further if

$$\psi(p(z), zp'(z); z) \in \Omega$$

then $p(z) \prec q(z)$ in Δ.

Suppose that $p \in \mathscr{A}'$ with $p(z) = 1 + p_k z^k + \cdots$, and $q(z) = (1+z)/(1-z)$. Then the condition (6) reduces to

$$\psi(ix, y; z) \notin \Omega \tag{7}$$

when x is real and $y \leqslant -k(1 + x^2)/2$. Except for Theorems 5 and 6, we, in our results, consider the situations where $k = 1$.

3. Main results

We now state and prove our main results.

Lemma 1. *Let $p \in \mathscr{A}'$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \geqslant 0$ $(\alpha \neq 0)$, $\beta < 1$ be such that*

$$\operatorname{Re}\{p(z) + \alpha z p'(z)\} > \beta, \tag{8}$$

then

$$\operatorname{Re} p(z) > \beta + (1 - \beta)[2\delta - 1], \tag{9}$$

where

$$\delta = \delta(\operatorname{Re}\alpha) = \int_0^1 \frac{dt}{1 + t^{\operatorname{Re}\alpha}} \tag{10}$$

and $\delta(\operatorname{Re}\alpha)$ is an increasing function of $\operatorname{Re}\alpha$ with $(1 + \operatorname{Re}\alpha)/(1 + 2\operatorname{Re}\alpha) \leqslant \delta < 1$. The estimate cannot be improved in general.

Proof. We use the well-known result of Hallenbeck an Ruscheweyh [2], namely,

$$p(z)+\alpha zp'(z)\prec h(z) \text{ implies } p(z)\prec\frac{1}{\alpha}z^{-1/\alpha}\int_0^z h(t)t^{1/\alpha-1}\,dt \tag{11}$$

for $p\in\mathscr{A}'$ and h a convex(univalent) function with $h(0)=1$. If we let

$$h(z)=2\beta-1+\frac{2(1-\beta)}{1-z}$$

then h is convex and univalent on Δ, $h(0)=1$ and $\operatorname{Re} h(z)>\beta$. For this choice of h, the condition that (8) implies – in fact is equivalent to –

$$p(z)+\alpha zp'(z)\prec h(z).$$

Therefore from a straightforward calculation, Inequality (8) implies

$$p(z)\prec 2\beta-1+2(1-\beta)\phi(z) \tag{12}$$

where ϕ defined by

$$\phi(z)=1+\sum_{n=1}^{\infty}\frac{z^n}{n\alpha+1}=\int_0^1\frac{dt}{1-zt^{\alpha}}$$

is convex in Δ.

Let

$$W=\frac{1}{1-zt^{\alpha}},\quad z\in\Delta$$

so that

$$1-\frac{1}{W}=zt^{\alpha}.$$

Then, for $|z|=r$ and $0\leqslant t\leqslant 1$, we have

$$\left|1-\frac{1}{W}\right|\leqslant rt^{\operatorname{Re}\alpha}.$$

This implies that

$$\left|W-\frac{1}{1-r^2t^{2\operatorname{Re}\alpha}}\right|\leqslant\frac{rt^{\operatorname{Re}\alpha}}{1-r^2t^{2\operatorname{Re}\alpha}}$$

and so

$$\frac{1}{1+rt^{\operatorname{Re}\alpha}}\leqslant\operatorname{Re}W\leqslant\frac{1}{1-rt^{\operatorname{Re}\alpha}}.$$

(Note that if $\operatorname{Re}\alpha<0$, $rt^{\operatorname{Re}\alpha}$ need not be less than one and the above will not work.)

Therefore, we have

$$\operatorname{Re}\phi(z)\geqslant K(r)=\int_0^1\frac{dt}{1+rt^{\operatorname{Re}\alpha}},\quad \text{for } |z|=r,\quad 0<r<1.$$

Observe that the series $K(r)$ is absolutely convergent for $0<r<1$. Suitably rearranging the pairs of terms in $K(r)$ it can be shown that $1/2 \leqslant K(r) < 1$.

In particular for $r \to 1^-$ the above inequality reduces to

$$\operatorname{Re}\phi(z) \geqslant K(r) > K(1) = \delta(\operatorname{Re}\alpha),$$

where δ is as in (10).

Next we show that δ satisfies the inequality $(1+\operatorname{Re}\alpha)/(1+2\operatorname{Re}\alpha) \leqslant \delta < 1$. Since $2\beta - 1 + 2(1-\beta)\phi(z)$ is the best dominant for (8), we obtain taking $\lambda(z) = \alpha$, with $\operatorname{Re}\alpha > \eta$ in (1),

$$\operatorname{Re}\{2\beta - 1 + 2(1-\beta)\phi(z)\} > 2\beta - 1 + 2(1-\beta)K(1) \geqslant \frac{2\beta+\eta}{2+\eta},$$

$$\text{i.e., } \operatorname{Re}\phi(z) > K(1) \geqslant \frac{1+\eta}{2+\eta}.$$

Thus making $\eta \to \operatorname{Re}\alpha^+$, we get

$$\operatorname{Re}\phi(z) > K(1) = \delta(\operatorname{Re}\alpha) \geqslant \frac{1+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}.$$

This from (12) proves (9).

To complete the proof we need only to show that the bound in (9) cannot be improved in general. For this we let

$$q(z) = 2\beta - 1 + 2(1-\beta)\int_0^1 \frac{dt}{1-zt^\alpha}.$$

Then q is the best dominant for (8), because it satisfies the differential equation

$$q(z) + \alpha z q'(z) - 2\beta - 1 + \frac{2(1-\beta)}{1-z}h(z).$$

Therefore the function $q(z)$ shows that the bound in (9) cannot be improved. □□

Remark. In fact the second assertion, namely,

$$\frac{1+\operatorname{Re}\alpha}{1+2\operatorname{Re}\alpha} \leq \delta,$$

can be seen directly. If $\operatorname{Re}\alpha \geq 1$, then

$$\begin{aligned}\delta(\operatorname{Re}\alpha) &= \int_0^1 \left[1 - t^{\operatorname{Re}\alpha} + \frac{t^{2\operatorname{Re}\alpha}}{1+t^{\operatorname{Re}\alpha}}\right] dt \\ &> \frac{\operatorname{Re}\alpha}{1+\operatorname{Re}\alpha} + \frac{1}{2(1+2\operatorname{Re}\alpha)} \\ &\geq \frac{1+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}.\end{aligned}$$

Similarly if $0 \leqslant \operatorname{Re}\alpha < 1$, then

$$\begin{aligned}\delta(\operatorname{Re}\alpha) &= \frac{1}{2} + \frac{1}{2}\int_0^1 \left(\frac{1-t^{\operatorname{Re}\alpha}}{1+t^{\operatorname{Re}\alpha}}\right) dt \\ &= \frac{1}{2} + \frac{1}{2}\int_0^1 (1-t^{\operatorname{Re}\alpha})\left[\frac{1}{2} + \frac{1-t^{\operatorname{Re}\alpha}}{2(1+t^{\operatorname{Re}\alpha})}\right] dt \\ &\geqslant \frac{1}{2} + \frac{\operatorname{Re}\alpha}{4(1+\operatorname{Re}\alpha)} + \frac{(\operatorname{Re}\alpha)^2}{4(1+\operatorname{Re}\alpha)(1+2\operatorname{Re}\alpha)} \\ &\geqslant \frac{1}{2} + \frac{\operatorname{Re}\alpha}{2(2+\operatorname{Re}\alpha)} = \frac{1+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}.\end{aligned}$$

Using Lemma 1 in particular for $\operatorname{Re}\alpha \to 0$, $\alpha \neq 0$, $\beta = 0$, one has

$$\operatorname{Re}\{p(z) + \alpha z p'(z)\} > 0 \text{ implies } \operatorname{Re} p(z) > 0.$$

In the next result we improve this relation by showing that the same conclusion may be obtained under a weaker hypothesis on p.

Theorem 1. *Let α be a purely imaginary number, i.e., $\alpha = i\alpha_2$, α_2 real. Let Q be the unique-function that maps Δ onto the complement of the ray $\{it: t \leqslant 2^{-1}(\alpha_2^{-1} - \alpha_2)\}$ whenever $\alpha_2 > 0$ ($\{it: t \geqslant 2^{-1}(\alpha_2^{-1} - \alpha_2)\}$ whenever $\alpha_2 < 0$). If $p \in \mathscr{A}'$ satisfies*

$$p(z) + \alpha z p'(z) \prec Q(z)$$

then $\operatorname{Re} p(z) > 0$.

Proof. If we let $\psi(r, s) = r + \alpha s$, then $\psi(p(z), zp'(z))$ is analytic in Δ and the above subordination becomes

$$\psi(p(z), zp'(z)) \prec Q(z).$$

The conclusion of the theorem will follow from Lemma B and (7) if we can show that $\psi(ix, y) \notin Q(\Delta)$ when $y \leqslant -(1+x^2)/2$ and x-real. Suppose that $\alpha = i\alpha_2$, with $\alpha_2 > 0$, then $\psi(ix, y) = i(x + \alpha_2 y)$ and

$$x + \alpha_2 y \leqslant x - \alpha_2(1+x^2)/2 \leqslant 2^{-1}[\alpha_2^{-1} - \alpha_2]$$

for all x-real.

This shows that for $\alpha_2 > 0$, $\psi(ix, y) \notin Q(\Delta)$. A similar conclusion holds for the case $\alpha_2 < 0$. Hence the theorem. □

However the special case of the following lemma improves the conclusion of Lemma 1 further at least for $\alpha \in \mathbb{C}$ such that $|\operatorname{Im}\alpha| \leqslant \sqrt{3}(\operatorname{Re}\alpha - \eta)$ for a suitable fixed $\eta > 0$.

Lemma 2. Let λ be a function defined on Δ satisfying

$$|\arg(\lambda(z) - \eta)| < \pi/3 \tag{13}$$

such that

$$\eta = \inf_{z\in\Delta}\left(\operatorname{Re}\lambda(z) - \frac{|\operatorname{Im}\lambda(z)|}{\sqrt{3}}\right) \quad (>0) \tag{14}$$

and let

$$\beta'(\eta) = \left(\frac{6+5\eta^2+2\sqrt{9+15\eta^2}}{25\eta^2}\right)^{1/3}\left[\frac{9-2\sqrt{9+15\eta^2}}{10}\right] \tag{15}$$

be such that $2\beta'(\eta)+\eta \geqslant 0$. *If* $p\in\mathscr{A}'$ *satisfies*

$$\operatorname{Re}[p(z)+\lambda(z)zp'(z)] > \beta'(\eta) \tag{16}$$

then $|\arg p(z)| < \pi/3$.

Proof. Note that $\beta'(\eta)\leqslant 0$ if, and only if, $\eta \geqslant \sqrt{3}/2$. Now using (1), (16) implies

$$\operatorname{Re} p(z) > \frac{2\beta'(\eta)+\eta}{2+\eta}.$$

Since $2\beta'(\eta)+\eta \geqslant 0$, this Inequality further implies $\operatorname{Re} p(z) > 0$ in Δ.

If we let $\Omega = \{\omega\in\mathbb{C}:\operatorname{Re}\omega > \beta'(\eta)\}$ and $q(z) = [(1+z)/(1-z)]^{2/3}$, then $q(\Delta)$ equals $\{\omega\in\mathbb{C}: |\arg\omega| < \pi/3\}$. Then for $\psi(r,s;z) = r+\lambda(z)s$, (16) can be rewritten as

$$\{\psi(p(z), zp'(z); z): |z|<1\} \subset \Omega.$$

So to prove the lemma we need only to show that $p \prec q$.

If p is not subordinate to q, then by Lemma A there exist points $z_0\in\Delta$ and $\zeta_0\in\partial\Delta$, and $m\geqslant 1$ such that

$$p(|z|<|z_0|)\subset q(\Delta), \quad p(z_0)=q(\zeta_0) \text{ and } z_0p'(z_0) = m\zeta_0q'(\zeta_0).$$

We first discuss the case $p(z_0)\neq 0$ which corresponds to a point on one of the rays on the sector $q(\Delta)$. Since $p(z_0)\neq 0$, $\zeta_0 \neq \pm 1$. Next by letting X and Y be the real and imaginary parts of $\lambda(z_0)$, respectively, from (13) and (14), we find that

$$\left.\begin{matrix} X+Y/\sqrt{3} \\ X-Y/\sqrt{3}\end{matrix}\right\} \geqslant X-|Y|/\sqrt{3} \geqslant \eta > 0. \tag{17}$$

Further if we set $ix = (1+\zeta_0)/(1-\zeta_0)$ and use the above observations, we obtain

$$\psi(p(z_0), z_0p'(z_0); z_0) = (ix)^{2/3}\left[1+\operatorname{im}(X+iY)\frac{1+x^2}{3x}\right].$$

For $x\neq 0$,

$$\operatorname{Re}\psi(p(z_0), z_0p'(z_0); z_0) = \operatorname{Re}\begin{cases} |x|^{2/3}\left(\dfrac{1+i\sqrt{3}}{2}\right)\left(1-\dfrac{m(Y-iX)(1+x^2)}{3|x|}\right), & \text{if } x>0 \\ |x|^{2/3}\left(\dfrac{1-i\sqrt{3}}{2}\right)\left(1+\dfrac{m(Y-iX)(1+x^2)}{3|x|}\right), & \text{if } x<0 \end{cases}$$

$$= \begin{cases} |x|^{2/3}\left[1-\dfrac{m(1+x^2)}{\sqrt{3}|x|}\left(X+\dfrac{Y}{\sqrt{3}}\right)\right]\Big/2, & \text{if } x>0 \\ |x|^{2/3}\left[1-\dfrac{m(1+x^2)}{\sqrt{3}|x|}\left(X-\dfrac{Y}{\sqrt{3}}\right)\right]\Big/2, & \text{if } x<0. \end{cases}$$

Therefore, for $x \neq 0$, since $\lambda(z_0)$ satisfies (17) and $m \geqslant 1$, we obtain

$$\operatorname{Re}\psi(p(z_0), z_0p'(z_0); z_0) \leqslant |x|^{2/3}\left[1-\frac{\eta}{\sqrt{3}}\left(|x|+\frac{1}{|x|}\right)\right]\frac{1}{2} = f(|x|),$$

where

$$f(t) = t^{2/3}\left[1-\frac{\eta(t^2+1)}{\sqrt{3}t}\right]\frac{1}{2}, \quad \text{with } t=|x|.$$

Since

$$t_0 = \frac{\sqrt{3}+\sqrt{3+5\eta^2}}{5\eta}$$

is the maximum for $f(t)$, we have

$$\operatorname{Re}\psi(p(z_0), z_0p'(z_0); z_0) \leqslant f(|x|) \leqslant f(t_0) \equiv \beta'(\eta).$$

This implies that $\psi(p(z_0), z_0p'(z_0); z_0)$ lies outside Ω, contradicting (16). Hence we must have $p \prec q$ when $p(z_0) \neq 0$.

Now consider the case $p(z_0) = 0$ which corresponds to the corner of the sector $q(\Delta)$. Observe that the sector angle of $q(\Delta)$ is $2\pi/3$ and so $p(|z| = |z_0|)$ cannot pass through such a corner without itself having a corner and hence the case $p(z_0) = 0$ cannot occur for the present form of our lemma. This completes the proof. □

Lemmas 1 and 2 yield improvements on most of the results of [8]. As an equivalent form of Lemma 2 we state

Theorem 2. *Let $\beta'(\eta)$ be as defined by (15) so that $2\beta'(\eta)+\eta \geqslant 0$. Let $M(z) = z^n + \cdots$ and $N(z) = z^n + \cdots$ be analytic in Δ and such that for some $\alpha \in \mathbb{C}$, N satisfies*

$$\left|\operatorname{Im}\frac{\alpha N(z)}{zN'(z)}\right| \leqslant \sqrt{3}\left(\operatorname{Re}\frac{\alpha N(z)}{zN'(z)} - \eta\right), \quad (0 < \eta \leqslant \operatorname{Re}\alpha - |\operatorname{Im}\alpha|/\sqrt{3}).$$

Then

$$\operatorname{Re}\left[(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}\right] > \beta'(\eta) \text{ implies } \left|\arg\frac{M(z)}{N(z)}\right| < \frac{\pi}{3}.$$

Proof. Consider the function $p(z) = M(z)/N(z)$ and let $\lambda(z) = \alpha N(z)/zN'(z)$. Then by hypothesis, $p \in \mathscr{A}'$ and all the conditions of Lemma 2 are satisfied. Now it is elementary to show that

$$(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)} = p(z) + \lambda(z)zp'(z)$$

and hence Theorem 2 follows from Lemma 2.

COROLLARY 1.

If $p \in \mathscr{A}'$ and if λ is a function defined on Δ such that

$$|\operatorname{Im} \lambda(z)| \leqslant \sqrt{3}(\operatorname{Re} \lambda(z) - \sqrt{3}/2)$$

then

$$\operatorname{Re}\{p(z) + \lambda(z)zp'(z)\} > 0 \text{ implies } |\arg p(z)| < \pi/3.$$

Proof. If we let $\eta = \sqrt{3}/2$ then in this case $\beta'(\eta) = 0$ in (15) and the corollary now follows from Lemma 2. □

COROLLARY 2.

Let $f \in B_1(\gamma, 0)$. Then we have

(i) $\operatorname{Re}\left(\dfrac{f(z)}{z}\right)^{\gamma} > 2\delta(1/\gamma) - 1$, *for* $\gamma > 0$.

(ii) $\left|\arg\left(\dfrac{f(z)}{z}\right)^{\gamma}\right| < \pi/3$, *for* $0 < \gamma \leqslant 2/\sqrt{3}$.

For the function F defined by $F^{\gamma}(z) = \dfrac{\gamma + c}{z^c} \displaystyle\int_0^z t^{c-1} f^{\gamma}(t)\,\mathrm{d}t$, *we have*

(iii) $\operatorname{Re} F'(z)\left(\dfrac{F(z)}{z}\right)^{\gamma - 1} > 2\delta(1/(\gamma + c)) - 1$, *for γ and c real with* $0 < \gamma + c$.

(iv) $\left|\arg F'(z)\left(\dfrac{F(z)}{z}\right)^{\gamma - 1}\right| < \pi/3$, *for γ and c such that* $0 < \gamma + c \leqslant 2/\sqrt{3}$.

Proof. Proofs of the above inequalities follow from Lemma 1 and Lemma 2 using the techniques of [8]. □

Theorem 3. *Let $\eta > 0$ be such that*

$$\beta'(\eta) + \frac{2\delta(\eta) - 1}{2(1 - \delta(\eta))} \geqslant 0. \tag{18}$$

where $\beta'(\eta)$ is as defined in (15). *Let $p \in \mathscr{A}'$ and $\alpha \geqslant \eta$. Suppose*

$$\operatorname{Re}[p(z) + \alpha zp'(z)] > \frac{\beta'(\eta) - (1 - (\eta/\alpha))(2\delta(\alpha) - 1)}{1 - (1 - (\eta/\alpha))(2\delta(\alpha) - 1)}. \tag{19}$$

Then we have

$$\operatorname{Re}[p(z) + \eta zp'(z)] > \beta'(\eta), |\arg p(z)| < \pi/3$$

and

$$\operatorname{Re} p(z) > 2(1 - \delta(\eta))\beta'(\eta) + 2\delta(\eta) - 1.$$

Proof. Observe that

$$p(z) + \eta zp'(z) = \left(1 - \frac{\eta}{\alpha}\right)p(z) + \frac{\eta}{\alpha}[p(z) + \alpha zp'(z)], \quad (\alpha \geqslant \eta).$$

Now Lemmas 1 and (19) yield

$$\mathrm{Re}[p(z)+\eta zp'(z)]>\beta'(\eta). \tag{20}$$

Taking $\lambda(z)=\eta$ in Lemma 2 and $\alpha=\eta$ in Lemma 1, respectively, the theorem follows.

□

We note that, using Lemma 1 and Theorem 2, we can construct several new examples. The result even for the special case $\alpha\geqslant\sqrt{3}/2$ where $\beta'(\sqrt{3}/2)=0$ could not be found in the literature.

For $f\in\mathscr{A}$ and $\alpha\in\mathbb{C}$ with $|\mathrm{Im}\,\alpha|\leqslant\sqrt{3}(\mathrm{Re}\,\alpha-\sqrt{3}/2)$, we have

$$\mathrm{Re}\{f'(z)+\alpha zf''(z)\}>0$$

implies

$$|\arg f'(z)|<\pi/3 \text{ and } \mathrm{Re}\,f'(z)>2\delta(\mathrm{Re}\,\alpha)-1.$$

We can use Corollary 1 to improve the result obtained by Yoshikawa and Yoshikai in [12, Theorem 4] concerning the transformation

$$F(z)=f(z)\exp\left\{-z^{-c}\int_0^z t^c\left(\frac{f'(t)}{f(t)}-\frac{1}{t}\right)dt\right\}, \quad \text{for } \mathrm{Re}\,c\geqslant 0, c\neq 0, \tag{21}$$

of the well-known γ-spiral-like functions. His result proves that for $|\gamma|<\pi/2$,

$$\mathrm{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right)>0 \text{ implies } \mathrm{Re}\left(e^{i\gamma}\frac{zF'(z)}{F(z)}\right)>\frac{\mathrm{Re}(1/c)}{2+\mathrm{Re}(1/c)}\cos\gamma.$$

From Corollary 1, with $\lambda(z)=1/c$, we see that we can improve the above implication to

$$\frac{zf'(z)}{f(z)}\prec\frac{1-z}{1+z}$$

implies

$$\left|\arg\frac{zF'(z)}{F(z)}\right|<\pi/3 \text{ whenever } \left|\arg\left(\frac{1}{c}-\frac{\sqrt{3}}{2}\right)\right|<\pi/3;$$

or, equivalently, if $\left|\arg\left(\dfrac{1}{c}-\dfrac{\sqrt{3}}{2}\right)\right|<\pi/3$, then

$$e^{i\gamma}\frac{zf'(z)}{f(z)}\prec\frac{e^{i\gamma}-e^{-i\gamma}z}{1+z} \text{ implies } e^{i\gamma}\left[\frac{zF'(z)}{F(z)}\right]^{3/2}\prec\frac{e^{i\gamma}-e^{-i\gamma}z}{1+z}.$$

We next prove the following lemma and then apply this to derive Theorem 4.

Lemma 3. *Let* $\alpha^*\approx 0.407\cdots$ *be the root of the equation*

$$\alpha^*=\tan[(2\pi-3\pi\alpha^*)/6] \tag{22}$$

and $\theta = \alpha^*\pi/2$. *Suppose that* β *is the smallest positive root of the cubic equation*

$$12\beta^3 - [(6-4\sqrt{3})\cos^2\theta + 18 - 4\sqrt{3}]\beta^2 - [(10+4\sqrt{3})\cos^2\theta + 8\sqrt{3} - 13]\beta + (16-4\sqrt{3})\cos^2\theta - (2-\sqrt{3})^2 = 0.$$

Further let $F(z)$ *be a complex function that satisfies*

$$|\arg F(z)| < \alpha^*\pi/2. \tag{23}$$

If $p \in \mathscr{A}'$ *satisfies*

$$\operatorname{Re} F(z)\{\beta + (1-\beta)p(z) + (\sqrt{3}/2)[(\beta + (1-\beta)p(z)^2) + (1-\beta)zp'(z) - (\beta + (1-\beta)p(z))]\} > 0 \tag{24}$$

then $\operatorname{Re} p(z) > 0$ *in* Δ.

Proof. First, we write

$$\psi(r, s; z) = F(z)\{\beta + (1-\beta)r + (\sqrt{3}/2)[(\beta + (1-\beta)r)^2 + (1-\beta)s - (\beta + (1-\beta)r)]\}$$

and $F(z) = X + iY \equiv \operatorname{Re} F(z) + i \operatorname{Im} F(z)$. Let us now apply Lemma B. Then for all x, y reals and $z \in \Delta$, we have

$$\operatorname{Re}\psi(ix, y; z) = X[\beta + (\sqrt{3}/2)(\beta^2 - \beta - (1-\beta^2)x^2 + (1-\beta)y)] - Y(1-\beta)[1 + (\sqrt{3}/2)(2\beta - 1)]x.$$

From this it is easily verified that

$$\operatorname{Re}\psi(ix, y; z) \leqslant -(Rx^2 + Sx + T)$$

for all x real, $y \leqslant -(1+x^2)/2$ and all $z \in \Delta$, where

$$R = \sqrt{3}X(1-\beta)(3-2\beta),\ S = 2Y(1-\beta)(2-\sqrt{3}+2\sqrt{3}\beta), \text{ and}$$

$$T = [\sqrt{3}(1-\beta)(1+2\beta) - 4\beta] = 2\sqrt{3}\left(\frac{4-\sqrt{3}+\sqrt{43-8\sqrt{3}}}{4\sqrt{3}} - \beta\right)\left(\frac{4-\sqrt{3}-\sqrt{43-8\sqrt{3}}}{4\sqrt{3}} + \beta\right).$$

Therefore $\operatorname{Re}\psi(ix, y; z) \leqslant 0$ if, as usual, $Rx^2 + Sx + T \geqslant 0$ for all real x. The second inequality holds if and only if $S^2 \leqslant 4RT$. By performing further algebraic simplifications, it can be easily seen that this is indeed equivalent to

$$|Y| \leqslant (\tan(\alpha^*\pi/2))X, \quad \text{i.e., } |\arg F(z)| < \alpha^*\pi/2,$$

where the required identity to claim this is

$$\tan^2(\alpha^*\pi/2) = \frac{\sqrt{3}(3-2\beta)(\sqrt{3}(1-\beta)(1+2\beta) - 4\beta)}{(1-\beta)(2-\sqrt{3}+2\sqrt{3}\beta)^2}.$$

Since this automatically follows from the hypothesis, the desired conclusion now follows from Lemma B with $\Omega = \{\omega \in \mathbb{C}: \operatorname{Re}\omega > 0\}$ and (7). Therefore the proof is complete. □

Theorem 4. *Let $f \in \mathcal{A}$ and β be as stated in Lemma 3. Suppose that for $\alpha \geqslant \sqrt{3}/2$,*

$$\operatorname{Re}[f'(z) + \alpha z f''(z)] > \frac{-(1-(\sqrt{3}/2\alpha))(2\delta(\alpha)-1)}{1-(1-(\sqrt{3}/2\alpha))(2\delta(\alpha)-1)}. \tag{26}$$

This implies $f \in S^*(\beta)$.

Proof. Suppose that f satisfies (26). Then taking $p(z) = f'(z)$ and $\eta = \sqrt{3}/2$ in Theorem 3, we obtain

$$\operatorname{Re}\{f'(z) + (\sqrt{3}/2) z f''(z)\} > 0 \tag{27}$$

and $|\arg f'(z)| < \pi/3$. Thus from [6, Theorem 5] we get

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\alpha^*\pi}{2}$$

where α^* is as in (22).

Now we need only to show that (27) implies $f \in S^*(\beta)$. For this we let

$$p(z) = \left(\frac{zf'(z)}{f(z)} - \beta\right)(1-\beta)^{-1} \text{ and } F(z) = \frac{f(z)}{z}.$$

Then by performing differentiation and some algebraic simplifications, (27) deduces to

$$\operatorname{Re}\psi(p(z), zp'(z); z) > 0$$

where

$$\psi(r, s; z) = F(z)\{\beta + (1-\beta)r + (\sqrt{3}/2)[(\beta + (1-\beta)r)^2 + (1-\beta)s - (\beta + (1-\beta)r)]\}.$$

The theorem now follows from Lemma 3. □

Taking $\alpha = 1$ in the above theorem we obtain the following.

COROLLARY 3.

Let β be as in Lemma 3. If $g \in \mathcal{A}$ satisfies

$$\operatorname{Re} g'(z) > -\frac{(2-\sqrt{3})(2\ln 2 - 1)}{2-(2-\sqrt{3})(2\ln 2 - 1)},$$

then the Alexander Operator $I(g)$ defined by

$$[I(g)](z) = \int_0^z \frac{g(t)}{t} dt,$$

is in $S^*(\beta)$, where β is as in Lemma 3.

Observe that a little computation shows that β is slightly bigger than the value btained in [9, Corollary 3]. Further the above corollary favours the existence of family of analytic functions, containing non-univalent functions, mapping onto $^*(\beta) \subset S^*$ under the Alexander Operator.

Theorem 5. *Let α be a real number with $\alpha < -2/k\delta$. Let $M(z) = z^n + a_{n+k}z^{n+k} + \cdots$ and $N(z) = z^n + \cdots$ be analytic in Δ $(n \geqslant 1, k \geqslant 1)$ and let N satisfy*

$$\mathrm{Re}(N(z)/zN'(z)) > \delta, \quad (0 < \delta < 1/n). \tag{28}$$

f

$$\mathrm{Re}\left[(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}\right] < \beta, \tag{29}$$

hen

$$\mathrm{Re}\left[\frac{M(z)}{N(z)}\right] > \frac{2\beta + k\delta\alpha}{2 + k\delta\alpha} \text{ and } \mathrm{Re}\left[\frac{M'(z)}{N'(z)}\right] > \frac{\beta(2+k\delta) - k\delta(1-\alpha)}{2+k\delta\alpha}. \tag{30}$$

Proof. If we let $\Omega = \{\omega \in \mathbb{C}: \mathrm{Re}\,\omega < \beta\}$, $\beta_1 = 2\beta + k\delta\alpha/2 + k\delta\alpha$, $\lambda(z) = N(z)/zN'(z)$ and $(z) = (1-\beta_1)^{-1}(M(z)/N(z)) - \beta_1)$, then $p(z) = 1 + p_k z^k + \cdots$ is analytic in Δ and the ondition (29) implies

$$\psi(p(z), zp'(z); z) \in \Omega$$

here $\psi: \mathbb{C}^2 \times \Delta \to \mathbb{C}$ is $\psi(r, s; z) = \beta_1 + (1-\beta_1)[r + \alpha\lambda(z)s]$.

Since N satisfies (28), we have $\mathrm{Re}\,\lambda(z) > \delta$ in Δ. If x is real and $y \leqslant -k(1+x^2)/2$ then or this ψ we have

$$\begin{aligned}\psi(ix, y; z) &= \beta_1 + (1-\beta_1)\alpha[\mathrm{Re}\,\lambda(z)]y \\ &\geqslant \beta_1 - [(1-\beta_1)\alpha\delta k]/2 \equiv \beta,\end{aligned}$$

ince $\alpha < 0$, i.e. $\psi(ix, y; z) \notin \Omega$. Hence (7) is satisfied and Lemma B leads to $\mathrm{Re}\,p(z) > 0$. his shows the first part of (30). Since $1-\alpha > 0$, this proves $(1-\alpha)\mathrm{Re}(M(z)/N(z)) > 1-\alpha)\beta_1$. Moreover, from this and (29) we easily have the second inequality of (30). Hence the theorem. □

COROLLARY 4.

Let $|\lambda| < 1$ and $f \in \mathcal{A}_k$. (i) *If*

$$\mathrm{Re}\{(1+\lambda z)[(1+\alpha\lambda z)f'(z) + \alpha(1+\lambda z)zf''(z)]\} < \beta \tag{31}$$

hen for $k\alpha(1-|\lambda|) < -2$,

$$\mathrm{Re}(1+\lambda z)f'(z) > \frac{2\beta + (1-|\lambda|)\alpha k}{2 + (1-|\lambda|)\alpha k}.$$

ii) *If*

$$\mathrm{Re}\left\{e^{-\lambda z}\left[\left(1 - \frac{\lambda\alpha z}{1+\lambda z}\right)f'(z) + \frac{\alpha z}{1+\lambda z}f''(z)\right]\right\} < \beta \tag{32}$$

then for $k\alpha + 2(1+|\lambda|) < 0$,

$$\operatorname{Re} e^{-\lambda z} f'(z) > \frac{2\beta(1+|\lambda|)+\alpha k}{2(1+|\lambda|)+\alpha k}.$$

Proof. For the proof of (i) we choose $M(z) = zf'(z)$ and $N(z) = z/(1+\lambda z)$. Then

$$\frac{M'(z)}{N'(z)} = (1+\lambda z)^2[f'(z)+zf''(z)], \quad \frac{M'(0)}{N'(0)} = 1, \quad \frac{N(z)}{zN'(z)} = 1+\lambda z$$

and

$$(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)} = (1+\lambda z)[(1+\alpha\lambda z)f'(z) + \alpha(1+\lambda z)zf''(z)].$$

Since $f \in \mathscr{A}$ satisfies (31), we have

$$\operatorname{Re}\frac{M(z)}{N(z)} > \frac{2\beta+\alpha\delta k}{2+\alpha\delta k}, \quad \text{whenever } \delta < 1-|\lambda|.$$

But δ can be chosen as close to $1-|\lambda|$ as we please and so we can allow $\delta \to 1-|\lambda|$ from below. Thus making $\delta \to 1-|\lambda|$ we establish our claim. The proof for the case (ii) follows on similar lines taking $M(z) = zf'(z)$ and $N(z) = ze^{\lambda z}$. □

Similar arguments used in Theorem 5 would help us to prove the following more general result.

Theorem 6. *Let* α *be a complex number with* $\operatorname{Re}\alpha < -2n/k\delta$. *Let* $M(z) = z^n + a_{n+k}z^{n+k} + \cdots$ *and* $N(z) = z^n + \cdots$ *be analytic in* Δ $(n \geqslant 1, k \geqslant 1)$ *and let* N *satisfy*

$$\operatorname{Re}(\alpha N(z)/zN'(z)) < \delta, \quad (\operatorname{Re}\alpha/n < \delta < -2/k).$$

Then

$$\operatorname{Re}\left[(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}\right] < \beta \quad \textit{implies} \quad \operatorname{Re}\left[\frac{M(z)}{N(z)}\right] > \frac{2\beta+k\delta}{2+k\delta}.$$

COROLLARY 5.

Let $\alpha \in \mathbb{C}$ *be such that* $\operatorname{Re}\alpha < -2m/k$, *where* m *is a positive integer and let* $\beta > 1$. *If* $f \in \mathscr{A}$ *satisfy*

$$\operatorname{Re}\left\{(1-\alpha)\left(\frac{f(z)}{z}\right)^m + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{m-1}\right\} < \beta$$

then

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^m > \frac{2\beta m + k\operatorname{Re}\alpha}{2m + k\operatorname{Re}\alpha}.$$

Proof. The corollary follows from Theorem 6 taking $M(z) = (f(z))^m$ and $N(z) = z^m$. □

In the following theorem we generalize the concept of α-close-to-convexity [1] when α is a complex number.

Theorem 7. *Let* $M(z)=z^n+\cdots$ *and* $N(z)=z^n+\cdots$ *be analytic in* Δ *and suppose that* N *satisfies*

$$\operatorname{Re}(N(z)/zN'(z))>\delta, \quad (0<\delta<1/n). \tag{33}$$

Further let k *be a complex number satisfying*

$$|\operatorname{Im} k|\leqslant\sqrt{D\delta}, \quad 0<D\leqslant(\delta+2\operatorname{Re} k). \tag{34}$$

Then

$$\operatorname{Re}\left[(k-1)\frac{M(z)}{N(z)}+\frac{M'(z)}{N'(z)}\right]>\beta, \quad (\operatorname{Re} k>\beta)$$

implies

$$\operatorname{Re}\frac{M(z)}{N(z)}>\frac{\delta+2\beta-D}{\delta+2\operatorname{Re} k-D}. \tag{35}$$

Proof. If we let $\beta_1=(\delta+2\beta-D)/(\delta+2\operatorname{Re} k-D)$ so that $D=[\delta+2\beta-\beta_1(\delta+2\operatorname{Re} k)]/(1-\beta_1)$ and define

$$p(z)=(1-\beta_1)^{-1}\left(\frac{M(z)}{N(z)}-\beta_1\right) \tag{36}$$

then $p\in\mathscr{A}'$. From (36) and (35), we obtain, as before, $\operatorname{Re}\psi(p(z), zp'(z); z)>0$, where

$$\psi(r,s;z)=-\beta+k\beta+(1-\beta_1)[kr+s(N(z)/zN'(z))].$$

If we can show that $\operatorname{Re}\psi(ix, y; z)\leqslant 0$ when $y\leqslant-(1+x^2)/2$ and x any real, the required conclusion is immediate from Lemma B and (7). But for this ψ we obtain

$$\begin{aligned}\operatorname{Re}\psi(ix,y;z)&=-\beta+\beta_1\operatorname{Re} k+(1-\beta_1)[y\operatorname{Re}(N(z)/zN'(z))-x\operatorname{Im} k]\\&\leqslant-\beta+\beta_1\operatorname{Re} k+(1-\beta_1)\left[-\frac{(1+x^2)}{2}\delta-x\operatorname{Im} k\right]\\&=-(1-\beta_1)[\delta x^2+2x\operatorname{Im} k+D]/2.\end{aligned}$$

By (34), we deduce that $\operatorname{Re}\psi(ix, y; z)<0$ and so the proof is complete. □

Examples. Let $M(z)=z^n+\cdots$ and $N(z)=z^n+\cdots$ be analytic in Δ. Then for $k\in\mathbb{C}$ with $|\operatorname{Im} k|\leqslant\sqrt{\delta(\delta+2\beta)}$, and $\operatorname{Re}(N(z)/zN'(z))>\delta>0$, Theorem 7 shows

$$\operatorname{Re}\left(\frac{M'(z)}{N'(z)}+(k-1)\frac{M(z)}{N(z)}\right)>\beta \text{ implies } \operatorname{Re}\left(\frac{M(z)}{N(z)}\right)>0.$$

As a special case of Theorem 7, let $f\in\mathscr{A}$ and $k\in\mathbb{C}$ with $|\operatorname{Im} k|\leqslant\sqrt{D}<\sqrt{1+2\operatorname{Re} k}$. In this case, Theorem 7 leads to

$$\operatorname{Re}(kf'(z)+zf''(z))>\beta \text{ implies } \operatorname{Re} f'(z)>\frac{1+2\beta-D}{1+2\operatorname{Re} k-D}.$$

In particular, this yields

$$\mathrm{Re}(kf'(z)+zf''(z))>\beta \text{ implies } \mathrm{Re}\, f'(z)>0 \quad (\beta<\mathrm{Re}\, k)$$

provided $|\mathrm{Im}\, k|\leqslant\sqrt{1+2\beta}$. This simple fact for $\beta=0$ has been used in [9, Theorem 3] to obtain an affirmative answer to a problem of Mocanu (for details see [9]).

Problems

Suppose that $p\in\mathscr{A}'$, $\beta<1$, $\rho=\beta+(1-\beta)[2\delta(\mathrm{Re}\,\alpha)-1]$ and H be defined by

$$H(z)=\frac{1-2[(1+\alpha)\rho-\alpha]z-(1-2\rho)z^2}{(1-z)^2}=$$

$$-(1-2\rho)+2(1-\rho)\left[(1-\alpha)\frac{1}{1-z}+\alpha\frac{1}{(1-z)^2}\right].$$

Now by setting $|z|=1$, i.e., $z=e^{i\theta}$ and $H(|z|=1)=U+iV$, we easily obtain

$$U=U(\theta)=\rho-\frac{\mathrm{Re}\,\alpha(1-\rho)}{1-\cos\theta} \text{ and } V=V(\theta)=(1-\rho)\left[\frac{\sin\theta-\mathrm{Im}\,\alpha}{1-\cos\theta}\right].$$

This, upon simplification for the case α real, yields the parabola

$$V^2=\frac{-2(1-\rho)}{\alpha}\left[U-\rho+\frac{\alpha(1-\rho)}{2}\right]=\frac{-4(1-\beta)(1-\delta(\alpha))}{\alpha}$$

$$[U-\beta+(\alpha+1-(\alpha+2)\delta(\alpha))]$$

and so for real α, the function H maps the unit disc $|z|<1$ into the convex domain, say D, bounded by the above parabola. Observe that the domain D contains $\{\omega\in\mathbb{C}: \mathrm{Re}\,\omega>\beta+((\alpha+2)\delta(\alpha)-(\alpha+1))\}$ for $\beta<1$.

Also from the sharp subordination relation (11) and a little manipulation we have the following implication

$$p\in\mathscr{A}' \text{ and } p(z)+\alpha zp'(z)\prec H(z) \text{ implies } p(z)\prec\frac{1+(1-2\rho)z}{1-z}$$

provided $\mathrm{Re}\,\alpha\geqslant 0$. From this, it is interesting to note that the same bound in Lemma 1 may be obtained under weaker hypothesis, though the images of Δ under p, respectively under the stated conditions on h and H, are different. Here h is as in the proof of Lemma 1 and H as above.

Problem 1. Find a (convenient) function $G(z)$ such that $G(\Delta)\subset H(\Delta)$ for which

$$f\in\mathscr{A} \text{ and } f'(z)+\alpha zf''(z)\prec G(z) \quad \text{implies} \quad f\in S^*? \qquad \square$$

For $\alpha<-2$, let

$$P(\alpha,\beta)=\{f\in\mathscr{A}: \mathrm{Re}(f'(z)+\alpha zf''(z))<\beta\}, \quad (\beta>1).$$

For $f\in\mathcal{A}$, $\alpha<-2$ and $\operatorname{Re}\left\{(1-\alpha)\dfrac{f(z)}{z}+\alpha f'(z)\right\}<\beta$, by Theorem 5, we have

$$\operatorname{Re}\frac{f(z)}{z}>\frac{2\beta+\alpha}{2+\alpha} \text{ and } \operatorname{Re} f'(z)>\frac{3\beta+\alpha-1}{2+\alpha}.$$

However, for $\alpha\in\mathbb{C}$, $\operatorname{Re}\alpha<-2$, Theorem 6 yields

$$f\in\mathcal{A} \text{ and } \operatorname{Re}(f'(z)+\alpha z f'(z))<\beta \text{ implies } \operatorname{Re} f'(z)>\frac{2\beta+\operatorname{Re}\alpha}{2+\operatorname{Re}\alpha}.$$

In particular for $\alpha<-2$ and $\beta\leqslant-\alpha/2$,

$$f\in P(\alpha,\beta) \text{ implies } \operatorname{Re} f'(z)>0$$

and further it is easy to show that

$$P(\alpha,\beta)\subset P\left(\alpha',\frac{\beta(2+\alpha')+(\alpha-\alpha')}{2+\alpha}\right), \quad \text{for } -2>\alpha>\alpha'.$$

Although a function $f\in\mathcal{A}$ such that $\operatorname{Re} f'(z)>0$ in Δ is univalent, Krzyż [3] showed that such a function need not be starlike in Δ. As pointed out in [10] there are functions, say f in $\mathcal{A}$ satisfying the condition $|f'(z)-1|<1$ in Δ, but they are not in general starlike in Δ. However the natural problem is the following:

Problem 2. Find certain subsets Ω of the left half plane, such that $f\in S^*$, whenever $f'(z)+\alpha z f''(z)$ belongs to Ω for all $z\in\Delta$ and $\alpha<-2$. In particular, under what conditions on β and α, $z(F*G)'(z)$ is starlike in Δ whenever F and G belong to $P(\alpha,\beta)$. Here * between two functions denotes Hadamard convolution. □

For $\delta\geqslant 0$, define a δ-neighborhood of $f(z)=z+a_2z^2+\cdots\in\mathcal{A}$ by

$$N_\delta(f)=\left\{g\colon g(z)=z+a_2z^2+\cdots\in\mathcal{A},\ \sum_{k=2}^{\infty}k|a_k-b_k|<\delta\right\}.$$

δ-neighborhoods were introduced by Ruscheweyh [11], who used this to generalize the result that $N_1(z)\subset S^*$. Now for $\alpha\geqslant 0$, let

$$R(\alpha)=\{f\in\mathcal{A}\colon R(f'(z)+\alpha z f''(z))>0,\ z\in\Delta\}.$$

It is known [9] that, $R(\alpha)\subset S^*$ at least when $\alpha\geqslant 0.4269\cdots$. Using Lemma 1, it is seen that if $f\in R(1)$ then $\operatorname{Re} f'(z)>2\ln 2-1$ and hence proceeding as in [11], it is not difficult to show that $N_{2\ln 2-1}(R(1))\subset R(0)$.

Interestingly Ruscheweyh proved that if f is in $S^*(\beta)$ then there is no value of $\delta>0$ such that $N_\delta(S^*(\beta))\subset S^*$ for any $0\leqslant\beta<1$.

In spite of this, it seems reasonable to ask the following:

Problem 3. Do there exist some conditions on α and δ such that $N_\delta(R(\alpha))\subset S^*$? If so, what is the best possible δ for a suitable fixed α? □

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Convolution integral equations involving a general class of polynomials and the multivariable H-function

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Abstract. In this paper we first solve a convolution integral equation involving product of the general class of polynomials and the H-function of several variables. Due to general nature of the general class of polynomials and the H-function of several variables which occur as kernels in our main convolution integral equation, we can obtain from it solutions of a large number of convolution integral equations involving products of several useful polynomials and special functions as its special cases. We record here only one such special case which involves the product of general class of polynomials and Appell's function F_3. We also give exact references of two results recently obtained by Srivastava *et al* [10] and Rashmi Jain [3] which follow as special cases of our main result.

Keywords. The convolution integral equation; multivariable H-function; general class of polynomials; Laplace transform.

1. Introduction

On account of the usefulness of convolution integral equations, a large number of authors, notably Srivastava [5], Kalla [4], Buschman *et al* [1], Srivastava and Buschman [8], Srivastava *et al* [10] and Rashmi Jain [3], have done significant work on this topic. In the present paper we develop generalizations of results of the last two papers referred to above. Also, Srivastava and Buschman [7, pp. 34–42 and §4.3] have discussed extensively such family of convolution integral equations as those considered here and in the works cited above.

We start by giving the following definitions and results which will be required later on.

(i) A general class of polynomials [6, p. 1, eq. (1)]

$$S_N^M[x] = \sum_{k=0}^{[M/N]} \frac{(-N)_{Mk} A_{N,k}}{k!} x^k, \quad N = 0, 1, 2, \ldots \tag{1.1}$$

where M is an arbitrary positive integer and the coefficient $A_{N,k}$ $(N, k \geqslant 0)$ are arbitrary constants real or complex. On suitably specializing the coefficient $A_{N,k}$, $S_N^M[x]$ yields a number of known polynomials as its special cases. These include, among others, Laguerre polynomials, Hermite polynomials and several others [12, pp. 158–161].

(ii) A special case of the H-function of r variables [11, p. 271, eq. (4.1)]

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H^{0,0:1,n_1;\ldots;1,n_r}_{p,q:p_1,q_1+1;\ldots;p_r,q_r+1}$$

$$\times \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} \Bigg| \begin{matrix} (a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p}: & (c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\ldots; & (c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (b_j:\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q}: & (0,1),(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\ldots; & (0,1),(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix} \Bigg]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \phi_1(\xi_1)\cdots\phi_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)$$

$$\times \Gamma(-\xi_1)\cdots\Gamma(-\xi_r)z_1^{\xi_1}\cdots z_r^{\xi_r}\,d\xi_1\cdots d\xi_r, \quad \omega=\sqrt{-1}. \tag{1.2}$$

Or equivalently [10, p. 64, eq. (1.3)]

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = \sum_{k_1,\ldots,k_r=0}^{\infty} \phi_1(k_1)\cdots\phi_r(k_r)\psi(k_1,\ldots,k_r)\frac{(-z_1)^{k_1}}{k_1!}\cdots\frac{(-z_r)^{k_r}}{k_r!} \tag{1.3}$$

where

$$\phi_i(k_i) = \frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}k_i)}{\prod_{j=1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}k_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}k_i)} \quad (i=1,\ldots,r) \tag{1.4}$$

$$\psi(k_1,\ldots,k_r) = \left\{\prod_{j=1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}k_i\right)\prod_{j=1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}k_i\right)\right\}^{-1}. \tag{1.5}$$

For the convergence, existence conditions and other details of the multivariable H-function refer the book [9, pp. 251–253, eqs. (C.2)–(C.8)].

(iii) The following property of the Laplace transform [2, p. 131]

$$L\{f^{(n)}(x);s\} = s^n\bar{f}(s) \tag{1.6}$$

holds provided that $f^{(i)}(0)=0, \quad i=0,1,2,\ldots,n-1, n$ being a positive integer, where

$$L\{f(x);s\} = \int_0^\infty e^{-sx}f(x)\,dx = \bar{f}(s). \tag{1.7}$$

(iv) The well-known convolution theorem for Laplace transform

$$L\left\{\int_0^x f(x-u)g(u)\,du;s\right\} = L\{f(x);s\}L\{g(x);s\} \tag{1.8}$$

holds provided that the various Laplace transforms occurring in (1.8) exist.

2. Main result

The convolution integral equation

$$\int_0^x (x-u)^{\rho-1}S_N^M[-z_{r+1}(x-u)]H\begin{bmatrix} z_1(x-u) \\ \vdots \\ z_r(x-u) \end{bmatrix} f(u)\,du = g(x) \tag{2.1}$$

has the solution given by

$$f(x)=\int_0^x (x-u)^{l-\rho-\mu-1}\sum_{j=0}^{\infty}\frac{E_j(x-u)^j}{\Gamma(j+l-\rho-\mu)}g^{(l)}(u)\,du \tag{2.2}$$

where $\mathrm{Re}(l-\rho-\mu)>0$, $\mathrm{Re}(\rho)>0$

$g^{(i)}(0)=0\,(i=0,1,\ldots,l-1)$, l being a positive integer and E_j is given by the recurrence relation

$$E_0\lambda_\mu=1,\quad \sum_{t=0}^{q}E_t\lambda_{q+\mu-t}=0,\quad q=1,2,3,\ldots \tag{2.3}$$

or by

$$E_j=(-1)^j(\lambda_\mu)^{-j-1}\det\begin{bmatrix}\lambda_{\mu+1} & \lambda_\mu & 0 & 0 & \cdots & 0\\ \lambda_{\mu+2} & \lambda_{\mu+1} & \lambda_\mu & 0 & \cdots & 0\\ \vdots & \vdots & & & & \\ \lambda_{\mu+j} & \lambda_{\mu+j-1} & & & \cdots & \lambda_{\mu+1}\end{bmatrix} \tag{2.4}$$

and μ is least B for which $\lambda_B\neq 0$

$$\lambda_B=(-1)^B\sum_{k_1+\cdots+k_{r+1}=B}\Delta(k_1,\ldots,k_{r+1})\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_{r+1}^{k_{r+1}}}{k_{r+1}!} \tag{2.5}$$

where

$$\Delta(k_1,\ldots,k_{r+1})=\phi_1(k_1)\cdots\phi_{r+1}(k_{r+1})\psi(k_1,\ldots,k_{r+1}) \tag{2.6}$$

$$\psi(k_1,\ldots,k_{r+1})=\Gamma(\rho+k_1+\cdots+k_{r+1})\times\left\{\prod_{j=1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}k_i\right)\prod_{j=1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}k_i\right)\right\}^{-1} \tag{2.7}$$

$$\phi_i(k_i)=\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}k_i)\left\{\prod_{j=n_i+1}^{p_i}\Gamma(c_j-\gamma_j^{(i)}k_i)\prod_{j=1}^{q_i}\Gamma(1-d_j+\delta_j^{(i)}k_i)\right\}^{-1} \tag{2.8}$$

$(i=1,\ldots,r)$

and

$$\phi_{r+1}(k_{r+1})=\begin{cases}(-N)_{Mk_{r+1}}A_{N,k_{r+1}}, & 0\leqslant k_{r+1}\leqslant\left[\dfrac{N}{M}\right]\\ 0, & k_{r+1}>\left[\dfrac{N}{M}\right]\end{cases} \tag{2.9}$$

Proof. To solve the convolution integral equation (2.1) we first take the Laplace transform of its both sides. We easily obtain by the definition of Laplace transform and its convolution property stated in (1.8), the following result

$$\left[\int_0^\infty e^{-sx}x^{\rho-1}S_N^M[(-z_{r+1})x]H\begin{bmatrix}z_1x\\ \vdots\\ z_rx\end{bmatrix}dx\right]\bar{f}(s)=\bar{g}(s). \tag{2.10}$$

Now expressing the $S_N^M[(-z_{r+1})x]$ and $H\begin{bmatrix} z_1 x \\ \vdots \\ z_r x \end{bmatrix}$ involved in (2.10) in series using (1.1) and (1.3), changing the order of series and integration and evaluating the x-integral, we obtain

$$\left[\sum_{k_1,\ldots,k_{r+1}=0}^{\infty} \Delta(k_1,\ldots,k_{r+1}) \frac{(-z_1)^{k_1}}{k_1!} \cdots \frac{(-z_{r+1})^{k_{r+1}}}{k_{r+1}!} \times s^{-\rho-(k_1+\cdots+k_{r+1})}\right] \bar{f}(s) = \bar{g}(s) \quad (2.11)$$

where $\Delta(k_1,\ldots,k_{r+1})$ is defined by (2.6). Now making use of the known formula [10, p. 67, eq. (2.3)], we easily obtain from (2.11)

$$\left[\sum_{B=0}^{\infty} \lambda_B s^{-B}\right] s^{-\rho} \bar{f}(s) = \bar{g}(s) \quad (2.12)$$

where λ_B is defined by (2.5).

Again, (2.12) is equivalent to

$$\bar{f}(s) = s^{\rho}\left[\sum_{B=0}^{\infty} \lambda_B s^{-B}\right]^{-1} \bar{g}(s). \quad (2.13)$$

If μ denotes the least B for which $\lambda_B \neq 0$, the series given by (2.13) can be reciprocated. Writing

$$\left[\sum_{B=0}^{\infty} \lambda_{B+\mu} s^{-B}\right]^{-1} = \sum_{j=0}^{\infty} E_j s^{-j} \quad (2.14)$$

eq. (2.13) takes the following form:

$$\bar{f}(s) = s^{\rho-l+u} \sum_{j=0}^{\infty} E_j s^{-j}[s^l \bar{g}(s)]. \quad (2.15)$$

(2.15) can be written as

$$L\{f(x); s\} = L\left\{\sum_{j=0}^{\infty} E_j \frac{x^{j+l-\mu-\rho-1}}{\Gamma(j+l-\mu-\rho)}; s\right\} L\{g^{(l)}(x); s\} \quad (2.16)$$

[on using (1.6)].

Now using the convolution theorem in the RHS of (2.16) we get

$$L\{f(x); s\} = L\left\{\int_0^x \sum_{j=0}^{\infty} \frac{E_j (x-u)^{j+l-\rho-\mu-1}}{\Gamma(j+l-\rho-\mu)} g^{(l)}(u)\,du; s\right\}. \quad (2.17)$$

Finally, on taking the inverse of the Laplace transform of both sides of (2.17) we arrive at the desired result (2.2).

3. Special cases

If we put $r = 2$ in (2.1) and reduce the H-function of two variables thus obtained to Appell's function F_3 [9, p. 89, eq. (6.4.6)] we find after a little simplification that the convolution equation given by

$$\int_0^x (x-u)^{\rho-1} S_N^M[-z_3(x-u)] F_3[c_1^{(1)}, c_1^{(2)}, c_2^{(1)}, c_2^{(2)}; b; -z_1(x-u), -z_2(x-u)] \times f(u)\,du = g(x) \tag{3.1}$$

has the solution

$$f(x) = \frac{\Gamma(c_1^{(1)})\Gamma(c_1^{(2)})\Gamma(c_2^{(1)})\Gamma(c_2^{(2)})}{\Gamma(b)} \int_0^x (x-u)^{l-\rho-\mu-1} \sum_{j=0}^{\infty} \frac{E_j (x-u)^j}{\Gamma(j+l-\rho-\mu)} g^{(l)}(u)\,du \tag{3.2}$$

where $\mathrm{Re}(l-\rho-\mu)>0$, $\mathrm{Re}(\rho)>0$, $|z_1(x-u)|<1$, $|z_2(x-u)|<1$, $g^{(i)}(0)=0$ $(i=0,1,\ldots,l-1)$, l being a positive integer and E_j are given by recurrence relation (2.9) or (2.4) and μ is least B for which $\lambda_B \neq 0$

$$\lambda_B = (-1)^B \sum_{k_1+k_2+k_3=B} \Delta(k_1,k_2,k_3) \frac{z_1^{k_1}}{k_1!} \frac{z_2^{k_2}}{k_2!} \frac{z_3^{k_3}}{k_3!} \tag{3.3}$$

where in (3.3)

$$\Delta(k_1,k_2,k_3) = \frac{\Gamma(c_1^{(1)}+k_1)\Gamma(c_1^{(2)}+k_2)\Gamma(c_2^{(1)}+k_1)\Gamma(c_2^{(2)}+k_2)\Gamma(\rho+k_1+k_2+k_3)}{\Gamma(b+k_1+k_2)} \times \phi_3(k_3) \tag{3.4}$$

and

$$\phi_3(k_3) = \begin{cases} (-N)_{Mk_3} A_{N,k_3}, & 0 \leqslant k_3 \leqslant \left[\dfrac{N}{M}\right] \\ 0, & k_3 > \left[\dfrac{N}{M}\right] \end{cases} \tag{3.5}$$

In the main result if we take $N=0$ (the polynomial S_0^M will reduce to $A_{0,0}$ which can be taken to be unity without loss of generality), we arrive at a result given by Srivastava *et al* [10, p. 64, eq. (1.1)].

Again, if we put $r=1$, $p=q=0$, $z_2=-1$ in the main result, and further reduce the Fox's H-function thus obtained to $\exp(-z_1)$ [9, p. 18, eq. (2.6.2)] and let $z_1 \to 0$, the Fox's H-function reduces to unity and we arrive at a result which in essence is the same as that given by Rashmi Jain [3, pp. 102–103, eqs (3.5), (3.6)].

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On L^1-convergence of modified complex trigonometric sums

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Abstract. We study here L^1-convergence of a complex trigonometric sum and obtain a new necessary and sufficient condition for the L^1-convergence of Fourier series.

Keywords. L^1-convergence of modified complex trigonometric sums; L^1-convergence of Fourier series; Dirichlet kernel; Fejér kernel.

1. Introduction

It is well known that if a trigonometric series converges in L^1 to a function $f \in L^1$, then it is the Fourier series of the function f. Riesz [1, Vol. II, Ch. VIII §22] gave a counter example showing that in a metric space L we cannot expect the converse of the abovesaid result to hold true. This motivated the various authors to study L^1-convergence of trigonometric series. During their investigations some authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that they converge in L^1-metric to the sum of the trigonometric series whereas the classical series itself may not.

Let the partial sums of the complex trigonometric series

$$\sum_{|n| \leqslant \infty} c_n e^{int}$$

be denoted by

$$S_n(C, t) = \sum_{|k| \leqslant n} c_k e^{ikt}, t \in T = \mathbb{R}/2\pi Z.$$

If the trigonometric series is the Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$ for all n and $S_n(C, t) = S_n(f, t) = S_n(f)$.

If $a_k = o(1)$ as $k \to \infty$, and $\sum_{k=1}^{\infty} k^2 |\Delta^2(a_k/k)| < \infty$, then we say that the series $\sum_{k=1}^{\infty} a_k \Phi_k(x)$, where $\Phi_k(x)$ is $\cos kx$ or $\sin kx$, belongs to the class $\mathbb{R}$. Kano [2] proved that if $\sum_{k=1}^{\infty} a_k \Phi_k(x)$ belongs to the class $\mathbb{R}$, then it is a Fourier series or equivalently, it represents an integrable function. Ram and Kumari [3] introduced modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \sin kx$$

and studied their L^1-convergence. The aim of this paper is to study the L^1-convergence of the complex form of the above sums.

Let

$$D_n(t) = \frac{1}{2} + \sum_{m=1}^{n} \cos mt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

$$\tilde{D}_n(t) = \sum_{m=1}^{n} \sin mt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

and

$$\tilde{K}_n(t) = \frac{1}{n+1}\sum_{m=0}^{n} \tilde{D}_m(t) = \frac{1}{4\sin^2\frac{t}{2}}\left[\sin t - \frac{\sin(n+1)t}{n+1}\right]$$

denote the Dirichlet's kernel, the conjugate Dirichlet's kernel, and the conjugate Fejér's kernel respectively. Let $E_n(t) = \sum_{k=0}^{n} e^{ikt}$. Then the first differentials $D_n'(t)$ and $\tilde{D}_n'(t)$ of $D_n(t)$ and $\tilde{D}_n(t)$ can be written as

$$D_n'(t) = E_n'(t) + E_{-n}'(t)$$

$$2i\tilde{D}_n'(t) = E_n'(t) - E_{-n}'(t),$$

where $E_n'(t)$ denotes the first differential of $E_n(t)$. The complex form of the above modified sums is

$$g_n(C,t) = S_n(C,t) + \frac{i}{n+1}[c_{n+1}E_n'(t) - c_{-(n+1)}E_{-n}'(t)].$$

We introduce here a new class R^* of sequence as follows:

Definition. A null sequence $\langle c_n \rangle$ of complex numbers belongs to the class R^* if

$$\sum_{k=1}^{\infty} \left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right| k\log k < \infty, \tag{1.1}$$

$$\sum_{k=1}^{\infty} k^2 \left|\Delta^2\left(\frac{c_k}{k}\right)\right| < \infty. \tag{1.2}$$

2. Lemmas. The proof of our result is based upon the following lemmas, of which the first three are due to Sheng [4]:

Lemma 1. $\|D_n'(t)\|_1 = 4/\pi(n\log n) + o(n)$

Lemma 2. $\|\tilde{D}_n'(t)\|_1 = o(n\log n)$.

Lemma 3. For each non-negative integer n, there holds

$$\| c_n E'_n(t) + c_{-n} E'_{-n}(t) \|_1 = o(1), \quad n \to \infty$$

if and only if $nc_n \log|n| = o(1)$, $|n| \to \infty$, *where* $\langle c_n \rangle$ *is a complex sequence.*

Lemma 4. (i) There exist positive constants α *and* β *such that*

$$\alpha(\log n) \leqslant \| \tilde{K}_n(t) \|_1 \leqslant \beta(\log n)$$

(ii) $\| \tilde{K}'_n(t) \|_1 = o(n)$.

Proof. The existence of β follows from the fact that $\| \tilde{D}_n(t) \|_1 = o(\log n)$. Further, we have

$$\begin{aligned} 2\pi \| \tilde{K}_n(t) \|_1 &\geqslant \int_0^{\pi} \tilde{K}_n(t)\, dt \\ &= \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \int_0^{\pi} \sin kt\, dt \\ &= \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) (1 - \cos k\pi)/k \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \left[\sum_{j=0}^{k} (1 - \cos j\pi)/j \right] \\ &\geqslant M(\log n!)/(n+1) \end{aligned}$$

for some constant M, the last step being the consequence of the relation $\sum_{v=1}^{n} \log v = \log n!$. Using Sterling's asymptotic formula $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$, we then have

$$\| \tilde{K}_n(t) \|_1 \geqslant \alpha \log n.$$

This completes the proof of (i). To prove (ii) we have,

$$|\tilde{D}'_n(t)| = \left| \sum_{k=0}^{n} k \cos kt \right| \leqslant n(n+1)/2$$

and so

$$|\tilde{K}'_n(t)| \leqslant (n+1)^{-1} \sum_{k=0}^{n} |\tilde{D}'_k(t)| = o(n^2).$$

This implies that

$$\int_{|t| \leqslant \pi/n} |\tilde{K}'_n(t)|\, dt = o(n).$$

Differentiating $\tilde{K}_n(t)$ we get

$$\tilde{K}'_n(t) = \Sigma_{1n}(t) - \Sigma_{2n}(t) + \Sigma_{3n}(t),$$

where

$$\Sigma_{1n}(t) = \{\cos t - \cos(n+1)t\} \Big/ \left(4 \sin^2 \frac{t}{2}\right),$$

$$\Sigma_{2n}(t) = (2\sin^2 t)\Big/\left(2\sin\frac{t}{2}\right)^4,$$

$$\Sigma_{3n}(t) = \{2\sin t \sin(n+1)t\}/(n+1)\left(2\sin\frac{t}{2}\right)^4.$$

Obviously, $|\Sigma_{jn}(t)| = o(|t|^{-2})$ for $j = 1, 2$, and $(n+1)|\Sigma_{3n}(t)| = o(|t|^{-3})$. Using these estimates, we get

$$\int_{\pi/n \leqslant |t| \leqslant \pi} |\tilde{K}_n'(t)|\,dt = o\left(\int_{\pi/n \leqslant |t| \leqslant \pi} t^{-2}\,dt\right) + o\left(\frac{1}{n+1}\int_{\pi/n \leqslant |t| \leqslant \pi} t^{-3}\,dt\right)$$

$$= o(n).$$

Combining the above estimates, we infer that $\|\tilde{K}_n'(t)\|_1 = o(n)$.

Lemma 5. Let $n \geqslant 1$ and $0 < \varepsilon < \pi$. Then there exists $A_\varepsilon > 0$ such that for all $\varepsilon \leqslant |t| \leqslant \pi$

(i) $|E_n'(t)| \leqslant A_\varepsilon n/|t|$,
(ii) $|E'_{-n}(t)| \leqslant A_\varepsilon n/|t|$,
(iii) $|D_n'(t)| \leqslant 2A_\varepsilon n/|t|$, *and*
(iv) $|\tilde{D}_n'(t)| \leqslant A_\varepsilon n/|t|$.

Proof. We have

$$i^{-1}E_n'(t) = \sum_{k=0}^{n} k\,e^{ikt} = \sum_{k=0}^{n} k(E_k(t) - E_{k-1}(t))$$

$$= \sum_{k=0}^{n} (\Delta k)E_k(t) + (n+1)E_n(t).$$

Since $|E_n(t)| \leqslant A_\varepsilon/|t|$ for some constant A_ε, we have

$$|E_n'(t)| \leqslant \frac{A_\varepsilon}{|t|}\left[\sum_{k=0}^{n} 1 + (n+1)\right] \leqslant A_\varepsilon n/|t|.$$

Since

$$E'_{-n}(t) = (-1)E_n'(-t),$$

we obtain $|E'_{-n}(t)| \leqslant A_\varepsilon n/|t|$. The other two inequalities follow from $D_n'(t) = E_n'(t) + E'_{-n}(t)$ and $2i\tilde{D}_n'(t) = E_n'(t) - E'_{-n}(t)$.

3. Main theorem

We prove the following result.

Theorem. *Let $c_n \in R^*$. Then there exists $f(t)$ such that*

$$\lim_{n\to\infty} g_n(C, t) = f(t) \text{ for all } 0 < |t| \leqslant \pi, \tag{3.1}$$

$$f(t) \in L^1(T) \text{ and } \|g_n(C, t) - f(t)\|_1 = o(1) \text{ as } n \to \infty, \tag{3.2}$$

$$\|S_n(f, t) - f(t)\|_1 = o(1) \text{ as } n \to \infty \text{ if and only if } \hat{f}(n)\log|n| = o(1) \text{ as } |n| \to \infty. \tag{3.3}$$

Proof. We have, by using Abel's transformation,

$$g_n(C,t) = S_n(C,t) + \frac{\mathrm{i}}{n+1}[c_{n+1}E'_n(t) - c_{-(n+1)}E'_{-n}(t)]$$

$$= 2\sum_{k=1}^{n} \Delta\left(\frac{c_k}{k}\right)\tilde{D}'_k(t) + \sum_{k=1}^{n} \Delta\left(\frac{c_{-k}-c_k}{k}\right)\mathrm{i}E'_{-k}(t).$$

By Lemma 5, we get

$$\sum_{k=1}^{\infty}\left|\Delta\left(\frac{c_k}{k}\right)\tilde{D}'_k(t)\right| \leqslant (A_1/|t|)\sum_{k=1}^{\infty}\left\{k\left|\Delta\left(\frac{c_k}{k}\right)\right|\right\}$$

$$\leqslant (A_1/|t|)\left\{\sum_{k=1}^{\infty}\sum_{j=k}^{\infty} k\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right\}$$

$$= (A_1/|t|)\left\{\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j} k\right)\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right\}$$

$$= O\left\{(A_1/|t|)\left(\sum_{j=1}^{\infty} j^2\right)\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right\} < \infty,$$

and

$$\sum_{k=3}^{\infty}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t)\right| \leqslant (A_1/|t|)\left\{\sum_{k=3}^{\infty} k\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|\right\}$$

$$\leqslant (A_1/|t|)\left\{\sum_{k=3}^{\infty} k(\log k)\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|\right\} < \infty,$$

where A_1 is a suitable constant. These imply that

$$f(t) = 2\left\{\sum_{k=1}^{\infty}\Delta\left(\frac{c_k}{k}\right)\tilde{D}'_k(t)\right\} + \mathrm{i}\left\{\sum_{k=1}^{\infty}\Delta\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t)\right\}$$

exists and thus (3.1) follows.

Further, for $t \neq 0$, we have

$$f(t) - g_n(C,t) = 2\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_k}{k}\right)\tilde{D}'_k(t) + \mathrm{i}\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t)$$

$$= 2\sum_{k=n+1}^{\infty}(k+1)\Delta^2\left(\frac{c_k}{k}\right)\tilde{K}'_k(t) - 2(n+1)\Delta\left(\frac{c_{n+1}}{n+1}\right)\tilde{K}'_n(t)$$

$$+ \mathrm{i}\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t).$$

Thus

$$\|f(t) - g_n(C,t)\|_1 \leqslant 2\sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\left(\frac{c_k}{k}\right)\right|\int_{-\pi}^{\pi}|\tilde{K}'_k(t)|\,\mathrm{d}t$$

$$+ 2(n+1)\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}|\tilde{K}'_{n+1}(t)|\,\mathrm{d}t$$

$$+\sum_{k=n+1}^{\infty}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|\int_{-\pi}^{\pi}|E'_{-k}(t)|\,dt.$$

But, by Lemma 4,

$$\int_{-\pi}^{\pi}|\tilde{K}'_k(t)|\,dt=o(k).$$

Also

$$\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right|=\left|\sum_{k=n+1}^{\infty}\Delta^2\left(\frac{c_k}{k}\right)\right|$$

$$\leqslant\sum_{k=n+1}^{\infty}\frac{k^2}{k^2}\left|\Delta^2\left(\frac{c_k}{k}\right)\right|\leqslant(n+1)^{-2}\sum_{k=n+1}^{\infty}k^2\left|\Delta^2\left(\frac{c_k}{k}\right)\right|$$

$$=o((n+1)^{-2}),$$

by the hypothesis of the theorem. Lemma 1 and Lemma 2 imply that

$$\int_{-\pi}^{\pi}|E'_{-k}(t)|\,dt=o(k\log k).$$

Therefore,

$$\|f(t)-g_n(C,t)\|_1=o\left(\sum_{k=n+1}^{\infty}(k+1)^2\left|\Delta^2\left(\frac{c_k}{k}\right)\right|\right)+o(1)$$

$$+o\left(\sum_{k=n+1}^{\infty}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|k\log k\right)$$

$$=o(1),\text{ by the hypothesis of the theorem.}$$

Since $g_n(C,t)$ is a polynomial, it follows that $f\in L^1(T)$, which proves the assertion (3.2).

We notice further that

$$\|f-S_n(f)\|_1=\|f-g_n(C,t)+g_n(C,t)-S_n(f)\|_1$$

$$\leqslant\|f-g_n(C,t)\|_1+\|g_n(C,t)-S_n(f)\|_1$$

$$=\|f-g_n(C,t)\|_1+\left\|\frac{i}{n+1}(\hat{f}(n+1))E'_n(t)-\hat{f}(-(n+1))E'_{-n}(t))\right\|_1$$

and

$$\left\|\frac{i}{n+1}(\hat{f}(n+1)E'_n(t)-\hat{f}(-(n+1))E'_{-n}(t))\right\|_1=\|g_n(C,t)-S_n(f)\|_1$$

$$\leqslant\|f-S_n(f)\|_1+\|f-g_n(C,t)\|_1.$$

Since $\|f-g_n(C,t)\|=o(1)$, $n\to\infty$, by (3.2), and by Lemma 3, $\|\hat{f}(n)E'_n(t)-\hat{f}(-n)E'_{-n}(t)\|_1=o(n)$, $n\to\infty$ if and only if $\hat{f}(n)\log|n|=o(1)$, $|n|\to\infty$, the assertion (3.3) follows.

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Absolute summability of infinite series

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Abstract. It is shown in [4] that if a normal matrix A satisfies some conditions then $|C,1|_k$ summability implies $|A|_k$ summability where $k \geqslant 1$. In the present paper, we consider the converse implication.

Keywords. Normal matrix; $|C,1|_k$ summability; $|A|_k$ summability.

1. Introduction

By u_n^α and t_n^α we denote, respectively, the Cesàro means of order $\alpha(\alpha > -1)$ of the sequences (s_n) and (r_n), where (s_n) is the partial sums of the series Σx_n and $r_n = nx_n$. The series Σx_n is then called absolutely summable (C,α) with index k, or simply summable $|C,\alpha|_k, k \geqslant 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|u_n^\alpha - u_{n-1}^\alpha|^k < \infty, [1]. \tag{1}$$

Since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$, [3], condition (1) can be written in the form

$$\sum_{n=1}^{\infty} n^{-1}|t_n^\alpha|^k < \infty. \tag{2}$$

Let $A = (a_{nv})$ be a normal matrix, i.e., lower-semi matrix with non-zero diagonal entries. By (T_n) we denote the A-transform of the sequence (s_n), i.e.,

$$T_n = \sum_{v=0}^{n} a_{nv}s_v; \quad n = 0, 1, \ldots$$

We say that the series Σx_n is summable $|A|_k, k \geqslant 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|T_n - T_{n-1}|^k < \infty. \tag{3}$$

Given a normal matrix $A = (a_{nv})$, we associate two lower-semi matrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}; \quad n, v = 0, 1, \ldots, \quad \hat{a}_{00} = \bar{a}_{00} = a_{00},$$

$$\hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v} \quad \text{for } n = 1, 2, \ldots$$

If A is a normal matrix, then $A' = (a'_{nv})$ will denote the inverse of A. Clearly, if A is normal then $\hat{A} = (\hat{a}_{nv})$ is normal and it has two-sided inverse $\hat{A}' = (\hat{a}'_{nv})$, which is also normal (see [2]).

Note that, if A is normal then

$$T_n = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \sum_{i=v}^{n} a_{ni} x_v = \sum_{v=0}^{n} \bar{a}_{nv} x_v$$

and

$$\Delta T_{n-1} = T_n - T_{n-1} = \sum_{v=0}^{n} (\bar{a}_{nv} - \bar{a}_{n-1,v}) x_v = \sum_{v=0}^{n} \hat{a}_{nv} x_v \quad (a_{n-1,n} = 0),$$

which implies

$$x_n = \sum_{v=0}^{n} \hat{a}'_{nv} \Delta T_{v-1} \quad (T_{-1} = 0). \tag{4}$$

In connection with the absolute summability we have the following theorem.

Theorem A. *Suppose that, for* $k \geqslant 1$,

$$\sum_{n=1}^{\infty} n^{k-1} |\hat{a}_{n0}|^k < \infty, \quad \sum_{v=1}^{n} |\Delta \hat{a}_{nv}| = O(1/n), \quad \sum_{n=v}^{\infty} |\Delta \hat{a}_{nv}| = O(1/v),$$

$$\sum_{v=1}^{n} (1/v) |\hat{a}_{nv}| = O(1/n) \quad \text{and} \quad \sum_{n=v}^{\infty} |\hat{a}_{nv}| = O(1),$$

then if Σx_n *is summable* $|C, 1|_k$, *it is also summable* $|A|_k$, *where* $\Delta \hat{a}_{nv} = \hat{a}_{nv} - \hat{a}_{n,v+1}$ [4].

Furthermore it is shown in [4] that the conditions of Theorem A are satisfied whenever A is (C, α), $\alpha \geqslant 1$. This deduces that $|C, 1|_k$ summability implies $|C, \alpha|_k$, $k \geqslant 1, \alpha \geqslant 1$, summability which is a well-known result.

We may now ask what conditions should be imposed on $A = (a_{nv})$ so that the converse implication holds in Theorem A. It is the object of this paper to answer this question.

2. The main result

Theorem B. *Let* $A = (a_{nv})$ *be a normal matrix such that*

(i) $1 = O(v a_{vv})$,

(ii) $(a_{vv} - a_{v+1,v}) = O(a_{vv} a_{v+1,v+1})$

(iii) $\sum_{v=i}^{\infty} (v+2) |\hat{a}'_{v+2,i}| = O(i+1)$. (5)

If Σx_n *is summable* $|A|_k$, *then it is also summable* $|C, 1|_k, k \geqslant 1$.

Proof. By T_n and t_n we denote the A-transform and $(C, 1)$-mean of the series Σx_n and the sequence (nx_n), respectively. Then it follows from (3) that

$$\begin{aligned}
t_n &= (n+1)^{-1} \sum_{v=1}^{n} v x_v = (n+1)^{-1} \sum_{v=1}^{n} v \left\{ \sum_{r=0}^{v} \hat{a}'_{vr} \Delta T_{r-1} \right\} \\
&= (n+1)^{-1} \sum_{v=1}^{n} v \left\{ \hat{a}'_{v,v-1} \Delta T_{v-2} + \hat{a}'_{vv} \Delta T_{v-1} + \sum_{r=0}^{v-2} \hat{a}'_{vr} \Delta T_{r-1} \right\} \\
&= (n+1)^{-1} \left\{ \sum_{v=1}^{n-1} v \hat{a}'_{vv} \Delta T_{v-1} + n \hat{a}'_{nn} \Delta T_{n-1} + \sum_{v=2}^{n} v \hat{a}'_{v,v-1} \Delta T_{v-2} + \hat{a}'_{10} \Delta T_{-1} \right. \\
&\quad \left. + \sum_{v=2}^{n} v \sum_{r=0}^{v-2} \hat{a}'_{vr} \Delta T_{r-1} \right\} \\
&= (n+1)^{-1} \sum_{v=1}^{n-1} \{ v \hat{a}'_{vv} + (v+1) \hat{a}'_{v+1,v} \} \Delta T_{v-1} + n(n+1)^{-1} \hat{a}'_{nn} \Delta T_{n-1} \\
&\quad + (n+1)^{-1} \hat{a}'_{10} \Delta T_{-1} + (n+1)^{-1} \sum_{v=0}^{n-2} (v+2) \sum_{r=0}^{v} \hat{a}'_{v+2,r} \Delta T_{r-1}.
\end{aligned}$$

By considering the equality

$$\sum_{k=v}^{n} \hat{a}'_{nk} \hat{a}_{kv} = \delta_{nv},$$

where δ_{nv} is the Kronecker delta, we have

$$\begin{aligned}
v \hat{a}'_{vv} + (v+1) \hat{a}'_{v+1,v} &= v/a_{vv} + (v+1)(-\hat{a}_{v+1,v}/a_{vv} a_{v+1,v+1}) \\
&= v/a_{vv} + (v+1)[-(a_{v+1,v} + a_{v+1,v+1} - a_{vv})/a_{vv} a_{v+1,v+1}] \\
&= (v+1)[1/a_{v+1,v+1} - a_{v+1,v}/a_{vv} a_{v+1,v+1}] - 1/a_{vv}
\end{aligned}$$

and so

$$\begin{aligned}
t_n &= (n+1)^{-1} \sum_{v=1}^{n-1} \{ (v+1)[1/a_{v+1,v+1} - a_{v+1,v}/a_{vv} a_{v+1,v+1}] - 1/a_{vv} \} \Delta T_{v-1} \\
&\quad + n(n+1)^{-1} (1/a_{nn}) \Delta T_{n-1} + \hat{a}'_{10} \Delta T_{-1} (n+1)^{-1} \\
&\quad + (n+1)^{-1} \sum_{r=0}^{n-2} \Delta T_{r-1} \sum_{v=r}^{n-2} (v+2) \hat{a}'_{v+2,r}
\end{aligned}$$

which implies, by virtue of (5i), (5ii) and (5iii), that

$$\begin{aligned}
t_n &= O\left\{ (n+1)^{-1} \sum_{v=1}^{n-1} v |\Delta T_{v-1}| + n |\Delta T_{n-1}| + (n+1)^{-1} \right\} \\
&= w_{n,1} + w_{n,2} + w_{n,3}, \text{ say.}
\end{aligned}$$

To prove the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} n^{-1} |w_{n,i}|^k < \infty, \quad \text{for } i = 1, 2, 3.$$

Now it follows from Hölder's inequality that

$$\sum_{n=2}^{m+1} n^{-1}|w_{n,1}|^k = O\left\{\sum_{n=2}^{m+1} n^{-k-1}\left\{\sum_{v=1}^{n-1} v|\Delta T_{v-1}|\right\}^k\right\}$$

$$= O\left\{\left(\sum_{n=2}^{m+1} n^{-2}\sum_{v=1}^{n-1} v^k|\Delta T_{v-1}|^k\right)\left(n^{-1}\sum_{v=1}^{n-1} 1\right)^{k-1}\right\}$$

$$= O\left\{\sum_{v=1}^{m} v^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} n^{-2}\right\}$$

$$= O\left\{\sum_{v=1}^{\infty} v^{k-1}|\Delta T_{v-1}|^k\right\} < \infty,$$

and

$$\sum_{n=1}^{\infty} n^{-1}|w_{n,2}|^k = O\left\{\sum_{n=1}^{\infty} n^{-1}|n\Delta T_{n-1}|^k\right\} < \infty.$$

Finally,

$$\sum_{n=1}^{\infty} n^{-1}|w_{n,3}|^k = O\left\{\sum_{n=1}^{\infty} n^{-k-1}\right\} < \infty, \quad k \geqslant 1.$$

Hence the proof of the theorem is completed.

3. Applications

Let (p_n) be a sequence of positive real numbers such that $P_n = p_0 + p_1 + \cdots + p_n$, $P_{-1} = p_{-1} = 0$. The Riesz (weighted mean) matrix is defined by $a_{nv} = p_v/P_n$ for $0 \leqslant v \leqslant n$ and $a_{nv} = 0$ for $v > n$. From now on, we suppose that $A = (a_{nv})$ is a weighted mean matrix with $P_n \to \infty$ and $n \to \infty$. Hence if no confusion is likely to arise, we say that Σx_n is summable $|R, p_n|_k$, $k \geqslant 1$, if (3) holds.

With this notation we have

COROLLARY 1

Let (p_n) be a sequence of positive real numbers such that $P_n = O(np_n)$. Then if Σx_n is summable $|R, p_n|_k$, it is also summable $|C, 1|_k$, $k \geqslant 1$.

Proof. Applying Theorem B with $A = (a_{nv})$, a weighted mean matrix, we see that (5ii) clearly holds and (5i) is reduced to the condition $P_n = O(np_n)$. On the other hand, a small calculation reveals that

$$\bar{a}_{nv} = (P_n - P_{v-1})/P_n, \quad \hat{a}_{nv} = p_n P_{v-1}/P_n P_{n-1}$$

and

$$\hat{a}'_{nv} = \begin{cases} -P_{n-2}/p_{n-1} & \text{if } v = n-1 \\ P_n/p_n & \text{if } v = n \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get

$$\sum_{v=i}^{\infty} (v+2)|\hat{a}'_{v+2,i}| = 0 \text{ for all } i,$$

and so the proof is completed.

COROLLARY 2

Let (p_n) be a sequence of positive real numbers with $np_n = O(P_n)$. Then if Σx_n is summable $|C,1|_k$, it is also summable $|R,p_n|_k$, $(k \geqslant 1)$.

Proof. Apply Theorem A.

Now the next result which appears in [5] is a consequence of Corollaries 1 and 2.

COROLLARY 3

Suppose that (p_n) is a sequence of positive real numbers such that

$$np_n = O(P_n) \quad \textit{and} \quad P_n = (nP_n).$$

Then the summability $|C,1|_k$ is equivalent to the summability $|R,p_n|_k, k \geqslant 1$.

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Solution of a system of nonstrictly hyperbolic conservation laws

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Abstract. In this paper we study a special case of the initial value problem for a 2×2 system of nonstrictly hyperbolic conservation laws studied by Lefloch, whose solution does not belong to the class of L^∞ functions always but may contain δ-measures as well. Lefloch's theory leaves open the possibility of nonuniqueness for some initial data. We give here a uniqueness criteria to select the entropy solution for the Riemann problem. We write the system in a matrix form and use a finite difference scheme of Lax to the initial value problem and obtain an explicit formula for the approximate solution. Then the solution of initial value problem is obtained as the limit of this approximate solution.

Keywords. System of conservation laws; delta waves; explicit formula

1. Introduction

The standard theory of hyperbolic systems of conservation laws assumes usually the systems to be strictly hyperbolic with genuinely nonlinear or linearly degenerate characteristic fields, see Lax [6] and Glimm [1]. But many of the hyperbolic systems which come in applications do not satisfy these assumptions and such cases were studied by many authors [3, 5, 8]. In all these papers solutions are found in the sense of distributions, say in the class of L^∞ functions. In a very interesting paper, Lefloch [7] considered a system of conservation laws, namely

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(a(u)v) = 0,$$

with initial conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \tag{1.2}$$

where $a(u) = f'(u)$ and $f: R \to R$ is a strictly convex function. For systems of this type generally there is neither existence nor uniqueness in the class of entropy weak solutions in the sense of distributions. He has shown that when $u_0 \in L^1(R) \cap BV(R)$ and $v_0 \in L^\infty(R) \cap L^1(R)$ (1.1) and (1.2) has at least one solution $(u, v) \in L^\infty(R_+, BV(R)) \times L^\infty(R_+, M(R))$ given by

$$u(x,t) = (f^*)'\left(\frac{x - y_0(x,t)}{t}\right),$$

$$v(x,t) = \frac{\partial}{\partial x}\int_{-\infty}^{y_0(x,t)} v_0(z)\,dz,$$

where $y = y_0(x,t)$ minimizes

$$\min_{-\infty<y<\infty}\left[\int_{-\infty}^{y} u_0(z)\,dz + tf^*\left(\frac{x-y}{t}\right)\right]$$

and f^* is the convex dual of $f(u)$ and $M(R)$ is the space of bounded borel measures on R. Further he proved that if u_0 satisfies

$$\frac{du_0}{dx} \leqslant K_0 \tag{1.3}$$

in the sense of distributions for some K_0, then the problem (1.1) and (1.2) has one and only one entropy solution. If we take

$$u_0(x) = \begin{cases} u_L & \text{if } x<0 \\ u_R & \text{if } x>0 \end{cases}$$

then (1.3) is equivalent to saying

$$(u_R - u_L)\varphi(0) \leqslant K_0 \quad \text{for all } \varphi \in C_0^\infty(R),\ \varphi \geqslant 0$$

and this will be true for some K_0 and for all $\varphi \in C_0^\infty(R)$, $\varphi \geqslant 0$, iff $u_L \geqslant u_R$. In fact for the Riemann problem, i.e., when the initial data for (1.1) is of the form

$$(u(x,0), v(x,0)) = \begin{matrix}(u_L, v_L) & \text{if } x<0 \\ (u_R, v_R) & \text{if } x>0,\end{matrix} \tag{1.4}$$

Lefloch [7] has given an infinite number of solution for the case $u_L < u_R$.

In this paper we study a criteria to choose the correct entropy solution. Classically, vanishing viscosity method or proper numerical approximations are used to choose the correct entropy solution. Following Hopf [2], vanishing viscosity method was used by Joseph [4] to pick up the unique solution for the Riemann problem when $f(u) = u^2/2$ in (1.1). It was shown that in the case, $u_L < u_R$, which is the case of non-uniqueness, the v component of the vanishing viscosity solution is

$$v(x,t) = \begin{cases} v_L, & \text{if } x < u_L t \\ 0, & \text{if } u_L t < x < u_R t \\ v_R, & \text{if } x > u_R t. \end{cases}$$

In other words in the rarefaction fan region of u component, the v component is zero.

In the present paper, we consider the special case $f(u) = \log[ae^u + be^{-u}]$, $a+b=1$, $a>0$, $b>0$ are constants in (1.1). Then we have

$$u_t + (\log(ae^u + be^{-u}))_x = 0, \tag{1.5}$$

$$v_t + \left(\frac{ae^u - be^{-u}}{ae^u + be^{-u}} v\right)_x = 0,$$

and study the unique choice of solution. Here we use a numerical approximation of Lax [6], which he used to pick the correct entropy solution for a scalar conservation law. For the Riemann problem, we show that in the rarefaction fan region of u, the v component is zero, see Theorem 1. These examples suggest a uniqueness criteria at least for the Riemann problem.

Before stating our main results let us introduce the difference approximation. To do this first we note that (1.5) can be written in the matrix form

$$A_t + [\log(ae^A + be^{-A})]_x = 0, \tag{1.6}$$

where

$$A = \begin{pmatrix} u & 0 \\ v & u \end{pmatrix}. \tag{1.7}$$

Let Δx and Δt be spatial and time mesh sizes and let

$$A_k^n \simeq A(k\Delta x, n\Delta t), \quad k = 0, \pm 1, \pm 2, \ldots, \quad n = 0, 1, 2, \ldots \tag{1.8}$$

and following Lax [6], define the difference approximation

$$A_k^n = A_k^{n-1} + \frac{\Delta t}{\Delta x}[g(A_{k-1}^{n-1}, A_k^{n-1}) - g(A_k^{n-1}, A_{k+1}^{n-1})], \tag{1.9}$$

where the numerical flux $g(A, B)$ is given by

$$g(A, B) = \log[ae^A + be^{-B}]. \tag{1.10}$$

Here we can take $\Delta t = \Delta x = \Delta$, since the characteristic speed of the eigenvalues $\lambda_1 = \lambda_2 = \dfrac{ae^u - be^{-u}}{ae^u + be^{-u}}$ of (1.5) which are less than one in modulus. Then we note that (1.9) and (1.10) become

$$A_k^n = A_k^{n-1} + \log[ae^{A_{k-1}^{n-1}} + be^{-A_k^{n-1}}] - \log[ae^{A_k^{n-1}} + be^{-A_{k+1}^{n-1}}] \tag{1.11}$$

with initial condition

$$A_k^0 = \begin{pmatrix} u_k^0 & 0 \\ v_k^0 & u_k^0 \end{pmatrix}. \tag{1.12}$$

When

$$A = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix},$$

(1.11) is nothing but the Lax scheme for the scalar equation $u_t + (\log[ae^u + be^{-u}])_x = 0$.

With the notations

$$s = (\log[ae^{u_R} + be^{-u_R}] - \log[ae^{u_L} + be^{-u_L}])/(u_R - u_L) \tag{1.13}$$

and

$$R(u_L, u_R, v_L, v_R) = s(v_R - v_L) - \frac{ae^{u_R} - be^{-u_R}}{ae^{u_R} + be^{-u_R}} v_R + \frac{ae^{u_L} - be^{-u_L}}{ae^{u_L} + be^{-u_L}} v_L, \tag{1.14}$$

we shall prove the following results.

Theorem 1. *Let* $(u^{\Delta}(x,t), v^{\Delta}(x,t))$ *be the approximate solution of* (1.1) *defined by* (1.11) *and* (1.12) *with Riemann initial data* (1.4), *then*

$$\lim_{\Delta \to 0} (u^{\Delta}(x,t), v^{\Delta}(x,t)) = (u(x,t), v(x,t))$$

exists in the sense of distributions and $(u(x,t), v(x,t))$ *is given by the following explicit formula:*

(i) *When* $u_L > u_R$, *then*

$$(u(x,t), v(x,t)) = \{u_L + (u_R - u_L)H(x - st), v_L + (v_R - v_L)H(x - st) + R(u_L, u_R, v_L, v_R)t\delta_{x=st}\},$$

where $H(x)$ *is the Heaviside function.*

(ii) *When* $u_L < u_R$, *then*

$$(u(x,t), v(x,t)) = \begin{cases} (u_L, v_L) & \text{if } x < \left(\dfrac{ae^{u_L} - be^{-u_L}}{ae^{u_L} + be^{-u_L}}\right)t \\ \left(\dfrac{1}{2}\log\left(\dfrac{b}{a}\cdot\dfrac{t+x}{t-x}\right), 0\right) & \text{if } \dfrac{ae^{u_L} - be^{-u_L}}{ae^{u_L} + be^{-u_L}}t < x < \dfrac{ae^{u_R} - be^{-u_L}}{ae^{u_R} + be^{-u_L}}t \\ (u_R, v_R) & \text{if } x > \dfrac{ae^{u_R} - be^{-u_R}}{ae^{u_R} + be^{-u_R}}t. \end{cases}$$

(iii) *When* $u_L = u_R = \bar{u}$, *then*

$$(u(x,t), v(x,t)) = \begin{cases} (\bar{u}, v_L), & \text{if } x < a(\bar{u})t, \\ (\bar{u}, v_R), & \text{if } x \geqslant a(\bar{u})t. \end{cases}$$

Theorem 2. *Let the initial data* $u^0(x)$ and $v^0(x) \in L^{\infty}(R) \cap L^1(R)$. *Then* $(u^{\Delta}(x,t), v^{\Delta}(x,t))$ *defined by* (1.11) *and* (1.12) *tends to* $(u(x,t), v(x,t))$ *in the sense of distributions and is given by*

$$u(x,t) = \frac{1}{2}\log\left[\frac{b}{a}\cdot\frac{t + x - y_0(x,t)}{t - x + y_0(x,t)}\right],$$

$$v(x,t) = -\frac{\partial}{\partial x}\int_{y_0(x,t)}^{\infty} v_0(z)\,dz,$$

where $y = y_0(x,t)$ *maximizes*

$$\max_{x-t \leqslant y \leqslant x+t}\left[\int_y^{\infty} u_0(z)dz - tf^*\left(\frac{x-y}{t}\right)\right].$$

Here $f^*(\lambda)$ is the convex dual of $f(u) = \log[ae^u + be^{-u}]$ and is given by

$$f^*(\lambda) = (1/2)\log(1+\lambda)^{1+\lambda}(1-\lambda)^{1-\lambda} - 1/2\log\{4a^{1+\lambda}b^{1-\lambda}\}.$$

2. Proof of Theorem 1

As a first step in the proof of Theorem 1, we obtain $(u^{\Delta}(x,t), v^{\Delta}(x,t))$ explicitly. In order to do this we recall from (1.11), (1.12) and (1.4),

$$A_k^n = A_k^{n-1} + \log[ae^{A_{k-1}^{n-1}} + be^{-A_k^{n-1}}] - \log[ae^{A_k^{n-1}} + be^{-A_{k+1}^{n-1}}] \tag{2.1}$$

for $n = 1, 2, 3, \ldots, k = 0, \pm 1, \pm 2, \ldots$, with

$$A_k^0 = \begin{cases} A_R = \begin{pmatrix} u_R & 0 \\ v_R & u_R \end{pmatrix} & \text{if } k \geqslant 0 \\ A_L = \begin{pmatrix} u_L & 0 \\ v_L & u_L \end{pmatrix} & \text{if } k < 0. \end{cases} \tag{2.2}$$

Let us set

$$C_k^n = A_k^n - A_R \tag{2.3}$$

then, (2.1) becomes

$$\begin{aligned} C_k^n = C_k^{n-1} &+ \log[ae^{A_R + C_{k-1}^{n-1}} + be^{-A_R - C_k^{n-1}}] \\ &- \log[ae^{A_R + C_k^{n-1}} + be^{-A_R - C_{k+1}^{n-1}}]. \end{aligned} \tag{2.4}$$

Let

$$D_k^n = \sum_{j=k}^{\infty} C_j^n. \tag{2.5}$$

Taking summation in (2.4) from k to ∞, we have

$$\begin{aligned} D_k^n = D_k^{n-1} &+ \log[ae^{A_R} e^{(D_{k-1}^{n-1} - D_k^{n-1})} + be^{-A_R} e^{-(D_k^{n-1} - D_{k+1}^{n-1})}] \\ &- \log(ae^{A_R} + be^{-A_R}). \end{aligned} \tag{2.6}$$

Following Lax [6], we use the nonlinear transformation,

$$D = \log E \tag{2.7}$$

in (2.6), and obtain

$$\begin{aligned} \log E_k^n = \log E_k^{n-1} &+ \log[ae^{A_R}(E_k^{n-1})^{-1} E_{k-1}^{n-1} + be^{-A_R}(E_k^{n-1})^{-1} E_{k+1}^{n-1}] \\ &- \log(ae^{A_R} + be^{-A_R}). \end{aligned}$$

Simplifying this we get

$$E_k^n = \alpha E_{k-1}^{n-1} + \beta E_{k+1}^{n-1}, \tag{2.8}$$

where

$$\alpha = ae^{A_R}(ae^{A_R} + be^{-A_R})^{-1} \quad \text{and} \quad \beta = be^{-A_R}(ae^{A_R} + be^{-A_R})^{-1}. \tag{2.9}$$

We note that $\alpha + \beta = I$. It can be easily seen that the solution E_k^n of (2.8) is given by

$$E_k^n = \sum_{q=0}^{n} \binom{n}{q} \alpha^q \beta^{n-q} E_{n+k-2q}^0. \tag{2.10}$$

From (2.2), (2.3), (2.5) and (2.7) we get

$$E_{n+k-2q}^0 = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } n+k-2q \geqslant 0 \\ e^{(n+k-2q)(u_L - u_R)} \begin{pmatrix} 1 & 0 \\ (n+k-2q)(v_L - v_R) & 1 \end{pmatrix}, & \text{if } n+k-2q < 0. \end{cases} \tag{2.11}$$

Using (2.11) in (2.10) we get

$$E_k^n = \begin{pmatrix} \theta_k^n & 0 \\ \eta_k^n & \theta_k^n \end{pmatrix}, \tag{2.12}$$

where

$$\theta_k^n = \frac{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} e^{(2q-n)u_R} e^{S(n,k,q)(u_L-u_R)}}{(ae^{u_R}+be^{-u_R})^n} \tag{2.13}$$

and

$$\eta_k^n = \frac{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} e^{(2q-n)u_R} e^{S(n,k,q)(u_L-u_R)} \cdot \left\{ \frac{2v_R(qbe^{-u_R}+(q-n)ae^{u_R})}{(ae^{u_R}+be^{-u_R})} + S(n,k,q)(v_L-v_R) \right\}}{(ae^{u_R}+be^{-u_R})^n}. \tag{2.14}$$

Here we used the notation $S(n,k,q) = \frac{1}{2}(n+k-2q-|n+k-2q|)$. Now

$$\log E_k^n = (\log \theta_k^n) I + \log \left[I + \begin{pmatrix} 0 & 0 \\ \eta_k^n/\theta_k^n & 0 \end{pmatrix} \right],$$

i.e.

$$\log E_k^n = \begin{pmatrix} \log \theta_k^n & 0 \\ \eta_k^n/\theta_k^n & \log \theta_k^n \end{pmatrix}. \tag{2.15}$$

By the transformations (2.3), (2.5) and (2.7) we get,

$$\begin{aligned} \Delta \log E_k^n &= \Delta D_k^n \\ &= \Delta \sum_k^\infty C_k^n \\ &= \Delta \sum_k^\infty (A_k^n - A_R). \end{aligned} \tag{2.16}$$

Componentwise this becomes

$$\Delta \sum_{j=k}^{\infty} (u_j^n - u_R) = \Delta \log \theta_k^n \tag{2.17}$$

$$\Delta \sum_{j=k}^{\infty} (v_j^n - v_R) = \Delta \frac{\eta^n k}{\theta_k^n}. \tag{2.18}$$

By using Stirling's formula,

$$n! \approx \left(\frac{n}{e}\right)^n (2\pi n)^{1/2}, \quad \text{as } n \to \infty,$$

we get

$$\binom{n}{q} = \frac{n!}{q!(n-q)!} \approx \frac{n^n n^{1/2}}{q^q q^{1/2} (n-q)^{n-q} (2\pi)^{1/2} (n-q)^{1/2}}, \quad \text{as } n, q, n-q \to \infty.$$

Let $t = n\Delta$, $x = k\Delta$, $y = (n+k-2q)\Delta$ be fixed, then

$$q\Delta = \frac{t+x-y}{2}, \quad (2q-n)\Delta = x-y. \tag{2.19}$$

We have,

$$\lim_{\Delta\to 0} \Delta \log \theta_k^n = \max_{0\leqslant (t+x-y)/2 \leqslant t} \left[\Delta \log \binom{n}{q} + \Delta q \log a + \Delta(n-q)\log b \right.$$
$$\left. + \Delta(2q-n)u_R - \tfrac{1}{2}(y-|y|)(u_L-u_R) - t\log(ae^{u_R}+be^{-u_R}) \right]. \tag{2.20}$$

Also as $\Delta\to 0$ in the above fashion, we have

$$\Delta \log \binom{n}{q} \approx \log \left[\frac{t^t}{\left(\frac{t+x-y}{2}\right)^{(t+x-y)/2} \cdot \left(\frac{t-x+y}{2}\right)^{(t-x+y)/2}} \right]. \tag{2.21}$$

and hence from (2.19)–(2.21), we get

$$\lim_{\Delta\to 0} \Delta \log \theta_k^n = \max_{x-t\leqslant y\leqslant x+t} [-1/2(y-|y|)(u_L-u_R)$$
$$+ (x-y)u_R - t\log(ae^{u_R}+be^{-u_R})]$$
$$+ \frac{(t+x-y)}{2}\log a + \frac{(t-x+y)}{2}\log b$$
$$+ \log \left[\frac{t^t}{\left(\frac{t+x-y}{2}\right)^{(t+x-y)/2} \left(\frac{t-x+y}{2}\right)^{(t-x+y)/2}} \right]. \tag{2.22}$$

Let $y_0(x,t)$ be the value of y for which maximum is attained on the RHS of (2.22). An easy calculation shows that the following is true.

Lemma. Let $y_0(x,t)$ be a point where maximum is attained on the RHS of (2.22), *then $y_0(x,t)$ is given by the following*:

(i) *Let $u_L > u_R$, then*

$$y_0(x,t) = \begin{cases} x - a(u_L)t, & \text{if } x < st \\ x - a(u_R)t, & \text{if } x > st. \end{cases}$$

(ii) *Let $u_L < u_R$, then*

$$y_0(x,t) = \begin{cases} x - a(u_L)t, & \text{if } x < a(u_L)t \\ 0, & \text{if } a(u_L)t < x < a(u_R)t \\ x - a(u_R)t, & \text{if } x > a(u_R)t, \end{cases}$$

where

$$a(u) = f'(u) = (ae^u - be^{-u})/(ae^u + be^{-u})$$

and s is given by (1.13).

From the above lemma and (2.22) we have if $u_L > u_R$, then $\lim_{\Delta\to 0} \Delta \log \theta_k^n = A_1(x,t)$, where

$$A_1(x,t) = \begin{cases} -(x-a(u_L)t)(u_L-u_R) + u_R a(u_L)t - t\log(ae^{u_R} + be^{-u_R}) \\ +\left(\frac{1+a(u_L)}{2}\right) t\log a + \left(\frac{1-a(u_L)}{2}\right) t\log b + t\log t \\ -\left(\frac{1+a(u_L)}{2}\right) t\log\left\{\left(\frac{1+a(u_L)}{2}\right)t\right\} - \left(\frac{1-a(u_L)}{2}\right) t\log\left(\frac{(1-a(u_L))}{2}t\right) \\ \qquad\qquad \text{if } x < st \\ a(u_R)tu_R - t\log(ae^{u_R} + be^{-u_R}) + \left(\frac{1+a(u_R)}{2}\right) t\log a \\ +\frac{1-a(u_R)}{2} t\log b + t\log t - \left(\frac{1+a(u_R)}{2}\right) t\log\left\{\left(\frac{1+a(u_R)}{2}\right)t\right\} \\ -\left(\frac{1-a(u_R)}{2}\right) t\log\left\{\left(\frac{1-a(u_R)}{2}\right)t\right\}, \quad \text{if } x > st. \end{cases}$$

If $u_L < u_R$, then $\lim_{\Delta\to 0} \Delta \log \theta_k^n = A_2(x,t)$, where

$$A_2(x,t) = \begin{cases} -(x-a(u_L)t)(u_L-u_R) + u_R a(u_L)t - t\log(ae^{u_R} + be^{u_R}) \\ +\frac{(1+a(u_L))}{2} t\log a + \frac{(1-a(u_L))}{2} t\log b + t\log t \\ -\frac{(1+a(u_L))}{2} t\log\left\{\left(\frac{1+a(u_L)}{2}\right)t\right\} - \left(\frac{1-a(u_L)}{2}\right) t\log\left\{\left(\frac{1-a(u_L)}{2}\right)t\right\}, \\ \qquad\qquad \text{if } x < a(u_L)t, \\ xu_R - t\log[ae^{u_R} + be^{-u_R}) + \frac{t+x}{2}\log a + \frac{t-x}{2}\log b + t\log t \\ -\frac{(t+x)}{2}\log\frac{(t+x)}{2} - \frac{(t-x)}{2}\log\left(\frac{t-x}{2}\right), \quad \text{if } a(u_L)t < x < a(u_R)t \\ u_R a(u_R)t - t\log(ae^{u_R} + be^{-u_R}) + \frac{1+a(u_R)}{2} t\log a \\ +\left(\frac{1-a(u_R)}{2}\right) t\log b + t\log t - \frac{(1+a(u_R))}{2} t\log\left\{\left(\frac{1+a(u_R)}{2}\right)t\right\} \\ -\left(\frac{1-a(u_R)}{2}\right) t\log\left\{\left(\frac{1-a(u_R)}{2}\right)t\right\} \quad \text{if } x > a(u_R)t. \end{cases}$$

Again from (2.13), (2.14) and (2.19), we get

$$\lim_{\Delta\to 0} \Delta\frac{\eta_k^n}{\theta_k^n} = \frac{v_R be^{-u_R}(t+x-y_0(x,t)) + v_R ae^{u_R}(x-y_0(x,t)-t)}{ae^{u_R} + be^{-u_R}}$$

$$-\tfrac{1}{2}(y_0(x,t)-|y_0(x,t)|)(v_L-v_R),$$

where $y_0(x,t)$ maximizes the RHS of (2.22). Using the lemma we have the following: If $u_L > u_R$, then

$$\lim_{\Delta\to 0}\Delta(\eta_k^n/\theta_k^n) = B_1(x,t),$$

where

$$B_1(x,t)=\begin{cases} -(x-a(u_L)t(v_L-v_R)) \\ \quad + \dfrac{v_R[ae^{u_R}(a(u_L)-1)t+be^{-u_R}(1+a(u_L))t]}{ae^{u_R}+be^{-u_R}}, & \text{if } x<st, \\ \dfrac{v_R[ae^{u_R}(a(u_R)-1)t+be^{-u_R}(1+a(u_R))t]}{ae^{u_R}+be^{-u_R}}, & \text{if } x>st. \end{cases}$$

If $u_L < u_R$, then

$$\lim_{\Delta\to 0}\Delta(\eta_k^n/\theta_k^n) = B_2(x,t),$$

where

$$B_2(x,t)=\begin{cases} -(x-a(u_L)t)(v_L-v_R) \\ \quad + \dfrac{v_R[ae^{u_R}(a(u_L-1))t+be^{-u_R}(1+a(u_L))t]}{ae^{u_R}+be^{-u_R}}, & \text{if } x<a(u_L)t \\ \dfrac{v_Rbe^{-u_R}(t+x)+v_Rae^{u_R}(x-t)}{ae^{u_R}+be^{-u_R}}, & \text{if } a(u_L)t<x<(u_R)t \\ \dfrac{v_R[ae^{u_R}(a(u_R)-1)t+be^{-u_R}(1+a(u_R))t]}{ae^{u_R}+be^{-u_R}}, & \text{if } x>a(u_R)t. \end{cases}$$

Now it follows that, if $u_L > u_R$

$$\lim_{\Delta\to 0}\int_x^\infty (u^\Delta(y,t)-u_R)\,dy = A_1(x,t),$$

$$\lim_{\Delta\to 0}\int_x^\infty (v^\Delta(y,t)-v_R)\,dy = B_1(x,t)$$

and if $u_L < u_R$

$$\lim_{\Delta\to 0}\int_x^\infty (u^\Delta(y,t)-u_R)\,dy = A_2(x,t),$$

$$\lim_{\Delta\to 0}\int_x^\infty (v^\Delta(y,t)-v_R)\,dy = B_2(x,t).$$

Hence

$$\left.\begin{aligned} u^\Delta(x,t) &\to \frac{-\partial A_1}{\partial x}+u_R \\ v^\Delta(x,t) &\to -\frac{\partial B_1}{\partial x}+v_R \end{aligned}\right\} \quad \text{if } u_L > u_R$$

$$\left.\begin{aligned} u^{\Delta}(x,t) &\to \frac{-\partial A_2}{\partial x} + u_R \\ v^{\Delta}(x,t) &\to -\frac{\partial B_2}{\partial x} + v_R \end{aligned}\right\} \quad \text{if } u_L < u_R$$

in the sense of distribution as $\Delta \to 0$. An easy calculation shows that

$$u_R - \frac{\partial A_1}{\partial x} = \begin{cases} u_L, & \text{if } x < st, \\ u_R, & \text{if } x > st, \end{cases}$$

$$v_R - \frac{\partial B_1}{\partial x} = t[s(v_R - v_L) - a(u_R)v_R + a(u_L)v_L] \cdot \delta_{x=st} + v_R + (v_L - v_R)[1 - H(x-st)],$$

$$u_R - \frac{\partial A_2}{\partial x} = \begin{cases} u_L, & \text{if } x < a(u_L)t \\ \dfrac{1}{2}\log\left[\dfrac{b}{a}\dfrac{t+x}{t-x}\right], & \text{if } a(u_L)t < x < a(u_R)t \\ u_R, & \text{if } x > a(u_R)t, \end{cases}$$

$$v_R - \frac{\partial B_2}{\partial x} = \begin{cases} v_L, & \text{if } x < a(u_L)t \\ 0, & \text{if } a(u_L)t < x < a(u_R)t \\ v_R, & \text{if } x > a(u_R)t. \end{cases}$$

Proof of (iii) is similar. The proof of Theorem 1 is complete.

3. Proof of Theorem 2

To prove Theorem 2, we first note that the approximate solutions are defined by

$$A_k^n = A_k^{n-1} + \log[ae^{A_{k-1}^{n-1}} + be^{-A_k^{n-1}}] - \log[ae^{A_k^{n-1}} + be^{-A_{k+1}^{n-1}}], \tag{3.1}$$

with

$$A_k^0 = \begin{pmatrix} u_k^0 & 0 \\ v_k^0 & u_k^0 \end{pmatrix} \tag{3.2}$$

where $u_k^0 = u^0(k\Delta)$, $v_k^0 = v^0(k\Delta)$. Following Lax [6], let us introduce

$$B_k^n = \sum_k^{\infty} A_k^n$$

and use the nonlinear transformation

$$B = \log E.$$

We get as before

$$E_k^{n+1} = aE_{k-1}^n + bE_{k+1}^n,$$

whose solution is

$$E_k^n = \sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} \exp\left\{ \sum_{j=n+k-2q}^{\infty} u_j^0 \begin{pmatrix} 1 & 0 \\ \sum_{j=n+k-2q}^{\infty} v_j^0 & 1 \end{pmatrix} \right\}.$$

In terms of the original variable A_k^n, we have

$$A_k^n = \log\left[E_k^n (E_{k+1}^n)^{-1}\right].$$

Carrying out the explicit calculations as before, we get

$$u_k^n = \log\left[\frac{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} \exp\left\{ \sum_{j=n+k-2q}^{\infty} u_j^0 \right\}}{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} \exp\left\{ \sum_{j=n+k+1-2q}^{\infty} u_j^0 \right\}} \right],$$

$$\sum_{j=k}^{\infty} v_j^n = \frac{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} \left(\sum_{j=n+k-2q}^{\infty} v_j^0 \right) \exp\left\{ \sum_{j=n+k-2q}^{\infty} u_j^0 \right\}}{\sum_{q=0}^{n} \binom{n}{q} a^q b^{n-q} \exp\left\{ \sum_{j=n+k-2q}^{\infty} u_j^0 \right\}}.$$

Now let $x = k\Delta$, $t = n\Delta$, $y = (n+k-2q)\Delta$ be fixed and let $\Delta \to 0$. Lax has shown that

$$u^\Delta(x,t) \to u(x,t) = \frac{1}{2} \log\left[\frac{b}{a} \frac{t+x-y_0(x,t)}{t-x+y_0(x,t)} \right],$$

where $y = y_0(x,t)$ maximizes

$$\max_{x-t \leqslant y \leqslant x+t} \left[\int_y^\infty u_0(z)\,dz - t f^*\left(\frac{x-y}{t} \right) \right], \tag{3.3}$$

where

$$f^*(\lambda) = \tfrac{1}{2} \log\left[(1+\lambda)^{1+\lambda} \cdot (1-\lambda)^{1-\lambda}\right] - \tfrac{1}{2} \log\left[4a^{1+\lambda} b^{1-\lambda}\right].$$

Again the same analysis of Lax [6] gives

$$\lim_{\Delta \to 0} \int_x^\infty v^\Delta(x,t)\,dy = \lim_{\Delta \to 0} \Delta \sum_k^\infty v_j^n = \int_{y_0(x,t)}^\infty v_0(z)\,dz.$$

Here again $y = y_0(x,t)$ maximizes (3.3). Since $\int_x^\infty v^\Delta(y,t)\,dy$ is a sequence of bounded function converging to $\int_{y_0(x,t)}^\infty v_0\,dx$ for a.e. (x,t), it follows that $v^\Delta(x,t)$ converges to

$$-\frac{\partial}{\partial x} \int_{y_0(x,t)}^\infty v_0(z)\,dz$$

in distribution. The proof of Theorem 2 is complete.

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Oscillation in odd-order neutral delay differential equations

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Abstract. Consider the odd-order functional differential equation

$$(x(t)-\alpha x(t-\tau))^{(n)}+p(t)f(x(t-\sigma))=0, \tag{*}$$

where $0\leqslant\alpha<1$, τ, $\sigma\in(0,\infty)$, $p\in C([0,\infty),(0,\infty))$, $f\in C^1(R,R)$ such that f is increasing, $xf(x)>0$ for $x\neq 0$ and f satisfies a generalized linear condition

$$\liminf_{x\to 0}\left|\left(\frac{df}{dx}\right)\right|=1.$$

Here we prove that every solution of (*) oscillates if

$$\liminf_{t\to\infty}\int_{t-\sigma/n}^{t}\sigma^{n-1}p(s)\,ds>\frac{1}{e}(1-\alpha)(n-1)!\left(\frac{n}{n-1}\right)^{n-1}.$$

This result generalizes a recent result of Gopalsamy *et al.* [6].

Keywords. Functional differential equations; oscillation of all solutions.

1. Introduction

In a remarkable result Ladas [4] proved that every solution of the first-order delay differential equation

$$x'(t)+px(t-\sigma)=0, \tag{1}$$

where $p,\sigma\in(0,\infty)$ oscillates (i.e., every solution has an unbounded set of zeros in $(0,\infty)$) if and only if

$$p\sigma>\frac{1}{e}. \tag{2}$$

The result was extended by authors in [5] for general *odd*-order differential equation

$$x^{(n)}(t)+px(t-\sigma)=0, \tag{3}$$

replacing (2) by

$$p^{1/n}\left(\frac{\sigma}{n}\right)>\frac{1}{e}. \tag{4}$$

The first result was further improved (see [7]) for equations with variable coefficients with the statement that

$$\liminf_{t\to\infty}\int_{t-\sigma}^{t} p(s)\,ds > \frac{1}{e},$$

and

$$\limsup_{t\to\infty}\int_{t-\sigma}^{t} p(s)\,ds > \frac{1}{e}$$

are respectively sufficient and necessary conditions for every solution of

$$x'(t) + p(t)x(t-\sigma) = 0,$$

where $p\in C([0,\infty),(0,\infty))$, to be oscillatory. But a similar extension for

$$x^{(n)}(t) + p(t)x(t-\sigma) = 0 \tag{5}$$

has not been proved yet.

Recently, Gopalsamy *et al* [6] proved that

$$\liminf_{t\to\infty}\int_{t-\sigma}^{t} (t-s)^{n-1}p(s)\,ds > (1-\alpha)(n-1)! \tag{6}$$

implies that every solution of the *odd*-order differential equation

$$(x(t)-\alpha x(t-\tau))^{(n)} + p(t)x(t-\sigma) = 0 \tag{7}$$

oscillates, where $0\leqslant\alpha<1$. Indeed, for $\alpha=0$ and $p(t)=p\in(0,\infty)$, (6) reduces to

$$p\sigma^n > n!,$$

that is,

$$p^{1/n}\left(\frac{\sigma}{n}\right) > \frac{1}{n}(n!)^{1/n}, \tag{8}$$

which is the sufficient condition for oscillation of (3). In view of the condition given in (4), the lower bound of $p^{1/n}(\sigma/n)$ in (8) is comparatively larger than that of $(1/e)$.

In this paper we prove a result, a particular case of which shows that all solutions of (7) are oscillatory if

$$\liminf_{t\to\infty}\int_{t-\sigma/n}^{t} \sigma^{n-1}p(s)\,ds > \frac{1}{e}(1-\alpha)(n-1)!\left\{\frac{n}{n-1}\right\}^{n-1}. \tag{9}$$

When $p(t)=p\in(0,\infty)$ and $\alpha=0$, the above condition reduces to

$$p^{1/n}\left(\frac{\sigma}{n}\right) > \frac{1}{n}\left(\frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1}\right)^{1/n}(n!)^{1/n}. \tag{10}$$

In view of the known inequality

$$\begin{aligned}\left\{\frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1}\right\}^{1/n} &= \left\{\frac{1}{e}\left(1+\frac{1}{n-1}\right)^{n-1}\right\}^{1/n}\\ &= \left\{\frac{1}{e}\left(\sum_{r=0}^{n-1} C(n-1,r)\left(\frac{1}{n-1}\right)^{r}\right)\right\}^{1/n}\\ &\leqslant \left\{\frac{1}{e}\left(\sum_{r=0}^{n-1}\frac{1}{r!}\right)\right\}^{1/n}\end{aligned}$$

$$\leqslant \left\{\frac{1}{e}\left(\sum_{r=0}^{\infty}\frac{1}{r!}\right)\right\}^{1/n} = 1, \tag{11}$$

where $C(n-1,r)$ is the $(r+1)$th binomial coefficient in the expansion of $(1+1/(n-1))^{n-1}$, our condition is weaker than that of (8). We give examples to support our claim.

2. Main results

Consider the *odd*-order nonlinear functional differential equation

$$(x(t)-\alpha x(t-\tau))^{(n)} + p(t)f(x(t-\sigma)) = 0, \tag{E}$$

with the assumptions that

$$\begin{aligned} &p\in C(R^+,R^+),\ f\in C(R,R) \text{ such that } f \text{ is increasing,}\\ &xf(x)>0 \text{ for } x\neq 0,\ |f(x)|\to\infty \text{ as } |x|\to\infty,\ 0\leqslant\alpha<1,\\ &n>1 \text{ is an odd integer and } \tau,\sigma\in(0,\infty). \end{aligned} \tag{H}$$

Let $\delta=\max\{\tau,\sigma\}$ and $\phi\in C([T-\delta,T],R)$. By a solution of (E) in $[T,\infty)$, we mean a function $x\in C([T,\infty),R)$ such that $x(t)=\phi(t)$, $T-\delta\leqslant t\leqslant T$, $(x(t)-\alpha x(t-\tau))\in C^{(n)}([T,\infty),R)$ and $x(t)$ satisfies (E) for $t\geqslant T$.

As usual, a solution $x(t)$ of (E) is called oscillatory if it has zeros for arbitrarily large t and nonoscillatory, otherwise.

We say (E) is *generalized sublinear* if f satisfies

$$\liminf_{x\to 0}\left|\left(\frac{df}{dx}\right)\right| > 1,$$

superlinear if

$$\liminf_{x\to 0}\left|\left(\frac{df}{dx}\right)\right| < 1 \tag{12}$$

and *linear* if

$$\liminf_{x\to 0}\left|\left(\frac{df}{dx}\right)\right| = 1, \tag{13}$$

which includes the cases $f(x)=x^{\alpha}$, $0<\alpha<1$, $1<\alpha<\infty$ and $\alpha=1$, respectively.

In what follows, we list the following two results for our use in sequel.

Theorem 1 ([3], *Lemma 1*). *Suppose that* $g\in C^{(n)}([T,\infty),(0,\infty))$ *such that* $g^{(i)}(t)$ *has no zeros in* $[T,\infty)$ $(i=1,2,\ldots(n-1))$ *and* $g^{(n)}(t)\leqslant 0$ *for* $t\geqslant T$. *If* $\beta\in(0,\infty)$ *then*

$$g(t-\beta)\geqslant\frac{\beta^{n-1}}{(n-1)!}g^{(n-1)}(t), \quad t\geqslant T+2\beta.$$

Theorem 2 ([7], *Theorem 2.1.1*). *If* $\beta\in(0,\infty)$, $Q\in C([T,\infty),(0,\infty))$, $T>0$ *and*

$$\liminf_{t\to\infty}\int_{t-\beta}^{t} Q(s)\,ds > \frac{1}{e},$$

then the first-order differential inequality

$$y'(t) + Q(t)y(t-\beta) \leqslant 0$$

has no eventually positive solutions.

Our main theorem is as follows.

Theorem 3. *Suppose that (H) holds and f satisfies* (13). *Then* (9) *implies that every solution of (E) oscillates.*

Proof. Since (9) holds, there exists $0 < \varepsilon < 1$ such that

$$(1-\varepsilon)^2 \liminf_{t\to\infty} \int_{t-\sigma/n}^{t} \sigma^{n-1} p(s)\mathrm{d}s > \frac{1}{e}(1-\alpha)(n-1)!\left(\frac{n}{n-1}\right)^{n-1}. \tag{14}$$

To the contrary, assume that $x(t)$ is a nonoscillatory solution of (E). Let $x(t) > 0$ for $t \geqslant t_0$. (The case for $x(t) < 0$, $t \geqslant t_0$ may be treated similarly.) Setting

$$z(t) = x(t) - \alpha x(t-\tau), \tag{15}$$

from (E) it may be observed that $z^{(n)}(t) \leqslant 0$ for $t \geqslant t_0 + \sigma$. Consequently, there exists $T \geqslant t_0 + \sigma$ such that $z^{(i)}(t)$ $(i = 0, 1, 2, 3 \ldots (n-1))$, has no zeros in $[T, \infty)$.

Suppose that $z(t) < 0$, $t \geqslant T$. Since n is odd, $z^{(n)}(t) \leqslant 0$, $t \geqslant T$ implies that $z'(t) < 0, t \geqslant T$. On the other hand, let

$$\limsup_{t\to\infty} x(t) = \mu.$$

If $\mu = \infty$, there exists a sequence of real numbers $\langle t_n \rangle_{n=1}^{\infty}$ such that $t_n \to \infty$, $x(t_n) \to \infty$ as $n \to \infty$ and $x(s) < x(t_n)$ for $s < t_n$. From (15) we see that

$$z(t_n) = x(t_n) - \alpha x(t_n - \tau) \geqslant (1-\alpha)x(t_n),$$

which further gives

$$\lim_{n\to\infty} z(t_n) = \infty,$$

a contradiction to the fact that $z(t) < 0, t \geqslant T$. In case μ is finite, there exists a sequence $\langle t_n \rangle_{n=1}^{\infty}$ such that $t_n \to \infty$, $x(t_n) \to \mu$ as $n \to \infty$. Since $\langle x(t_n - \tau) \rangle_{n=1}^{\infty}$ is a bounded sequence of real numbers, it admits a convergent subsequence. Let $\langle s_n \rangle_{n=1}^{\infty}$ be the subsequence for which $\langle x(s_n - \tau) \rangle_{n=1}^{\infty}$ converges to a real number λ. Clearly $\lambda \leqslant \mu$. Again $x(s_n) \to \mu$ as $n \to \infty$. Now

$$\lim_{n\to\infty} z(s_n) = \lim_{n\to\infty} (x(s_n) - \alpha x(s_n - \tau)) \geqslant (1-\alpha)\lambda,$$

that is,

$$\lim_{n\to\infty} z(s_n) \geqslant 0,$$

which is a contradiction to the fact that $z(t)$ is negative and decreasing function. Hence $z(t) < 0$, $t \geqslant T$ is impossible.

Let $z(t) > 0, t \geqslant T$. Clearly, it follows that $z'(t) < 0$, $t \geqslant T$. Indeed, otherwise, $z'(t) > 0$, $t \geqslant T$ implies that $\liminf_{t\to\infty} z(t) > 0$ and consequently

$$\liminf_{t\to\infty} x(t) = \liminf_{t\to\infty}(z(t) + \alpha x(t-\tau)) > 0.$$

Integrating (E) from T to t and using the above observation along with the fact that (9) implies

$$\int^{\infty} p(s)\,ds = \infty,$$

we see that $z^{(n-1)}(t) \to -\infty$ as $t \to \infty$. Consequently, $z(t) \to -\infty$ as $t \to \infty$, a contradiction. Further $z^{(n)}(t) \leqslant 0$ implies that

$$z^{(j)}(t)z^{(j+1)}(t) \leqslant 0, \quad 0 \leqslant j \leqslant (n-1).$$

Consequently,

$$\lim_{t\to\infty} z^{(j)}(t) = 0, \quad 1 \leqslant j \leqslant (n-1),$$

and

$$\lim_{t\to\infty} z(t) = k \geqslant 0.$$

If $k > 0$, then repeating the argument applied earlier we lead to a contradiction. Hence $k = 0$. From (13) it follows that

$$\liminf_{y\to 0}\left(\frac{df}{dy}\right) = 1.$$

Taking $y(t) = z^{(n-1)}(t)$, and from the definition of limit infimum it follows that for every $\varepsilon > 0$ there exists a large positive number M such that

$$\left(\frac{df}{dy}\right) > (1-\varepsilon) \quad \text{for } t \geqslant M. \tag{16}$$

From (15) we see that

$$x(t) = z(t) + \alpha x(t-\tau), \quad t \geqslant T. \tag{17}$$

The repeated application of (17) on it, as per the idea in the paper of Gopalsamy *et al* [6] results in

$$x(t) \geqslant z(t)\left(\sum_{n=0}^{N} \alpha^n\right), \quad t \geqslant T + N\tau.$$

From the above inequality it follows that there exists $M_1 \geqslant T + N\tau$ such that

$$x(t) > z(t)\frac{(1-\varepsilon)^2}{(1-\alpha)}, \quad t \geqslant M_1. \tag{18}$$

In Theorem 1, replacing $g(t)$ by $z(t-\sigma/n)$ and β by $\left(\frac{n-1}{n}\right)\sigma$ we get

$$z(t-\sigma) > \frac{1}{(n-1)!}\left(\frac{n-1}{n}\sigma\right)^{n-1} z^{(n-1)}(t-\sigma/n), \quad t \geqslant T + 3\sigma. \tag{19}$$

Using (19) in the inequality obtained by replacing t by $t-\sigma$ in (18) we get

$$x(t-\sigma) \geqslant Kz^{(n-1)}(t-\sigma/n), \quad t \geqslant \max\{M_1+\sigma, T+3\sigma\} = T_0,$$

where

$$K = \frac{(1-\varepsilon)^2}{(1-\alpha)} \frac{1}{(n-1)!} \left(\frac{n-1}{n}\sigma\right)^{n-1}. \tag{20}$$

Since f is increasing,

$$f(x(t-\sigma)) \geqslant f(Kz^{(n-1)}(t-\sigma/n)), \quad t \geqslant T_0. \tag{21}$$

From (E) and (21) it follows that

$$z^{(n)}(t) + p(t)f(Kz^{(n-1)}(t-\sigma/n)) \leqslant 0, \quad t \geqslant T_0. \tag{22}$$

Multiplying both sides of (22) by

$$\frac{\mathrm{d}}{\mathrm{d}y}(f(y)), \quad \text{where } y = z^{(n-1)}(t),$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(y(t))) + \left(p(t)\frac{\mathrm{d}f}{\mathrm{d}y}\right) f(Ky(t-\sigma/n)) \leqslant 0, \quad t \geqslant T_0. \tag{23}$$

Set

$$H(t) = f(Ky(t)), \quad t \geqslant T_0.$$

Now $z^{(n-1)}(t) > 0, t \geqslant T_0$ implies that $H(t) > 0, t \geqslant T_0$. From (23) and (16) it follows that

$$\frac{\mathrm{d}H}{\mathrm{d}t} + (1-\varepsilon)Kp(t)H(t-\sigma/n) \leqslant 0, \quad t \geqslant T_1 = \max\{T_0, M\}.$$

Hence $H(t)$ is an eventually positive solution of the differential inequality given in Theorem 2, where

$$Q(t) = (1-\varepsilon)Kp(t).$$

and

$$\beta = \sigma/n.$$

But, by (14),

$$\liminf_{t\to\infty} \int_{t-\sigma/n}^{t} Q(s)\,\mathrm{d}s = \liminf_{t\to\infty} \int_{t-\sigma/n}^{t} K(1-\varepsilon)p(s)\,\mathrm{d}s > \frac{1}{e},$$

a contradiction to Theorem 2. Hence (E) cannot have a nonoscillatory solution. This completes the proof of this theorem.

Example. Consider the equation

$$(x(t) - \tfrac{1}{2}x(t-1))^{(3)} + 3(\tfrac{19}{20} + e^{-t})x(t-1) = 0, \quad t \geqslant 1.$$

Since (6) fails to hold, Theorem 4.1 of Gopalsamy *et al* [6] is not applicable, but (9) holds and hence Theorem 3 shows that every solution of it oscillates.

Remark. In view of the inequality

$$\frac{1}{n}\left(\frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1}\right)^{1/n}(n!)^{1/n}<\frac{1}{2}\left(1+\frac{1}{n}\right), \tag{24}$$

it follows from (10) that

$$p^{1/n}\left(\frac{\sigma}{n}\right)>\frac{1}{2}\left(1+\frac{1}{n}\right),$$

or in particular,

$$p^{1/n}\left(\frac{\sigma}{n}\right)>\frac{2}{3} \tag{25}$$

implies that every solution of eq. (3) oscillates. Indeed, $n>1$ and odd gives that

$$n!=1(n-1)2(n-2)\cdots\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}\right).n.$$

Since the arithmetic mean exceeds the geometric mean $r(n-r)<\left(\frac{n}{2}\right)^2$ for every r and hence

$$n!\leqslant\left(\frac{n}{2}\right)^{n-1}n=\left(1-\frac{1}{2}\right)^{n-1}n^n.$$

Consequently, using Binomial theorem we get

$$\frac{1}{n}(n!)^{1/n}\leqslant\left(1-\frac{1}{2}\right)^{(1-(1/n))}\leqslant 1-\frac{1}{2}\left(1-\frac{1}{n}\right)=\frac{1}{2}\left(1+\frac{1}{n}\right). \tag{26}$$

Now (24) follows from (11) and (26). Since $n\geqslant 3$, (25) follows from (26).

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Surface waves due to blasts on and above inviscid liquids of finite depth

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Abstract. For the problem of waves due to an explosion above the surface of a homogeneous ocean of finite depth, asymptotic expressions of the velocity potential and the surface displacement are determined for large times and distances from the pressure area produced by the incident shock. It is shown that the first item in Sakurai's approximation scheme for the pressure field inside the blast wave as well as the results of Taylor's point blast theory can be used to yield realistic expressions of surface displacement. Some interesting features of the wave motion in general are described. Finally some numerical calculations for the surface elevation were performed and included as a particular case.

Keywords. Surface waves; inviscid liquid; asymptotic expansion; blast theory; surface elevation.

1. Introduction

The problem of surface waves caused by the interaction of a blast-generated shock wave with an ideal incompressible fluid has been analysed by Rumiantsev [9], Kisler [4] and Sen [11], mainly when the fluid is infinitely deep. The problem of waves produced by explosions above the surface of a shallow liquid has also been touched upon by Kranzer and Keller [5] as an application of the asymptotic Caucny-Poisson wave theory for fluids of finite depth. This treatment, however, did not include the effects of the time variation of the pressure distribution on the surface. Choudhuri [1] and Wen [14] considered the case where the disturbance is over any arbitrary region of the free surface and the water is of uniform finite depth by the method of multiple Fourier transforms. In both the cases the method of stationary phase was applied to obtain the approximate expression for the potential function and surface elevation for large values of time and distance. Mondal and Mukherjee [8] considered the corresponding problem by Hankel transform method and finally the approximate expressions for the potential function and inertial surface elevation were obtained for large distances and times by the method of stationary phase.

The basic simplifying assumption in this problem is that the large difference between the densities of the gas and the fluid make the fluid displacements too small to affect the motion of the gas, which is supposed to be known. Here we present the three-dimensional problem of the generation of waves due to explosions above the surface of a fluid of constant finite depth due to the incident shock and of the area on which it acts. After deriving the formal solution of the problem in terms of infinite integrals in the usual manner, we use the known asymptotic expansions of the Bessel function and Kummer's confluent hypergeometric function alongwith the method of stationary

phase to find approximate expressions of the velocity potential and the surface displacement ($=\zeta$) integrals of large times and distances from the pressure area. For the pressure field inside the blast wave, we first make use of an expression closely resembling the first term of Sakurai's [10] approximation scheme. It is also easy to see that the expressions of ζ in the form of infinite series may be obtained by the same methods as used by Sen [11], but these will not be deduced here. Instead, we describe the more tractable features of the asymptotic wave motion in its general form as well as special forms which use the results of the Taylor point blast theory, and then place our results on a more realistic footing.

2. Formulation of the problem

We assume that surface waves are excited when the spherical shock wave due to a point blast in the gas interacts with the fluid surface. An expanding circular region of pressure is formed on the free surface as a consequence. Using cylindrical coordinates (r, θ, z), we write the governing equations as follows:

For $t>0$,

$$\left.\begin{aligned} p_0(r,t) &= f(r,t), \quad r<r_0(t), \\ &= 0, \qquad r>r_0(t), \end{aligned}\right\} \tag{1}$$

$$\nabla^2\varphi(r,z,t)=0, \quad z<0, \quad t>0 \tag{2}$$

$$gp\zeta = -p_0(r,t)+\rho\left(\frac{\partial\varphi}{\partial t}\right)_{z=0} \tag{3}$$

$$\frac{\partial^2\varphi}{\partial t^2}+g\frac{\partial\varphi}{\partial z}=\rho^{-1}\frac{\partial p_0}{\partial t}, \quad z=0, \quad t>0 \tag{4}$$

$$\varphi(r,0,0)=0, \quad \varphi_t(r,0,0)=0 \tag{5}$$

$$\frac{\partial\varphi}{\partial z}=0 \quad \text{on } z=-h. \tag{6}$$

The conditions (5) are equivalent to the conditions

$$\varphi=0, \quad \zeta=0 \quad \text{at } t=0,$$

since p_0 is finite and $r\to 0$ as $t\to 0+$.

3. Formal solution

We assume a solution of (2) of the form

$$\varphi=\int_0^\infty A(k,t)J_0(kr)\cosh k(z+h)\operatorname{sech} kh\,dk$$

so that (6) is satisfied.

Substituting for φ in (4), we obtain the following differential equation for $A(k,t)$:

$$\ddot{A} + \sigma^2 A = k\rho^{-1}\frac{\partial}{\partial t}\int_0^{r_0(t)} \alpha f(\alpha,t) J_0(k\alpha)\mathrm{d}\alpha$$

here

$$\sigma^2 = gk\tanh kh.$$

he solution of this equation is

$$A = A_0(k)\cos(\sigma t + \varepsilon_k) + (k/\rho\sigma)\int_0^t \sin[\sigma(t-s)]\frac{\partial}{\partial s}\int_0^{r_0(s)} \alpha f(\alpha,s) J_0(k\alpha)\mathrm{d}\alpha\mathrm{d}s.$$

y (5), $A_0 = 0$.

An integration by parts then gives

$$A = (k/\rho)\int_0^t \cos[\sigma(t-s)]\mathrm{d}s \int_0^{r_0(s)} \alpha f(\alpha,s) J_0(k\alpha)\mathrm{d}\alpha. \tag{7}$$

ne velocity potential is therefore

$$\varphi = \rho^{-1}\int_0^\infty kJ_0(kr)\frac{\cosh k(z+h)}{\cosh kh}\mathrm{d}k \int_0^t \cos\sigma(t-s)\mathrm{d}s \int_0^{r_0(s)} \alpha f(\alpha,s) J_0(k\alpha)\mathrm{d}\alpha. \tag{8}$$

The surface displacement is then determined by (3):

$$\zeta = -(g\rho)^{-1}\int_0^\infty \sigma k J_0(kr)\mathrm{d}k \int_0^t \sin\sigma(t-s)\mathrm{d}s \int_0^{r_0(s)} \alpha f(\alpha,s) J_0(k\alpha)\mathrm{d}\alpha. \tag{9}$$

. Asymptotic representation of φ and ζ for a uniformly expanding pressure area

We adopt the following model for $f(r,t)$ because it closely resembles the first term of Sakurai's [8] approximation scheme for the determination of the pressure field nside the blast wave

$$f(r,t) = (t+t_1)^{-n}F(r/r_0(t)), \quad r < r_0(t) \tag{10}$$

where $n(>1)$ is non-integral, and t_1 is the time taken (from the moment of the explosion) by the shock front to just reach the surface. Also, at high pressures, $R(t+t_1) \equiv$ the radius of the shock front at time $(t+t_1) \propto (t+t_1)^{2/5}$ Ref. [6]

From this result, one can obtain the expression for $r_0(t)$:

$$r_0(t) = [\{R(t+t_1)\}^2 - \{R(t_1)\}^2]^{1/2}.$$

Here, however, we assume, alongwith the model (10), that

$$r_0(t) = ut, \quad u = \text{constant}, \tag{11}$$

for convenience of analysis.

Then, from (8)

$$\varphi = \rho^{-1} \int_0^1 \alpha F(\alpha) \mathrm{d}\alpha \int_0^\infty k J_0(kr) \frac{\cosh k(z+h)}{\cosh kh} \mathrm{d}k$$
$$\times \int_0^t r_0^2(s)(t_1+s)^{-n} \cos\sigma(t-s) J_0(k\alpha r_0(s)) \mathrm{d}s. \tag{12}$$

To evaluate (12) asymptotically for large r and t, we first replace $J_0(k\alpha r_0(s))$ by its integral representation,

$$J_0(k\alpha r_0(s)) = (2/\pi) \int_0^{\pi/2} \cos(k\alpha us \sin\theta) \mathrm{d}\theta, \tag{13}$$

and $J_0(kr)$ by the first term of its asymptotic expansion for large kr,

$$J_0(kr) \simeq (2/\pi kr)^{1/2} \cos(kr - \pi/4). \tag{14}$$

The resulting s-integral is expressed in terms of a function $T_n(ia, t, t_1)$ defined as follows:

$$T_n(ia,t,t_1) = e^{iat_1} \int_0^t s^2 (t_1+s)^{-n} e^{ias} \mathrm{d}s$$
$$= [(3-n)^{-1} s^{3-n} {}_1F_1(3-n; 4-n; ias)$$
$$- 2(2-n)^{-1} t_1 s^{2-n} {}_1F_1(2-n; 3-n; ias)$$
$$+ (1-n)^{-1} t_1^2 s^{1-n} {}_1F_1(1-n; 2-n; ias)]_{s=t_1}^{s=(t+t_1)}. \tag{15}$$

Here

$$a \equiv a_j = -\{\sigma + (-1)^j ku\alpha \sin\theta\}, \quad j = 1, 2. \tag{16}$$

and ${}_1F_1$ denotes Kummer's confluent hypergeometric function.

In place of (12), we have now

$$(\pi\rho u^{-2}/2)\varphi \simeq (8\pi r)^{-1/2} \mathrm{Re} \int_0^1 \alpha F(\alpha) \mathrm{d}\alpha \int_0^{\pi/2} \mathrm{d}\theta$$
$$\times \int_K^\infty k^{1/2} \cosh[k(z+h)] \operatorname{sech} kh \sum_{j=1}^{2} T_n(ia_j(k), t, t_1)$$
$$\times [\exp\{irP_j(k)\} + \exp\{irQ_j(k)\}] \mathrm{d}k, \tag{17}$$

where

$$P_j(k) = \sigma(t+t_1) r^{-1} + (-1)^j ku\alpha t_1 r^{-1} \sin[\theta - k + (\pi/4r)],$$
$$Q_j(k) = \sigma(t+t_1) r^{-1} + (-1)^j ku\alpha t_1 r^{-1} \sin[\theta + k - (\pi/4r)] \tag{18}$$

and $K > 0$ is such that $Kr \gg 1$, and the stationary point(s), if any, lies in (K, ∞).

The function $Q_j(k)$ has no stationary point for $j=2$ in $0<k<\infty$, and none either in the same interval for $j=1$, since $ut_1 \ll r$. Therefore, the part of the k-integral arising from $\exp\{irQ_j(k)\}$ in (17) is $O(r^{-1})$, as $r\to\infty$. The function $P_j(k)$, on the other hand, has one and only one stationary point, $k=k_j$ (say), when

$$\tau \equiv [(t+t_1)\sqrt{gh}/2r] > \frac{1}{2}+\frac{ut_1}{2r}\delta_{j1} \tag{19}$$

where δ_{j1} is Kronecker's delta function. To show this, we note that

(i) $P_j'(k)$ is continuous in $0<k<\infty$,

(ii) $P_j'(k)$ is strictly monotone decreasing in $0<k<\infty$, since $P_j''(k)<0$ therein,

(iii) $P_j'(k)\to 2\tau-1+(-1)^j u\alpha t_1 r^{-1}\sin\theta \equiv a_1$(say) as $k\to 0+$

$P_j'(k)\to(-1)^j u\alpha t_1 r^{-1}\sin\theta - 1 \equiv a_2$(say), as $k\to\infty$,

<0 for both j, since $ut_1 r^{-1}\ll 1$.

These conditions make $P_j'(k)$ vanish once and only once in $0<k<\infty$, when $(a_1)_{\min}>0$, that is, when $\tau > \frac{1}{2}+\frac{ut_1}{2r}\delta_{j1}$, as stated above.

A similar argument shows that the equation $[P_j'(k)]_{u=0}=0$ has one and only one non-negative real root $k=k_0$ (say), independent of j, when $\tau>1/2$ and hence, under the condition (19) as well.

Since $ut_1 r^{-1}\ll 1$, an approximate value of k_j may be obtained by putting

$$k_j = k_0+\varepsilon_j \tag{20}$$

in the equation $P_j'(k)=0$, whence

$$\varepsilon_j \simeq (-1)^{j+1} u\alpha t_1 r^{-1}\sin\theta/P_j''(k_0). \tag{21}$$

Applying the method of stationary phase to evaluate the k-integral of (17), we obtain

$$\begin{aligned}(\pi\rho u^{-2}/2)\varphi \simeq \int_0^1 \alpha F(\alpha)\,d\alpha \int_0^{\pi/2} d\theta\cdot(2r)^{-1}(k_j/|P''(k_j)|)^{1/2}\\ \times\cosh k_j(z+h)\operatorname{sech} k_j h\\ \times \operatorname{Re}\sum_j T_n(ia_j(k_j),t,t_1)\exp i\{rP_j(k_j)-\pi/4\}\\ \geqq O(r'^{-3/2}),\quad r'\equiv(r/h)\to\infty.\end{aligned} \tag{22}$$

The asymptotic expansions of the functions ${}_1F_1$ for large arguments [Erdélyi, [2] I, 6.13.1 (2)] show that

$$T_n(ia_j(k_j),t,t_1)\sim\frac{t^2}{ia_j(k_j)}(t+t_1)^{-n}e^{ia_j(k_j)(t+t_1)}, \tag{23}$$

where we suppose $n<2$, a restriction required for the Taylor point blast theory.

Using the approximations (20) and (23), we get for (22), the expression

$$(\pi\rho u^{-2}/2)\varphi \simeq \int_0^1 \alpha F(\alpha)\mathrm{d}\alpha \int_0^{\pi/2} \mathrm{d}\theta \frac{t^2}{2r(t+t_1)^n} (g\tanh k_0 h|P_j''(k_0)|)^{-1/2}$$
$$\times \cosh[k_0(z+h)]\operatorname{sech} k_0 h$$
$$\times \sum_j \sin\{k_0 r + (-1)^j k_0 u\alpha t \sin\theta\}. \tag{24}$$

By [Erdélyi, [2], I, 7.12. (45)], we have

$$(2/\pi)\int_0^{\pi/2} \sin\{k_0 r + (-1)^j k_0 ut\alpha \sin\theta\}\,\mathrm{d}\theta$$
$$= \sin(k_0 r) J_0(k_0 ut\alpha) + (-1)^j (2/\pi)\cos(k_0 r)\, s_{0,0}(k_0 ut\alpha) \tag{25}$$

when $\tau > 1/2 + \dfrac{ut_1}{2r}$ so that both k_1 and k_2 exist, the Lommel function $s_{0,0}(k_0 ut\alpha)$ cancels out in the j-sum of (24). The asymptotic expression for φ thus finally becomes

$$\rho\varphi \simeq \frac{u^2 t^2}{r(t+t_1)^n} \{g\tanh(k_0 h)|P_j''(k_0)|\}^{-1/2} \frac{\cosh k_0(z+h)}{\cosh k_0 h} \bar{F}(k_0 ut)\sin(k_0 r), \tag{26}$$

where

$$\bar{F}(x) = \int_0^1 \alpha F(\alpha) J_0(\alpha x)\mathrm{d}\alpha. \tag{27}$$

The result (26) holds under the conditions

$$r \gg r_0(t) \gg r_0(t_1), \quad k_0 r \gg 1, \quad \tau > \frac{1}{2} + \frac{ut_1}{2r} \tag{28}$$

A similar process applied to (9) gives for the surface displacement the asymptotic expression

$$\zeta \simeq -(g\rho)^{-1} \frac{u^2 t^2}{r} (t+t_1)^{-n} k_0^{1/2} |P_j''(k_0)|^{-1/2} \bar{F}(k_0 ut)\cos(k_0 r) \tag{29}$$

under the same conditions (28).

If $F(x) = D$, $0 < x < 1$, the limiting value of ζ, as $h \to \infty$, equals the corresponding value of ζ for the case of infinite depth [Sen [11], eqn. (69)].

4.1 *An illustrative case*

When a concentrated explosion of constant total energy E takes place in a still atmosphere of density ρ_0, Taylor's formula for the maximum pressure (which happens

to be on the shock front) is

$$p_{\max} = 0{\cdot}141\{E^2\rho_0^3(t+t_1)^{-6}\}^{1/5} \tag{30}$$

when the ratio of specific heats of air is about 1.4.

If we adopt this law of pressure for an approximation in the present case while retaining the hypothesis $r_0(t) = ut$ for a relatively small spread of the pressure area, we have

$$n = 6/5, \quad \text{and } F(R) = 0{\cdot}141\ (E^2\rho_0^3)^{1/5} \quad \text{for all } R$$

so that

$$\bar{F}(k) = 0{\cdot}141(E^2\rho_0^3)^{1/5}k^{-1}J_1(k).$$

Equation (29) then gives

$$\zeta \simeq -0{\cdot}141\frac{(E^2\rho_0^3)^{1/5}}{g\rho k_0^{1/2}}\cdot\frac{ut}{r(t+t_1)^{6/5}}\cdot|P_j''(k_0)|^{-1/2}J_1(k_0ut)\cos(k_0r) \tag{31}$$

subject to the conditions (28).

5. Wave elevation due to a Taylor point blast above the fluid surface

At the outset, we transform the general expression (9) for ζ as follows:

$$\zeta = -(g\rho)^{-1}t\,\mathrm{Im}\int_0^\infty \sigma k J_0(kr)e^{i\sigma(st+t_1)}\mathrm{d}k$$

$$\times\int_0^1 r_0^2(st)\cdot(t_1+st)^{-n}e^{-i\sigma(st+t_1)}\bar{F}(kr_0(st))\mathrm{d}s. \tag{32}$$

Writing

$$T_n(k,t,t_1) = e^{-i\sigma t_1}\int_0^1 r_0^2(st)(t_1+st)^{-n}e^{-i\sigma st}\bar{F}(kr_0(st))\mathrm{d}s \tag{33}$$

We follow the same procedure as shown in §4, it being assumed that the function $\bar{F}(kr_0(st))$ is sufficiently well behaved, and it does not make T_n strongly oscillatory or singular for large t. The latter is a pre-requisite for the applicability of the method of stationary phase [Stoker, [12], §6.8]. Then

$$\zeta \simeq -(k_0t/g^{1/2}\rho r)(\tanh k_0h/|P''(k_0)|)^{1/2}$$

$$\times\,\mathrm{Im}[T_n(k_0,t,t_1)\exp i\{rP(k_0)-\pi/4\}] + O(r'^{-3/2}),$$

$$\text{as } r' \equiv (r/h)\to\infty \tag{34}$$

where

$$P(k) = (t+t_1)r^{-1}(gk\tanh kh)^{1/2} - k + \pi/4r,$$

$P''(k) \equiv P_j''(k)$ as obtained from (18), and $k = k_0$

is the non-negative real root of $P'(k) = 0$.

This approximation holds under the conditions

$$(t+t_1)\sqrt{gh}\,r^{-1}>1, \quad r\gg r_0(t), \quad k_0 r\gg 1. \tag{35}$$

For large t, $T_n(k_0,t,t_1)$ approximates to

$$T_n(k_0,t,t_1)\simeq (i/t)(gk_0\tanh k_0h)^{-1/2}r_0^2(t)(t+t_1)^{-n} \times \bar{F}(k_0r_0(t))e^{-i(t+t_1)\sqrt{gk_0\tanh(k_0h)}}. \tag{36}$$

Therefore

$$\zeta\simeq -(g\rho r)^{-1}[k_0/|P''(k_0)|]^{+1/2}r_0^2(t)(t+t_1)^{-n}\bar{F}(k_0r_0(t))\cos(k_0r). \tag{37}$$

This result is used below to determine the wave height caused by a Taylor point blast above the fluid surface.

5.1 *Pressure inside a blast wave: Taylor's formula*

For an intense explosion of constant total energy E occurring at a point O' at a height H above the ground, the pressure $p(r,z,t)$ inside the expanding spherical blast wave and the radius $R(t)$ of the shock wave at time t from the moment of the explosion are given by the following formulae due to Taylor [13]:

$$p=0{\cdot}133R^{-3}Ef_1(\eta), \tag{38}$$

$$f_1(\eta)=\frac{2\gamma}{\gamma+1}\left[\frac{\gamma+1}{\gamma}-\frac{\eta^{n-1}}{\gamma}\right]^{-(2\gamma^2+7\gamma-3/7-\gamma)}, \tag{39}$$

$$\eta=(z^2+r^2)^{1/2}/R, \tag{40}$$

$$t=0{\cdot}926R^{5/2}\rho_0^{1/2}E^{-1/2}. \tag{41}$$

Here $n=(7\gamma-1)/(\gamma^2-1)$, $\gamma=$ ratio of specific heats of air $\simeq 1.4$.
$z=$ depth of a point vertically downwards from O'.
$r=$ distance of a point P from the perpendicular $O'O$ on the surface.

The surface pressure distribution in the present problem may therefore be taken as

$$p_0(r,t)=\begin{cases}0{\cdot}133R^{-3}Ef_1(\sqrt{H^2+r^2}/\sqrt{H^2+r_0^2}), & r<r_0(t),\\ 0, & r>r_0(t),\end{cases} \tag{42}$$

where

$$r_0^2(t)=R^2(t+t_1)-R^2(t_1) \tag{43}$$

and $t_1=$ time taken by the shock to reach the surface.

5.2 *Adjustment of Taylor's formula to the wave problem*

The pressure model (10) without the one for $r_0(t)$ results from the above when $H=0$. The same model may be retained when H is small compared with $r_0(t)$ or R. To this purpose, a Lagrange expansion (MacRobert [7], § 54) of $p_0(r,t)$ is useful.

Writing

$$\mu = (H^2 + r^2)/(H^2 + r_0^2),$$
$$\mu_0 = (r/r_0)^2, \tag{44}$$
$$f_1(\sqrt{\mu}) = f_2(\mu),$$

we get

$$\mu = \mu_0 + (H/r_0)^2(1-\mu),$$

and

$$f_2(\mu) = f_2(\mu_0) + \sum_1^\infty \frac{1}{m!}\left(\frac{H}{r_0}\right)^{2m} \frac{\mathrm{d}^{m-1}}{\mathrm{d}\mu_0^{m-1}}[(1-\mu_0)^m f_2'(\mu_0)]. \tag{45}$$

Also

$$\left(\frac{H}{r_0}\right)^{2m} = \left(\frac{R^2-H^2}{H^2}\right)^{-m} = (H/R)^{2m}\left(1-\frac{H^2}{R^2}\right)^{-m}$$
$$= (H/R)^{2m}\left[1 + m\left(\frac{H}{R}\right)^2 + \frac{m(m+1)}{2!}\left(\frac{H}{R}\right)^4 + \cdots\right]. \tag{46}$$

Consequently,

$$p_0(r,t) = 0{\cdot}121(E^2\rho_0^3)^{1/5}(t+t_1)^{-6/5}\left[f_2(\mu_0) + \sum_{m=1}^\infty \sum_{l=0}^\infty \frac{m(m+1)\cdots(m+l-1)}{m!\,l!} \times \{t_1/(t+t_1)\}^{4(m+l)/5} F_m(\mu_0)\right], \quad \text{when } \mu_0 < 1;$$
$$p_0(r,t) = 0, \qquad \text{when } \mu_0 > 1, \tag{47}$$

since

$$R(t+t_1) = (0{\cdot}926)^{-2/5}(E/\rho_0)^{1/5}(t+t_1)^{2/5}$$
$$= 1{\cdot}031(E/\rho_0)^{1/5}(t+t_1)^{2/5}. \tag{48}$$

Here

$$f_2(\mu_0) = \frac{2\gamma}{\gamma+1}\left[\frac{\gamma+1}{\gamma} - \frac{\mu_0^{(n-1)/2}}{\gamma}\right]^{-(2\gamma^2+7\gamma-3/7-\gamma)} \tag{49}$$

$$F_m(\mu_0) = \frac{d^{m-1}}{d\mu_0^{m-1}}[(1-\mu_0)^m f_2'(\mu_0)]. \tag{50}$$

5.3 *Asymptotic wave height*

Subject to the validity of the linearised wave theory, the asymptotic expression for ζ under the conditions (35) is

$$\zeta \simeq -0{\cdot}129(E^4\rho_0^{1/5}/g\rho r)\{k_0/|P''(k_0)|\}^{1/2}(t+t_1)^{-6/5} \times \{(t+t_1)^{4/5} - t_1^{4/5}\}\cos(k_0 r) \times \left[\tilde{f}_2(k_0 r_0(t)) + \sum_{m=1}^\infty \sum_{l=0}^\infty \frac{m(m+1)\cdots(m+l-1)}{m!\,l!} \times \{t_1/(t+t_1)\}^{4(m+l)/5} \tilde{F}_m(k_0 r_0(t))\right], \tag{51}$$

where

$$\tilde{f}_2(k) = \int_0^1 \alpha f_2(\alpha^2) J_0(k\alpha) \mathrm{d}\alpha. \tag{52}$$

The last result shows that a good approximation to ζ for small H is obtained if only the first term $\tilde{f}_2(k_0 r_0(t))$ in the square bracket is retained. Further evaluation of ζ can be accomplished, it seems, only by numerical methods.

6. Some characteristics of the motion

In both the expressions (29) and (51) for the wave elevation ζ, $k_0 r \gg 1$. The factor $\cos k_0 r$ in both therefore changes its sign rapidly so that we may regard $k_0 r$ as the phase and the co-factor of $\cos k_0 r$ as the amplitude of ζ in either expression. The phase is not directly affected by the velocity parameter u in (29) or by $r_0(t)$ in (29) and (51).

Since $\mathrm{d}k_0/\mathrm{d}t$ is positive as per (18), the degree of oscillation of level at any point becomes more rapid with time. Since $\dfrac{\mathrm{d}}{\mathrm{d}t}(\tau^2 - k_0 h)$ at first diminishes with τ (up to the value given by the equation $-2\tau^2 P_j''(k_0) = h$) and then increases with it, the oscillation at any point in shallow water is somewhat more rapid at first and less rapid thereafter than what would happen if the sea were deep.

Denoting $t\sqrt{gh}/(2r)$ by τ_0, we have for

$$k_0 \sim \kappa(\tau_0), \quad P_j''(k_0)[\text{or } P''(k_0)] \sim (\tau_0/\tau)[P_j''(\kappa) \text{ or } P''(\kappa)] \equiv P_0''(\kappa),$$

(say) and equation $(P_j'(k))_{u=0} = 0)$ may be written as $t/\psi(\kappa) = 1$, where

$$[\psi(k)]^{-1} = (\sqrt{gh}/2r)[\{\tanh kh/kh\}^{1/2} + \{kh/\tanh kh\}^{1/2} \operatorname{sech}^2 kh]. \tag{53}$$

The amplitude of ζ in (29) then varies as

$$r^{-1} t^{2-n} \kappa^{1/2} |P_0''(\kappa)|^{-1/2} \bar{F}(\kappa u t).$$

From (53), it appears that $kh = O(\tau_0^2)$ and $P''(\kappa) = O(\kappa^{-1})$ when $\kappa h (\text{or } \tau) \gg 1$. As n is usually > 1, one finds that the amplitude $\to 0$ as $t \to \infty$ when $\bar{F}(x)$ is $O(x^{-1})$ or of a higher order of smallness as $x \to \infty$.

The times of maximum amplitude at any point are given by

$$t_n = \frac{2r}{\sqrt{gh}} \frac{(a_n \tanh a_n)^{1/2}}{\tanh a_n + a_n \operatorname{sech}^2 a_n}, \quad n = 1, 2, 3, \ldots,$$

where

$$\kappa = a_n, \quad n = 1, 2, 3, \ldots$$

satisfy the equation

$$\left[2x\left(\frac{1}{\kappa P_0''(\kappa)} - 1\right)\frac{\bar{F}'(x)}{\bar{F}(x)}\right]_{x = \kappa u \psi(\kappa)} = 3 - 2n + \left\{\frac{P_0'''(\kappa)}{P_0''^2(\kappa)} - \frac{1}{\kappa P_0''(\kappa)}\right\}.$$

Therefore, the points of maximum amplitude at a distance r travel outwards with the corresponding constant velocities

$$\frac{1}{2}\sqrt{gh}\,\frac{\tanh a_n + a_n \operatorname{sech}^2 a_n}{(a_n \tanh a_n)^{1/2}}.$$

The amplitude at any point becomes nearly zero at times

$$\tau_n = \frac{2r}{\sqrt{gh}}\,\frac{(b_n \tanh b_n)^{1/2}}{\tanh b_n + b_n \operatorname{sech}^2 b_n}$$

where

$$\kappa = b_n, \quad n = 1, 2, 3, \ldots$$

satisfy the equation $\bar{F}(\kappa u \psi(\kappa)) = 0$. These points of minimum amplitude travel outwards with the corresponding constant velocities

$$\frac{1}{2}\sqrt{gh}\,\frac{\tanh b_n + b_n \operatorname{sech}^2 b_n}{(b_n \tanh b_n)^{1/2}}.$$

The values of a_n and b_n increase with n. Hence, the outer rings spread out faster than the inner ones. A similar discussion may be given for (51).

7. A particular case

Let

$$f(r,t) = D(t+t_1)^{-n} = F(r)(t+t_1)^{-n}, \quad r < r_0(t) = ut.$$

Therefore

$$\begin{aligned}\bar{F}(k_0 ut) &= \int_0^1 \alpha F(\alpha) J_0(\alpha k_0 ut)\,\mathrm{d}\alpha \\ &= \frac{D}{k_0 ut}\int_0^1 (k_0 \alpha ut) J_0(\alpha k_0 ut)\,\mathrm{d}\alpha \\ &= \frac{D}{k_0 ut} J_1(k_0 ut).\end{aligned}$$

Then (29) gives

$$\frac{g\rho t_1}{D}\zeta = -\frac{(ut)t_1}{k_0^{1/2} r}\,\frac{J_1(k_0 ut)}{(t+t_1)^n}\,|p_j''(k_0)|^{-1/2}\cos(k_0 r), \quad r \gg r_0(t) = ut, \ k_0 r \gg 1. \tag{54}$$

By (18), we have

$$\begin{aligned}P_j''(k) = \frac{g^{1/2}(t+t_1)}{4r(k\tanh kh)^{3/2}}&[4kh(\tanh kh)(\operatorname{sech}^2 kh)(1 + k\tanh kh) \\ &- (\tanh kh + kh\operatorname{sech}^2 kh)^2].\end{aligned} \tag{55}$$

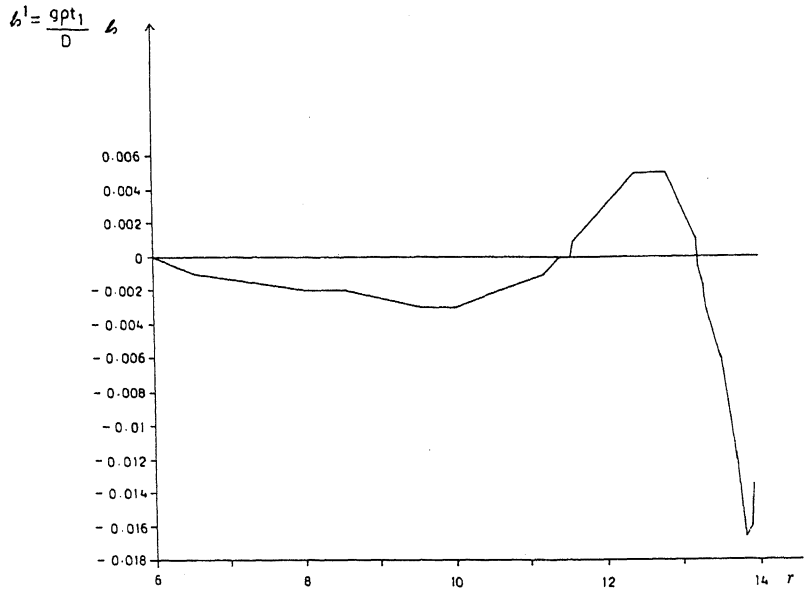

Figure 1. Variation of ζ^1 with r. $u = 0{\cdot}05$, $n = 1$, $g = 32$, $t_1 = 0{\cdot}5$, $t = 2$, $h = 1$.

Now let us take

$$J_1(k_0 ut) \simeq \left(\frac{2}{\pi k_0 ut}\right)^{1/2} \cos\left(k_0 ut - \frac{3}{4}\pi\right). \tag{56}$$

Using (55) and (56) in (54), we get

$$\begin{aligned}\frac{\sqrt{2\pi} g\rho t_1}{D}\zeta = &\left[\frac{(ut)t_1}{rk_0^{1/2}}\right]\left[\frac{(k_0 ut)^{-1/2}}{(t+t_1)^n}\right]\left[\frac{g^{1/2}(t+t_1)}{r(k_0 \tanh k_0 h)^{3/2}}\right]^{-1/2} \\ &\times [(4k_0 h \tanh k_0 h \operatorname{sech}^2 k_0 h)(1 + k_0 \tanh k_0 h) \\ &- (\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h)^2]^{-1/2} \\ &\times \cos(k_0 r)\cos(k_0 ut - 3\pi/4), \quad r > ut. \end{aligned} \tag{57}$$

The variation of

$$\zeta' = \frac{\sqrt{2\pi} g\rho t_1}{D}\zeta$$

with r as shown in figure 1.

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Generation and propagation of *SH*-type waves due to stress discontinuity in a linear viscoelastic layered medium

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Abstract. In this paper the generation and propagation of *SH*-type waves due to stress discontinuity in a linear viscoelastic layered medium is studied. Using Fourier transforms and complex contour integration technique, the displacement is evaluated at the free surface in closed form for two special types of stress discontinuity created at the interface. The numerical result for displacement component is evaluated for different values of non-dimensional station (distance) and is shown graphically. Graphs are compared with the corresponding graph of classical elastic case.

Keywords. *SH*-type waves; stress discontinuity;

1. Introduction

The usefulness of surface waves and its investigations in isotropic elastic medium have been well recognised in the study of earthquake waves, seismology and geophysics. Wave propagation in a layered medium has been studied extensively by many people, especially in the last two decades. Various approximate theories have been proposed to predict the dynamic response of layered medium and one of them is due to Sun *et al* [10]. Nag and Pal [7] have considered the disturbance of *SH*-type waves due to shearing stress discontinuity in an isotropic elastic medium. In another paper, Pal and Debnath [8] have considered the propagation of *SH*-type waves due to uniformly moving stress discontinuity at the interface of anisotropic elastic layered media.

Due to the effect of viscosity, gravity plays an important role in the propagation of surface waves (Love, Rayleigh, etc.). The viscoelastic behaviour of the material is described by the mechanical behaviour of solid materials with small voids. The linear viscoelasticity generally displayed by linear elastic materials is termed as 'standard linear solid', if elastic materials are having voids.

Kanai [5] has discussed the Love-type waves propagating in a singly stratified viscoelastic layer residing on the semi-infinite viscoelastic body under the conditions of the surface of discontinuity. Sarkar [9] considered the effect of body forces and stress discontinuity on the motion of *SH*-type waves in a semi-infinite viscoelastic medium. The propagation of *SH*-waves in nonhomogeneous viscoelastic layer over a semi-infinite voigt medium due to irregularity in the crustal layer has been discussed by Chattopadhyay [1]. He has followed the perturbation technique as indicated by

Eringen and Samuels [4]. The viscoelastic behaviour of linear elastic materials with voids has been considered by Cowin [2].

The present paper considers the generation and propagation of *SH*-type wave due to shearing stress discontinuity at the interface of two homogeneous viscoelastic media. Fourier transform method combined with complex contour integration technique is used to evaluate the displacement function at the free surface for two different types of stress discontinuity. Numerical results are obtained for a case only with the aid of viscoelastic model as considered by Martineĉk [6]. Results are shown graphically and are found to be in good agreement with classical elastic case.

Since the material of the earth is viscoelastic of a standard linear type, certain seismic observations and calculations may be explained on this basis. Thus the problem considered here is of interest in the theory of seismology.

2. Formulation of the problem and basic equations

Let us consider a viscoelastic layer of standard linear type (I) of thickness h lying over a viscoelastic half-space (II). The origin of the rectangular co-ordinate system is taken at the interface. The wave-generating mechanism is a shearing stress discontinuity which is assured to be created suddenly at the interface. The geometry of the problem is depicted in figure 1. As the *SH*-type of motion is being considered here, we have $u = w = 0$ and $v = v(x, z, t)$. The displacement v is also assumed to be continuous, bounded and independent of y. The only equation of motion in two-layered viscoelastic media in terms of stress components is given by

$$\rho_i \frac{\partial^2 v_i}{\partial t^2} = \frac{\partial}{\partial x}(\tau_{xy})_i + \frac{\partial}{\partial z}(\tau_{yz})_i, \quad i = 1, 2 \tag{2.1}$$

where

$$(\tau_{xy})_i = \left(\mu_i + \mu_i' \frac{\partial}{\partial t}\right)\frac{\partial v_i}{\partial x}$$

$$(\tau_{yz})_i = \left(\mu_i + \mu_i' \frac{\partial}{\partial t}\right)\frac{\partial v_i}{\partial z} \tag{2.2}$$

μ_i are related to shear moduli and μ_i' to viscoelastic parameters. Substituting (2.2) in (2.1), the resulting equations of motion become

$$\rho_i \frac{\partial^2 v_i}{\partial t^2} = \left(\mu_i + \mu_i' \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial z^2}\right). \tag{2.3}$$

Assuming that the stress functions are harmonic and decrease with time, we have

$$\tau_{xy} = \theta(x, z)e^{-\omega t}, \quad \tau_{yz} = \psi(x, z)e^{-\omega t}, \tag{2.4}$$

and correspondingly

$$v(x, z) = V(x, z)e^{-\omega t}, \tag{2.5}$$

where ω is the frequency parameter.

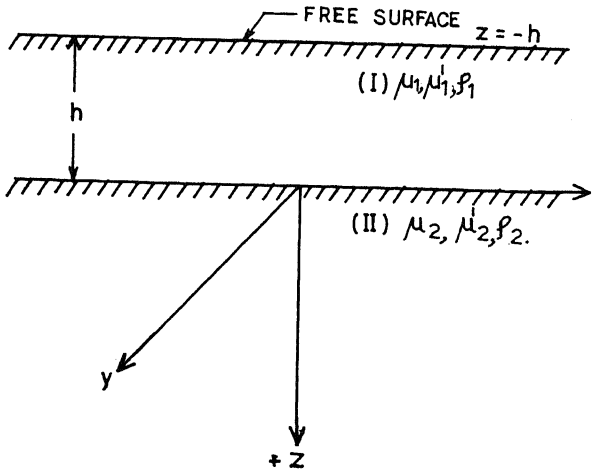

Figure 1. Standard linear viscoelastic layered model.

With the help of (2.5), (2.3) becomes

$$\omega^2 V = \sigma_j^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} \right), \tag{2.6}$$

ere

$$\sigma_j^2 = \frac{(\mu_j - \mu_j' \omega)}{\rho_j}, \quad j = 1, 2.$$

Method of solution

us define the Fourier transform $\bar{V}(\xi, z)$ of $V(x, z)$ by

$$\bar{V}(\xi, z) = \int_{-\infty}^{\infty} V(x, z) e^{-i\xi x} \mathrm{d}x. \tag{3.1}$$

refore

$$V(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{V}(\xi, z) e^{i\xi x} \mathrm{d}\xi. \tag{3.2}$$

pplying the above transformation into (2.6), it is found that $\bar{V}(\xi, z)$ satisfies the ation

$$\omega^2 \bar{V}(\xi, z) = \sigma_j^2 \left(-\xi^2 \bar{V} + \frac{\partial^2 \bar{V}}{\partial z^2} \right)$$

$$\frac{\partial^2 \bar{V}}{}$$

where

$$\eta_j^2 = \xi^2 + \frac{\omega^2}{\sigma_j^2}, \quad j = 1, 2. \tag{3.3}$$

Thus for the layers (I) and (II) we have

$$V_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} (A_1 \cosh \eta_1 z + B_1 \sinh \eta_1 z) e^{i\xi x} \mathrm{d}\xi, \tag{3.4}$$

$$V_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} B_2 e^{(-\eta_2 z + i\xi x)} \mathrm{d}\xi. \tag{3.5}$$

The boundary conditions of the problem under consideration are

(i) stress component must vanish on the free surface i.e.

$$(\tau_{yz})_1 = 0 \text{ at } z = -h \quad \text{for all } t > 0 \tag{3.6}$$

(ii) displacements must be continuous at the interface i.e.

$$V_1 = V_2 \text{ at } z = 0 \quad \text{for } t > 0 \tag{3.7}$$

(iii) stress components (shearing) must be discontinuous at the interface $z = 0$ i.e.

$$(\tau_{yz})_1 = (\tau_{yz})_2 = S(x) e^{-\omega t} \text{ at } z = 0, \quad \text{for all } x \text{ and } t, \tag{3.8}$$

where $S(x)$ is some continuous function of x to be chosen later.

The above boundary conditions determine the unknown constants A_1, A_2 and B_2. After simplifying we have at the free surface ($z = -h$)

$$V_1(x, -h) = \int_{-\infty}^{\infty} \exp(-\eta_1 h + i\xi x) \frac{U(\xi)}{\eta_1} [\Sigma \{K^m e^{-2m\eta} s_1^h + K^{m+1} e^{-(2m+1)\eta} s_1^h\}] \mathrm{d}\xi, \tag{3.9}$$

where $U(\xi)$ is an unknown function related to $S(x)$ by

$$U(\xi) = \frac{1}{2\pi\sigma_1^2 \rho_1}\int_{-\infty}^{\infty} S(x) \exp(i\xi x) \mathrm{d}x \tag{3.10}$$

and

$$K = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} < 1$$

which is associated with the reflection coefficient in the two media.

4. Determination of unknown function $U(\xi)$

We now consider two different forms of the function $S(x)$ to determine $U(\xi)$.

Case I. Let

$$S(x) = P, \quad |x| \leqslant a$$
$$= 0, \quad \text{elsewhere.} \tag{4.1}$$

This case implies that the stress discontinuity is created in the region $-a \leqslant x \leqslant a$. Hence

$$U(\xi) = \frac{P}{2\pi\sigma_1^2\rho_1}\int_{-a}^{a} \exp(-i\xi x)\mathrm{d}x \tag{4.2}$$

$$= \frac{Pi}{2\pi\sigma_1^2\rho_1}\left(\frac{e^{i\xi a} - e^{-i\xi a}}{\xi}\right).$$

From (3.9) and (4.2) we have

$$V_1(x,-h) = \frac{P}{\pi\sigma_1^2\rho_1} I_m \int_0^{\infty} \left(\frac{e^{i\xi x_1} - e^{i\xi x_2}}{\eta_1 \xi}\right) e^{-\eta_1 h}[\Sigma K^m e^{-2m\eta_1 h}$$

$$+ K^{m+1} e^{-(2m+1)\eta_1 h}]\mathrm{d}\xi$$

$$(x_1 = x - a, x_2 = x + a). \tag{4.3}$$

Here we wish to evaluate the integral for a few values of m, say $m = 0, 1, 2$ only. So we have

$$V_1(x,-h) = \frac{P}{\pi\sigma_1^2\rho_1}[I_0 + I_1 + I_2 + \cdots], \tag{4.4}$$

where

$$I_0 = \int_0^{\infty}\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-\eta_1 h}\mathrm{d}\xi + \int_0^{\infty} K\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-3\eta_1 h}\mathrm{d}\xi$$

$$= I_{01} + I_{02}; \text{(say)} \tag{4.5}$$

$$I_1 = \int_0^{\infty} K\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-5\eta_1 h}\mathrm{d}\xi$$

$$+ \int_0^{\infty} K^2\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-7\eta_1 h}\mathrm{d}\xi$$

$$= I_{11} + I_{12}; \text{(say)} \tag{4.6}$$

$$I_2 = \int_0^{\infty} K^2\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-9\eta_1 h}\mathrm{d}\xi$$

$$+ \int_0^{\infty} K^3\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-11\eta_1 h}\mathrm{d}\xi$$

$$= I_{21} + I_{22}; \text{(say)}. \tag{4.7}$$

To evaluate $I_{02}, I_{11}, I_{12}, I_{21}, I_{22}$, we use the method of contour integration and I_{01} is directly evaluated from the Table of Integral Transforms by Eradelyi [3].

Thus, we have

$$I_{01} = \int_0^{x_1} \mathbb{K}_0\left[\frac{\omega h}{\sigma_1}\sqrt{\frac{x^2}{h^2} + \frac{1}{4}}\right]\mathrm{d}x + \int_0^{x_2} \mathbb{K}_0\left[\frac{\omega h}{\sigma_1}\sqrt{\frac{x^2}{h^2} + \frac{1}{4}}\right]\mathrm{d}x \tag{4.8}$$

where $\mathbb{K}_0(\theta)$ is a modified Bessel's function of argument θ and of order zero.

$$I_{02} = -2\int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{(\sin\xi\bar{x}_1 - \sin\xi\bar{x}_2)}{\zeta\left(\frac{\omega^2h^2}{\sigma_1^2} - \zeta^2\right)^{1/2}} e^{-(3/2)[(\omega^2h^2/\sigma_1^2)-\zeta^2]^{1/2}} d\zeta \tag{4.9}$$

where $\bar{x}_1 = (x_1/h), \bar{x}_2 = (x_2/h)$ and $\omega h/\sigma_1 > \omega h/\sigma_2$

$$I_{12} = 8\int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{(\sin\xi\bar{x}_1 - \sin\xi\bar{x}_2)}{\zeta\left(\frac{\omega^2h^2}{\sigma_1^2} - \zeta^2\right)^{1/2}} e^{-(5/2)[(\omega^2h^2/\sigma_1^2)-\zeta^2]^{1/2}} d\zeta \tag{4.10}$$

$$I_{02} = -2\int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{(\sin\xi\bar{x}_1 - \sin\xi\bar{x}_2)}{\zeta\left(\frac{\omega^2h^2}{\sigma_1^2} - \zeta^2\right)} e^{-(7/2)[(\omega^2h^2/\sigma_1^2)-\zeta^2)]^{1/2}} D(\zeta)d\zeta \tag{4.11}$$

where

$$D(\zeta) = \frac{\left(\frac{\omega^2h^2}{\sigma_1^2} + \frac{\omega^2h^2}{\sigma_2^2} - 2\zeta^2\right)\left(\zeta^2 - \frac{\omega^2h^2}{\sigma_2^2}\right)^{1/2}}{\zeta\left[\left(\frac{\omega^2h^2}{\sigma_1^2} + \frac{\omega^2h^2}{\sigma_2^2} - 2\zeta^2\right)^2 + 4\left(\frac{\omega^2h^2}{\sigma_1^2} - \zeta^2\right)\left(\zeta^2 - \frac{\omega^2h^2}{\sigma_2^2}\right)\right]} \tag{4.12}$$

where $\omega h/\sigma_1 > \omega h/\sigma_2$.

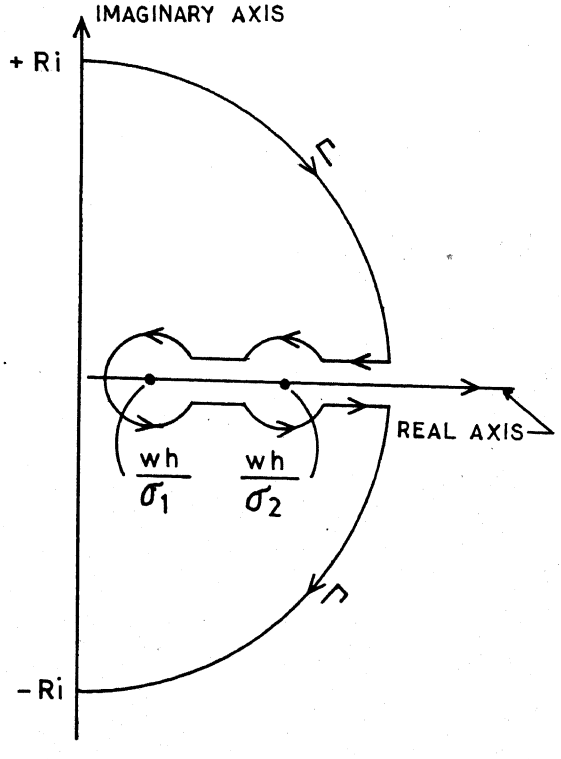

Figure 2. Complex contour integration in ζ-plane.

The integrals in $I_{01}, I_{02}, I_{11}, I_{12}, \ldots$ have branch points at $\zeta = \pm \omega h/\sigma_1$, $\pm \omega h/\sigma_2$ and a simple pole at $\zeta = 0$. The path of the contour integration is shown in figure 2. Hence

$$v_1(x, -h, t) = \frac{e^{-\omega t} P}{\pi \sigma_1^2 \rho_1} [I_{01} + I_{02} + I_{11} + I_{12} + \cdots]. \tag{4.13}$$

Case II. Let

$$S(x) = Ph\delta(x), \quad -\infty \leqslant x < \infty. \tag{4.14}$$

Factor h is multiplied on the right side because both sides should maintain the dimension of stress.

Now

$$U(\xi) = \frac{Ph}{2\pi \sigma_1^2 \rho_1}. \tag{4.15}$$

Therefore, in this case, we have

$$\begin{aligned} V_1(x, -h) &= \frac{Ph}{\pi \sigma_1^2 \rho_1} \operatorname{Re} \int_0^\infty \frac{e^{i\zeta x} e^{-\eta_1 h}}{\eta_1} [\Sigma K^m e^{-2m\eta_1 h} + K^{m+1} e^{-(2m+1)\eta_1 h}] \mathrm{d}\xi \\ &= \frac{Ph}{\pi \sigma_1^2 \rho_1} \int_0^\infty \frac{\cos \xi x \, e^{-\eta_1 h}}{\eta_1} [\Sigma K^m e^{-2m\eta_1 h} + K^{m+1} e^{-(2m+1)\eta_1 h}] \mathrm{d}\xi \end{aligned} \tag{4.16}$$

In this case also, we evaluate the integral on the right-hand side of (4.16) for a few values of m only, say $m = 0, 1, 2$. Hence

$$V_1(x, -h) = \frac{Ph}{\pi \sigma_1^2 \rho_1} [I_0 + I_1 + I_2 + \ldots], \tag{4.17}$$

$$\begin{aligned} I_0 &= \int_0^\infty \frac{\cos \xi x e^{-\eta_1 h}}{\eta_1} [1 + K e^{-\eta_1 h}] \mathrm{d}\xi \\ &= I_{01} + I_{02} \text{ (say)} \end{aligned} \tag{4.18}$$

$$\begin{aligned} I_1 &= \int_0^\infty \frac{\cos \xi x e^{-\eta_1 h}}{\eta_1} [K e^{-2\eta_1 h} + K^2 e^{-3\eta_1 h}] \mathrm{d}\xi \\ &= I_{11} + I_{12} \text{ (say)} \end{aligned} \tag{4.19}$$

$$\begin{aligned} I_2 &= \int_0^\infty \frac{\cos \xi x e^{-\eta_1 h}}{\eta_1} [K^2 e^{-4\eta_1 h} + K^3 e^{-5\eta_1 h}] \mathrm{d}\xi \\ &= I_{21} + I_{22} \text{ (say)}. \end{aligned} \tag{4.20}$$

Just like case I, we can evaluate $I_{01}, I_{02}, I_{11}, I_{12}, \ldots$ as follows:

$$I_{01} = K_0 \left[\frac{\omega h}{\sigma_1} \sqrt{\bar{x}^2 + \frac{1}{4}} \right] \quad (\bar{x} = x/h) \tag{4.21}$$

$$I_{02} = -4\int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{\left[e^{-\zeta\bar{x}} e^{-(3/2)[(\omega^2h^2/\sigma_1^2)-\zeta^2]^{1/2}[\zeta^2-[\omega^2h^2/\sigma_2^2]^{1/2}}\right]}{\left[\frac{\omega^2h^2}{\sigma_1^2}+\frac{\omega^2h^2}{\sigma_1^2}\right]} d\zeta \tag{4.22}$$

$$I_{11} = -4\int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{\left[e^{-\zeta\bar{x}} e^{-5/2[(\omega^2h^2/\sigma_1^2)-\xi^2]^{1/2}[\xi^2-(\omega^2h^2/\sigma_2^2)]^{1/2}}\right]}{\left[\frac{\omega^2h^2}{\sigma_1^2}+\frac{\omega^2h^2}{\sigma_1^2}\right]} d\zeta \tag{4.23}$$

$$I_{12} = -8\int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{\left[e^{-\xi\bar{x}} e^{-7/2[\omega^2h^2/\sigma_1^2)-\xi^2]^{1/2}[\xi^2-(\omega^2h^2/\sigma_2^2)]^{1/2}}\right]}{\left[\frac{\omega^2h^2}{\sigma_1^2}+\frac{\omega^2h^2}{\sigma_1^2}\right]} d\zeta \tag{4.24}$$

etc.

The integrals in (4.22), (4.23), (4.24) are valid only when $\omega h/\sigma_1 > \omega h/\sigma_2$.

Hence, in this case the displacement component on the free surface $z = -h$ is given by

$$v_1(x, -h, t) = \frac{e^{-\omega t}Ph}{\pi\sigma_1^2\rho_1}[I_{01} + I_{02} + I_{11} + I_{12} + \cdots]. \tag{4.25}$$

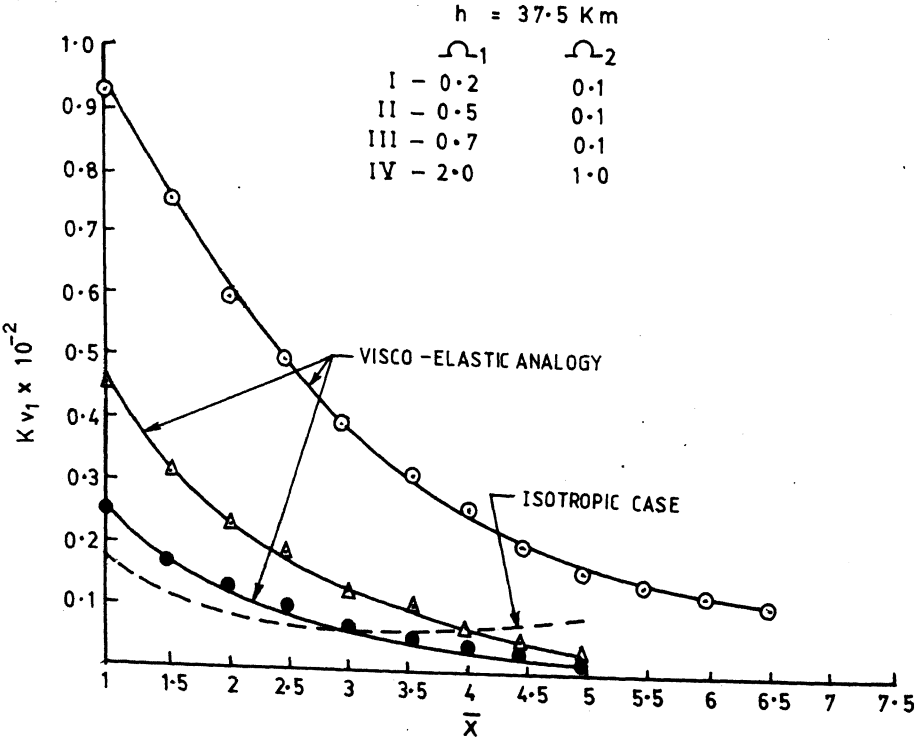

Figure 3. Variation of displacement with distance from the source.

Numerical results and discussion

umerical calculations are performed here for case II only using Gauss quadrature rmula and the table of integral transforms (Eradelyi [3]). The values of $Kv_1 \times 10^{-2}$, here $K = \pi\sigma_1^2\rho_1 e^{\omega t}/P$ are tabulated for different values of $\bar{x}$ and $\Omega_1 = \omega h/\sigma_1$ and eeping $\Omega_2 = \omega h/\sigma_2$ constant. The values of non-dimensional parameters Ω_1 and Ω_2 e taken from a viscoelastic model considered by Martineĉk [6]. For comparison graph corresponding to isotropic case is drawn (figure 3) and is found to be in good greement with viscoelastic analogy up to a certain value of $\bar{x}$. From the curves so awn, it is inferred that the displacement v_1 decreases as $\bar{x}$ increases and the rate of crease slows down after a certain distance.

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proof of Howard's conjecture in homogeneous parallel shear flows – I: Limitations of Fjortoft's necessary instability criterion

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Abstract. The present paper on the linear instability of nonviscous homogeneous parallel shear flows mathematically demonstrates the correctness of Howard's [4] prediction, for a class of velocity distributions specified by a monotone function U of the altitude y and a single point of inflexion in the domain of flow, by showing not only the existence of a critical wave number $k_c > 0$ but also deriving an explicit expression for it, beyond which for all wave numbers the manifesting perturbations attain stability. An exciting conclusion to which the above result leads to is that the necessary instability criterion of Fjortoft has the seeds of its own destruction in the entire range of wave numbers $k > k_c$—a result which is not at all evident either from the criterion itself or from its derivation and has thus remained undiscovered ever since Fjortoft enunciated [3].

Keywords. Shear flows;

Introduction

he point of inflexion theorem of Rayleigh [5] and the semicircle theorem of Howard ·] impose necessary restrictions on the basic velocity field $U(y)$ and the complex ave velocity field $c = c_r + ic_i$ which are accessible to an arbitrary unstable ($c_i > 0$) ave in the linear instability of nonviscous homogeneous parallel shear flows and it is interest to have a similar restriction on the growth rate kc_i possible for such an nstable wave, k being the wave number and y being the altitude. In his pioneering ontribution (1961; henceforth referred to as Ho), Howard established one such timate in the form

$$k^2 c_i^2 \leqslant \operatorname*{Max}_{\text{Flow domain}} \left(\frac{dU}{dy}\right)^2, \tag{1}$$

nd considering its inability to provide the correct qualitative result for the case of ane Couette flow with dU/dy constant, which is known to be neutrally stable with $c_i \to 0$ as $k \to \infty$ remarked "This estimate is not usually sharp—for example, the ouette flow with dU/dy constant, is known to be neutrally stable—but in most cases will probably give the correct order of magnitude of the maximum growth rate. It is fficient to show that c_i must approach zero as wavelength decreases to zero given e boundedness of dU/dy; but there is likelihood that infact $kc_i \to 0$ as $k \to \infty$, and ith sufficient assumptions the still stronger statement that all waves shorter than

some critical wavelength are stable is probably true, as illustrated by the examples of Drazin and Holmboe cited in I".

A rigorous mathematical proof of the first part of this conjectural assertion of Howard, namely that $kc_i \to 0$ as $k \to \infty$, was given in an earlier paper by Banerjee *et al* [1] under the restriction of the boundedness of d^2U/dy^2 in the concerned domain of flow and the present paper which is in continuation to the earlier one mathematically demonstrates the correctness of the latter part of this assertion, namely that all waves shorter than some critical wavelength are stable, that is $c_i = 0$ when $k > k_c$ where k_c is some critical value of k for the class of velocity distributions specified by a monotone function U of the altitude y and having a single point of inflexion in the domain of flow [2].

An exciting conclusion to which this latter part of Howard's assertion leads to is that the basic assumption $c_i \neq 0$ in Fjortoft's derivation of his necessary instability criterion breaks down, for the class of velocity distributions as specified in the preceding paragraph, in the wave number range $k > k_c$ where k_c has the same meaning as given in the abstract, thus rendering the derivation of the criterion invalid. This invalidity assumes striking proportions for the wave with wave length zero, that is $k \to \infty$, in which case Fjortoft's necessary criterion of instability is actually a sufficient criterion of stability as will be shown later. What is really surprising is that it has taken such a long time to discover this wave number dependence of Fjortoft's necessary instability criterion but it may, possibly, be expected on the ground that neither the Fjortoft's discriminant $(d^2U/dy^2)(U - U_s)$ which is to be negative somewhere in the domain of flow for any general velocity distribution $U(y)$ and negative everywhere in the domain of flow except being zero at the point of inflexion of $U(y)$ in the present context, involves any wave number implicitly or explicitly nor the derivation of the criterion itself shows any restrictivity with respect to some wave number in the set of all admissible wave numbers $k \geqslant 0$ where $U_s = U(y_s)$, $y_1 < y_s < y_2$ and $d^2U/dy^2 = 0$ at $y = y_s$ with U being twice continuously differentiable in $y_1 \leqslant y \leqslant y_2$.

Proof of Howard's Conjecture. To facilitate reference to Ho, we shall make use of the same notation here and denote the basic velocity field by $U(y)$ while the Rayleigh stability equation that governs the linear instability of nonviscous homogeneous parallel shear flows is

(Ho; equation (5.1) with $\beta = 0$ and $n = 1$)

$$\frac{d^2H}{dy^2} - k^2H - \frac{\left(\frac{d^2U}{dy^2}\right)H}{U - c} = 0. \tag{2}$$

The boundary conditions are that H must vanish on the rigid walls which may recede to $\pm\infty$ in the limiting cases and thus

$$H(y_1) = H(y_2) = 0. \tag{3}$$

Multiplying equation (2) by H^* (the complex conjugate of H) throughout and integrating the resulting equation over the vertical range of y with the help of the

boundary conditions (3), we derive

$$\int_{y_1}^{y_2} (|DH|^2 + k^2|H|^2)\mathrm{d}y + \int_{y_1}^{y_2} \frac{\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)|H|^2}{U-c}\,\mathrm{d}y = 0, \tag{4}$$

where D stands for $\mathrm{d}/\mathrm{d}z$.

Equating the real and the imaginary parts of both sides of equation (4), we obtain

$$\int_{y_1}^{y_2} (|DH|^2 + k^2|H|^2)\mathrm{d}y + \int_{y_1}^{y_2} \frac{\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)(U-c_r)|H|^2}{(U-c_r)^2 + c_i^2}\,\mathrm{d}y = 0, \tag{5}$$

and

$$c_i \int_{y_1}^{y_2} \frac{\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)|H|^2}{(U-c_r)^2 + c_i^2}\,\mathrm{d}y = 0. \tag{6}$$

Rayleigh's theorem, which states that a necessary criterion of instability ($c_i > 0$) is that the velocity distribution $U(y)$ must have at least one point of inflexion at some $y = y_s$ where $y_1 < y_s < y_2$ and $U_s = U(y_s)$ follows from equation (6) while Fjortoft's more stronger theorem, which states that a necessary criterion of instability is that

$$\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)(U-U_s) < 0 \quad \text{at some point } y = y_q \neq y_s \text{ (obviously)}$$
$$\text{where } y_1 < y_q < y_2 \text{ and } U_s = U(y_s), \tag{7}$$

follows from equation

$$\int_{y_1}^{y_2} (|DH|^2 + k^2|H|^2)\mathrm{d}y + \int_{y_1}^{y_2} \frac{\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)(U-U_s)|H|^2}{(U-c_r)^2 + c_i^2}\,\mathrm{d}y = 0, \tag{8}$$

which is obtained by multiplying equation (6) throughout by the constant factor $(c_r - U_s)$ after cancelling $c_i > 0$ from both sides of it and then adding the resulting equation to equation (5).

Further multiplying equation (2) by $\mathrm{d}^2H^*/\mathrm{d}y^2$ throughout, we get

$$\frac{\mathrm{d}^2H^*}{\mathrm{d}y^2}\left(\frac{\mathrm{d}^2H}{\mathrm{d}y^2} - k^2H\right) - \frac{\mathrm{d}^2H^*}{\mathrm{d}y^2}\cdot\frac{\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}H\right)}{U-c} = 0, \tag{9}$$

and substituting for $\mathrm{d}^2H^*/\mathrm{d}y^2$ from equation (2) in the last term of equation (9), we derive upon integrating this latter resulting equation over the range of y with the help of the boundary conditions (3)

$$\int_{y_1}^{y_2} (|D^2H|^2 + k^2|DH|^2)\mathrm{d}y - k^2 \int_{y_1}^{y_2} \frac{\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)|H|^2}{U-c} -$$

$$\int_{y_1}^{y_2} \frac{\left(\frac{d^2U}{dy^2}\right)^2 |H|^2}{(U-c_r)^2+c_i^2} dy = 0. \tag{10}$$

Equating the real part of both sides of equation (10), it follows that

$$\int_{y_1}^{y_2} (|D^2H|^2 + k^2|DH|^2)dy - k^2 \int_{y_1}^{y_2} \frac{\left(\frac{d^2U}{dy^2}\right)(U-c_r)|H|^2}{(U-c_r)^2+c_i^2} - \int_{y_1}^{y_2} \frac{\left(\frac{d^2U}{dy^2}\right)^2 |H|^2}{(U-c_r)^2+c_i^2} dy = 0, \tag{11}$$

and adding to equation (11), the equation

$$k^2(U_s - c_r) \int_{y_1}^{y_2} \frac{\left(\frac{d^2U}{dy^2}\right)|H|^2}{(U-c_r)^2+c_i^2} dy = 0, \tag{12}$$

which follows from equation (6) since $c_i > 0$, we obtain

$$\int_{y_1}^{y_2} (|D^2H|^2 + k^2|DH|^2)dy - k^2 \int_{y_1}^{y_2} \frac{\left(\frac{d^2U}{dy^2}\right)(U-U_s)|H|^2}{(U-c_r)^2+c_i^2} dy - \int_{y_1}^{y_2} \frac{\left(\frac{d^2U}{dy^2}\right)^2 |H|^2}{(U-c_r)^2+c_i^2} dy = 0, \tag{13}$$

U_s being the value of U at $y = y_s$ where $y_1 < y_s < y_2$. Writing equation (13) in the form

$$\int_{y_1}^{y_2} (|D^2H|^2 + k^2|DH|^2)dy - k^2 \int_{y_1}^{y_2} \frac{\left[\left(\frac{d^2U}{dy^2}\right)(U-U_s) + \left(\frac{d^2U}{dy^2}\right)^2 \Big/ k^2\right]}{(U-c_r)^2+c_i^2} |H|^2 dy = 0, \tag{14}$$

we derive that a necessary criterion of instability is that

$$\left(\frac{d^2U}{dy^2}\right)(U-U_s) + \frac{\left(\frac{d^2U}{dy^2}\right)^2}{k^2} > 0 \quad \text{at some point } y = y_p \neq y_s \text{ (obviously)}$$

$$\text{where } y_1 < y_p < y_2. \tag{15}$$

The necessary instability criterion expressed by inequality (15) imposes another independent restriction, one being imposed by Fjortoft on Fjortoft's discriminant $(d^2U/dy^2)(U - U_s)$, and is valid for any general velocity distribution $U(y)$.

We shall presently show the importance of this necessary instability criterion inestablishing the conjecture of Howard for a specific class of velocity distributions.

Consider the class of velocity distributions specified by a monotone function U of the altitude y and a single point of inflexion in the domain of flow $y_1 \leqslant y \leqslant y_2$. If instability is to manifest in such flows then Rayleigh's criterion implies that $y_1 < y_s < y_2$ and Fjortoft's more stronger criterion implies that

$$\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right)(U - U_s) \leqslant 0 \quad \text{everywhere in } y_1 \leqslant y \leqslant y_2, \tag{16}$$

with equality only where $y = y_s$ [2]. It may be noted that for a $U(y)$ belonging to this class $(\mathrm{d}^2 U/\mathrm{d}y^2)(U - U_s)$ can either be $\leqslant 0$ or $\geqslant 0$ everywhere in the domain of flow with equality only where $y = y_s$, and it is Fjortoft's criterion which shows that only those flows can possibly be unstable for which $(\mathrm{d}^2 U/\mathrm{d}y^2)(U - U_s) \leqslant 0$ everywhere in the domain of flow with equality only where $y = y_s$. Thus, a necessary criterion of instability can be derived from inequalities (15) and (16) in the form

$$-\left|\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right| |U - U_s| + \frac{\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right)^2}{k^2} > 0 \quad \text{at some point } y = y_p \neq y_s \text{ (obviously)}$$

$$\text{where } y_1 < y_p < y_2. \tag{17}$$

Hence, if

$$k^2 > k_c^2 = \max_{y(\neq y_s) \in \text{Flow Domain}} \left[\frac{\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right)^2}{\left|\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right| |U - U_s|} \right], \tag{18}$$

then the basic assumption $c_i > 0$ is not tenable and we must have $c_i = 0$ which implies stability since Rayleigh's equation (2) and boundary conditions (3) are invariant under complex conjugation.

It is clear from the above mathematical analysis that the conjecture of Howard remains valid even for a larger class of velocity distributions $U(y)$ which have a single point of inflexion at some $y = y_s$ where $y_1 < y_s < y_2$ and for which $(\mathrm{d}^2 U/\mathrm{d}y^2)(U - U_s) \leqslant 0$ everywhere in $y_1 \leqslant y \leqslant y_2$ with equality only where $y = y_s$.

The following two theorems are, thus true:

Theorem 1. *All nonviscous homogeneous parallel shear flows, with velocity distributions specified by a monotone function U of the altitude y and a single point of inflexion in the domain of flow, are stable against all infinitesimally small perturbations in the wave number range*

$$k > k_c = \max_{y(\neq y_s) \in \text{Flow Domain}} \sqrt{\left[\frac{\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right)^2}{\left|\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right| |U - U_s|} \right]}.$$

Theorem 2. *All nonviscous homogeneous parallel shear flows with velocity distributions $U(y)$ specified by a single point of inflexion in the domain of flow and the constraint $(d^2U/dy^2)(U-U_s)\leqslant 0$ everywhere in $y_1\leqslant y\leqslant y_2$ with equality only where $y=y_s$ are stable against all infinitesimally small perturbations in the wave number range*

$$k>k_c=\max_{y(\neq y_s)\in\text{Flow domain}}\sqrt{\left[\frac{\left(\frac{d^2U}{dy^2}\right)^2}{\left|\frac{d^2U}{dy^2}\right||U-U_s|}\right]}.$$

An Example. Consider a sinusoidal flow with $U(y)=\sin y (y_1\leqslant y\leqslant y_2)$ such that $y_1<0<y_2$. Rayleigh's necessary instability criterion is thus satisfied and hence we cannot draw any conclusion regarding stability or otherwise of the flow.

Now, let $y_2-y_1<\pi$. Then, since

$$\left(\frac{d^2U}{dy^2}\right)(U-U_s)=-\sin y(\sin y-\sin 0)=-\sin^2 y\leqslant 0$$

everywhere in $y_1\leqslant y\leqslant y_2$, with equality only where $y=y_s=0$ (origin being the only point of inflexion in the flow domain) Fjortoft's necessary instability criterion, in addition to Rayleigh's, is also satisfied and hence we cannot draw any conclusion, regarding stability or otherwise of the flow, as before.

Further, since according to the present criterion

$$\left(\frac{d^2U}{dy^2}\right)(U-U_s)+\frac{\left(\frac{d^2U}{dy^2}\right)^2}{k^2}=-(\sin^2 y)\left(1-\frac{1}{k^2}\right)$$

must be greater than zero at some point, other than the point of inflexion obviously, as a necessary criterion of instability, we see that it is satisfied only for $k^2<1$. Hence, for $k^2>1$ the flow must be stable. This simple counter-example to Rayleigh's necessary instability criterion was given by Tollmien [6] and incidentally it also serves the purpose of a counter-example to Fjortoft's necessary instability criterion in the light of our present work.

For velocity distributions $U(y)$ belonging to the class for which Theorem 1 is valid, we obtain a necessary criterion of instability for the wave with wave length zero (that is $k\to\infty$) from inequality (15) as

$$\left(\frac{d^2U}{dy^2}\right)(U-U_s)>0 \quad \text{at some point } y=y_p\neq y_s \text{ (obviously)}$$
$$\text{where } y_1<y_p<y_2, \tag{19}$$

and hence if

$$\left(\frac{d^2U}{dy^2}\right)(U-U_s)\leqslant 0 \quad \text{everywhere in } y_1\leqslant y\leqslant y_2 \text{ with}$$
$$\text{equality only where } y=y_s, \tag{20}$$

then we must have $c_i = 0$ which implies stability. It is to be noted that Fjortoft's necessary criterion of instability, which is given by

$$\left(\frac{d^2U}{dy^2}\right)(U - U_s) \leqslant 0 \quad \text{everywhere in } y_1 \leqslant y \leqslant y_2 \text{ with}$$

$$\text{equality only where } y = y_s, \tag{21}$$

in the present context, has actually become a sufficient criterion of stability for the wave with $k \to \infty$ and this is in accordance with Banerjee *et al*'s [1] theorem on the rate of growth of an arbitrary unstable perturbation. We state this result in the form of a mathematical theorem as follows:

Theorem 3. *Fjortoft's necessary criterion of instability, for all nonviscous homogeneous parallel shear flows with velocity distributions specified by a monotone U function U of the altitude y and a single point of inflexion in the domain of flow, is actually a sufficient condition of stability for the wave with $k \to \infty$ and this result is in accordance with the prediction of Howard* [4] *and its subsequent confirmation by Banerjee et al* [1].

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Lifting orthogonal representations to spin groups and local root numbers

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Abstract. Representations of D_k^*/k^* for a quaternion division algebra D_k over a local field k are orthogonal representations. In this note we investigate when these orthogonal representations can be lifted to the corresponding spin group. The results are expressed in terms of local root number of the representation.

Keywords. Orthogonal representations; spin groups; local root numbers.

Let D be a quaternion division algebra over a local field k. Then D_k^*/k^* is a compact topological group, and all its irreducible representations are finite dimensional. It can be seen that, in fact, all the irreducible representations are orthogonal, i.e. for any irreducible representation V of D_k^*/k^*, there exists a quadratic form q on V such that the representation takes values in $O(V)$. Using the natural embedding of $O(V)$ in $SO(V \oplus \mathbf{C})$ given by $g \mapsto (g, \det g)$, we get a homomorphism of D_k^*/k^* into $SO(V \oplus \mathbf{C})$. In this note we investigate when this can be lifted to the spin group of the quadratic space $V \oplus \mathbf{C}$. The results are expressed in terms of the local root number of the representation V, or of the corresponding two dimensional symplectic representation of the Weil–Deligne group. We recall that by a theorem of Deligne [D1] the local root number of an orthogonal representation of the Weil–Deligne group W_k' of a local field k is expressed in terms of the second Stiefel–Whitney number of the representation, or equivalently in terms of the obstruction to lifting the orthogonal representation to the spin group. In our case we have a symplectic two dimensional representation of the Weil–Deligne group and its root number is being related to the lifting problem for the orthogonal representation of the quaternion division algebra. The formulation of Deligne's theorem is very elegant and has important global consequences. We, however, have not succeeded in making such an elegant formulation of our results and have neither succeeded in any global application.

As the problem is trivial in the case of an archimedean field, we will confine ourselves to the non-archimedean case only. We have been able to treat the case of only those non-archimedean fields with odd residue characteristic; we will tacitly assume this to be the case all through, and let q denote the cardinality of the residue field of k, and ω the unique non-trivial quadratic character of $\mathbf{F}_q^*$.

Lemma 1. *Any finite dimensional irreducible representation of D_k^*/k^* is orthogonal.*

Proof. If $x \mapsto \bar{x}$ denote the canonical anti-automorphism of D_k^* such that $x \cdot \bar{x} = Nrd(x)$ where $Nrd(x)$ is the reduced norm of x, then as an element of D_k^*/k^*, $\bar{x} = x^{-1}$. By the

Skolem–Noether theorem, x and $\bar{x}$ are conjugate, and therefore x is conjugate to x^{-1} in D_k^*/k^*. By character theory, this implies that every representation of D_k^*/k^* is self-dual. Now it can be proved that for any irreducible representation V of D_k^*/k^*, there exists a quadratic extension L of k such that the trivial character of L^* appears in V; see Lemma 2 below for precise statement. Since every character of L^* appears with multiplicity $\leqslant 1$ in any irreducible representation of D_k^*, cf. Remark 3.5 in [P], the eigenspace corresponding to the trivial character of L^* is one-dimensional. The unique non-degenerate bilinear form on V must be non-zero on this one-dimensional subspace, and therefore the bilinear form must be symmetric.

The following Lemma follows easily from the construction of representations of D_k^*; it can also be proved using the theorem of Tunnell [Tu].

Lemma 2. *Let π be an irreducible representation of D_k^*/k^* associated to a character of a quadratic extension K of k. Let L be the quadratic unramified extension of k if K is ramified, and one of quadratic ramified extensions if K is unramified. Then the trivial representation of L^* appears in π. The trivial representation of K^* appears in π if and only if K/k is a ramified extension of k, and $q \equiv 3 \bmod (4)$.*

The proof of Lemma 1 shows more generally that a self-dual irreducible representation V of a group G must be orthogonal if we can find a subgroup H such that the restriction of V to H is completely reducible and contains the trivial representation of H with multiplicity one. From this remark, one gets the following Proposition.

PROPOSITION 1

Every irreducible, admissible, self-dual, generic representation V of $GL(n,k)$, k non-archimedean, is orthogonal for any $n \geqslant 1$.

Indeed, the theory of new vectors for generic representations of $GL(n,k)$ (cf. [J–PS–S]) gives the existence of an open compact subgroup C such that the space of C-invariant vectors in V is one-dimensional.

According to a program begun by Carayol in [C] for the $GL(2)$ case, representations of D^* where D is a division algebra over a non-archimedean field, together with corresponding representations of $GL(n)$ (assumed to be supercuspidal) and W_k are expected to appear in the middle dimension cohomology (H^{n-1}) of a certain rigid analytic space. Considerations with Poincare duality suggest the following conjecture generalising lemma 1.

Conjecture. Let D^* be the multiplicative group of a division algebra central over a non-archimedean local field k. Let σ_π be the representation of W_k' associated by the local Langlands correspondence to π. Then whenever σ_π is self-dual, symplectic, and trivial on the $SL(2,\mathbf{C})$ factor of W_k', π is orthogonal.

The following Proposition calculates the determinant of a representation of D_k^*/k^*, and implies in particular that the determinant is never trivial; this was the reason why we have to consider the representation $V \oplus \mathbf{C}$ of D_k^*/k^* instead of just V.

PROPOSITION 2

Let π be an irreducible representation of D_k^*/k^* associated to a character of a quadratic

extension K of k. Then

$$\det(\pi) = \omega_{L/k} \circ Nrd,$$

where $L = K$ if K is the quadratic unramified extension of k or if K is ramified with $q \equiv 1 \bmod (4)$; if K is ramified with $q \equiv 3 \bmod (4)$, then L is the other ramified quadratic extension.

Proof. Since the kernel of the reduced norm map is the commutator subgroup of D_k^*, we can write $\det(\pi)$ as $\mu \circ Nrd$ for a character μ of k^*. As π is self-dual, its determinant is of order $\leqslant 2$, and by class field theory, μ is either trivial or is $\omega_{E/k}$, for a quadratic extension E of k. For any quadratic extension M of k, write the decomposition of π as M^*-module as

$$\pi = \sum_{\mu \in X} \mu \oplus \sum_{\mu \in X} \mu^{-1} \oplus a \cdot 1 \oplus b \cdot \nu \tag{i}$$

where a and b are integers $0 \leqslant a, b \leqslant 1$, ν is the unique character of M^*/k^* of order 2, and X is a finite set of characters of M^*/k^* of order $\geqslant 3$. Since the dimension of π is known to be even, $a = b$.

It follows that the determinant of π restricted to M^*/k^* is trivial if and only if the trivial representation of M^* does not appear in π in which case μ is trivial on the norm subgroup $Nrd(M^*)$. Lemma 2 now easily completes the proof.

Remark 1. It should be noted that self-dual representations π of D^* not factoring through D^*/k^* need not be orthogonal. For instance, for $k = \mathbf{R}$, $\pi = \rho \otimes \det(\rho)^{-1/2}$, where ρ is the standard two-dimensional representation of D^*, is a symplectic representation of D^*. It will be interesting to characterize self-dual representations of D_k^* which are orthogonal.

Lemma 3. *Let $SO(2n+1, \mathbf{C})$ correspond to the quadratic form $q = x_1 x_2 + \ldots + x_{2n-1} x_{2n} + x_{2n+1}^2$, and T the associated maximal torus. For characters $(\chi_1, \ldots, \chi_n)$ of an abelian group G, let π be the representation of G with values in $SO(2n+1, \mathbf{C})$ given by $x \mapsto (\chi_1(x), \chi_1^{-1}(x), \chi_2(x), \chi_2^{-1}(x), \ldots, \chi_n(x), \chi_n^{-1}(x), 1)$. Then the representation π of G lifts to Spin $(2n+1, \mathbf{C})$ if and only if $\Pi_{i=1}^n \chi_i = \mu^2$ for some character μ of G, i.e. if and only if $\Pi_{i=1}^n \chi_i$ is trivial on the subgroup $G[2] = \{g \in G \mid 2g = 1\}$.*

Proof. The proof is a trivial consequence of the fact that the spin covering of $SO(2n+1, \mathbf{C})$ when restricted to the maximal torus $T = \{(z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}, 1) \mid z_i \in \mathbf{C}^*\}$ is the two-fold cover of T obtained by attaching $\sqrt{\Pi z_i}$.

Lemma 4. *A homomorphism $\pi: D_k^*/k^* \to SO(n)$ can be lifted to the corresponding spin group if and only if π restricted to K^*/k^* can be lifted for any quadratic extension K of k.*

Proof. As the two sheeted coverings of a group G are classified by $H^2(G, \mathbf{Z}/2)$, one needs to prove that an element of $H^2(D_k^*/k^*, \mathbf{Z}/2)$ is trivial if and only if its restriction to $H^2(K^*/k^*, \mathbf{Z}/2)$ is trivial for all quadratic extensions K of k. Let D_1^* be the image in D_k^*/k^* of the first congruence subgroup of D_k^* under the standard filtration. Then since

the residue characteristic of k is odd, $H^i(D_1^*, \mathbf{Z}/2) = 0$ if $i > 0$. It follows that $H^2(D_k^*/k^*, \mathbf{Z}/2) = H^2(D_k^*/k^* D_1^*, \mathbf{Z}/2)$. Now $D_k^*/k^* D_1^*$ is the dihedral group:

$$0 \to \mathbf{F}_{q^2}^*/\mathbf{F}_q^* \to D_k^*/k^* D_1^* \to \mathbf{Z}/2 \to 0,$$

where $\mathbf{F}_q$ is the residue field of k. Dividing $D_k^*/k^* D_1^*$ by the maximal subgroup H' of odd order of $\mathbf{F}_{q^2}^*/\mathbf{F}_q^*$, we again get the dihedral group $D_r = D_k^*/k^* D_1^* H'$ with $H^2(D_k^*/k^*, \mathbf{Z}/2) \cong H^2(D_k^*/k^* D_1^* H', \mathbf{Z}/2)$:

$$0 \to \mathbf{Z}/2^r \to D_r \to \mathbf{Z}/2 \to 0.$$

Clearly $\mathbf{Z}/2 \oplus \mathbf{Z}/2 \subseteq D_r$, and it can be seen from the explicit description of cohomology of dihedral groups, cf. [Sn, page 24], that $H^2(D_r, \mathbf{Z}/2)$ injects into $H^2(\mathbf{Z}/2 \oplus \mathbf{Z}/2, \mathbf{Z}/2) \oplus H^2(\mathbf{Z}/2^r, \mathbf{Z}/2)$ under restriction. An element of $H^2(\mathbf{Z}/2 \oplus \mathbf{Z}/2, \mathbf{Z}/2)$ is zero if and only if its restriction to all the three $\mathbf{Z}/2$'s in $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ is zero. These three $\mathbf{Z}/2$'s come from the three quadratic extensions; also, $\mathbf{Z}/2^r$ comes from the quadratic unramified extension, proving the proposition.

The following Lemma summarizes the information we need about the characters of irreducible representations π of D^*/k^*, for k non-archimedean, cf. [Si, pages 50–51] where he calculates the characters of representations of $PGL(2, k)$.

Lemma 5. For K a quadratic extension of k, let $\pi = \pi_\chi$ be the representation of D^*/k^* attached to a character χ of K^*. Then we have the following table

K/k	$\mathrm{cond}(\chi)$	$\dim(\pi)$	$\mathrm{cond}(\pi)$
unramified	f	$2q^{f-1}$	$2f$
ramified	$2f$	$(q+1)q^{f-1}$	$2f+1$

Let L be any quadratic extension of k, and x_0 the unique element of L^*/k^* of order 2. Denote by Θ_π the character of π. Then we have:

1. *If $L \neq K$, $\Theta_\pi(x_0) = 0$.*
2. *If $L = K$ and K/k unramified, $\Theta_\pi(x_0) = (-1)^{f+1} 2\chi(x_0)$*
3. *If $L = K$ and K/k ramified,*

$$\Theta_\pi(x_0) = -2G_\chi \omega(2)\omega(-1)^{f-1}\chi(x_0),$$

where

$$G_\chi = \frac{1}{\sqrt{q}} \sum_{x \in (\mathcal{O}_k/\pi_k)^*} \chi(1 + \pi_K^{2f-1} x)\omega(x).$$

We now begin analysing the lifting of orthogonal representations of D_k^*/k^* to spin groups.

PROPOSITION 3

Let π be an irreducible representation of D_k^*/k^* with values in $O(V)$ associated to a quadratic extension K of k. Then the associated representation with values in $SO(V \oplus \mathbf{C})$ lifts to the spin group, $Spin(V \oplus \mathbf{C})$, when restricted to L^*/k^* for L a quadratic extension of k different from K if and only if $\omega(-2) = -1$ if K is a ramified extension,

and $\omega(-1)^{f-1} = -1$ *if* K *is the unramified extension where* $2f$ *is the conductor of the representation* π. (*We recall that* ω *is the unique non-trivial quadratic character of* F_q^*.)

Proof. Let $L = k(x_0)$ with $x_0^2 \in k^*$. Clearly x_0 is the unique element of L^*/k^* of order 2. As π is self-dual, whenever a character μ of L^* appears in π, so does μ^{-1}. Let us now write the decomposition of π as L^*-module as

$$\pi = \sum_{\mu \in X} \mu \oplus \sum_{\mu \in X} \mu^{-1} + a \cdot 1 + b \cdot \nu \tag{i}$$

where a and b are integers $0 \leqslant a, b \leqslant 1$, ν is the unique character of L^*/k^* of order 2, and X is a finite set of characters of L^*/k^* of order $\geqslant 3$. Since the dimension of π is even, $a = b$. Note that $\nu(x_0) = -1$ except in the case when L is a quadratic unramified extension of k with $q \equiv 3 \bmod(4)$ in which case $\nu(x_0) = 1$.

By Lemma 3, the representation π of L^*/k^* with values in $SO(V \oplus \mathbf{C})$ lifts to the spin group, Spin $(V \oplus \mathbf{C})$, if and only if

$$\left(\nu^a \cdot \prod_{\mu \in X} \mu\right)(x_0) = 1.$$

As x_0 has order 2 in L^*/k^*, all the characters of L^*/k^* take the value ± 1 on x_0. Let r be the number of characters μ from X such that $\mu(x_0) = 1$, and let s be the number of characters μ from X such that $\mu(x_0) = -1$. From Lemma 5, the character of π at x_0 is zero. Assuming that L is not the quadratic unramified extension with $q \equiv 3 \bmod(4)$, so that $\nu(x_0) = -1$, we have from the decomposition of π as in (i)

$$\dim(\pi) = 2(r+s) + 2a \tag{ii}$$

$$\Theta_\pi(x_0) = 2(r-s) = 0, \tag{iii}$$

From (ii) and (iii),

$$\dim(\pi) = 4s + 2a. \tag{iv}$$

Also,

$$\left(\nu^a \cdot \prod_{\mu \in X} \mu\right)(x_0) = (-1)^{s+a}. \tag{v}$$

From (iv) and (v), and using Lemma 5 for the dimension of π, it follows that if K is a ramified extension of k, and L is not the quadratic unramified extension of k with $q \equiv 3 \bmod(4)$, the representation π restricted to L^*/k^* lifts to the spin group if and only if $q \equiv 5 \bmod(8)$ or $q \equiv 7 \bmod(8)$. Similarly, when K is the quadratic unramified extension of k, the representation π restricted to L^*/k^* lifts to the spin group if and only if $q \equiv 3 \bmod(4)$ and f even. Finally, if L is the quadratic unramified extension of k with $q \equiv 3 \bmod(4)$, then the representation π restricted to L^*/k^* lifts to the spin group if and only if $q \equiv 7 \bmod(8)$ as follows from a similar analysis. All these conclusions combine to prove the proposition.

We next consider the lifting of a representation π of D_k^*/k^* associated to a quadratic field K when restricted to K^*/k^*. In this case the obstruction to lifting is related to the epsilon factor of π. We will assume that the reader is familiar with the basic properties of the epsilon factor for which we refer to [T]. We, however, do want to state two theorems about epsilon factors which will be crucial to our calculations; the first due to Deligne

[D2, Lemma 4.1.6] describes how epsilon factor changes under twisting by a character of small conductor, and the second is a theorem of Frohlich and Queyrut [F-Q, Theorem 3].

Lemma 6. Let α and β be two multiplicative characters of a local field K such that $\mathrm{cond}(\alpha) \geqslant 2\,\mathrm{cond}(\beta)$. *For an additive character ψ of K, let y be an element of K such that $\alpha(1+x) = \psi(xy)$ for all $x \in K$ with $\mathrm{val}(x) \geqslant \frac{1}{2}\,\mathrm{cond}(\alpha)$ if conductor of α is positive; if conductor of α is 0, let $y = \pi_k^{-\mathrm{cond}(\psi)}$ where π_k is a uniformising parameter of k. Then*

$$\varepsilon(\alpha\beta, \psi) = \beta^{-1}(y)\varepsilon(\alpha, \psi).$$

Lemma 7. Let K be a separable quadratic extension of a local field k, and ψ an additive character of k. Let ψ_K be the additive character of K defined by $\psi_K(x) = \psi(\mathrm{tr}\, x)$. Then for any character χ of K^* which is trivial on k^*, and any $x_0 \in K^*$ with $\mathrm{tr}(x_0) = 0$

$$\varepsilon(\chi, \psi_K) = \chi(x_0).$$

In the next proposition we analyse the lifting of a representation π of D_k^*/k^* associated to a quadratic field K when restricted to K^*/k^*.

PROPOSITION 4

Let π be an irreducible representation of D_k^*/k^* with values in $O(V)$ associated to a character χ of K^* for a quadratic extension K of k. Then the associated representation with values in $SO(V \oplus \mathbf{C})$ lifts to the spin group, $Spin(V \oplus \mathbf{C})$, when restricted to K^*/k^* if and only if $\varepsilon(\pi) = -\omega(2)$ if K is ramified, and $\omega(-1)^f \varepsilon(\pi) = 1$ if K is unramified and the conductor of π is $2f$.

Proof. The proof of this proposition is very similar to that of Proposition 3. Since the proof is essentially the same in the case when K is unramified or ramified, and in fact since the unramified case is much simpler, we will assume in the rest of the proof that K is ramified.

Since k has odd residue characteristic, K^*/k^* has exactly one character of order 2 which is an unramified character of K^* taking the value -1 on a uniformising parameter π_K of K; denote this character by ν. We fix π_K such that $\pi_k = \pi_K^2$ belongs to k so that $K = k(\sqrt{\pi_k})$. Clearly π_K is the unique element of K^*/k^* of order 2.

Let us now write the decomposition of π as K^*-module as in Proposition 1:

$$\pi = \sum_{\mu \in X} \mu \oplus \sum_{\mu \in X} \mu^{-1} \oplus a \cdot 1 \oplus b \cdot \nu \qquad \text{(i)}$$

where a and b are integers $0 \leqslant a, b \leqslant 1$, and X is a finite set of characters of K^*/k^* of order $\geqslant 3$. Since the dimension of π is $(q+1)q^{f-1}$, it is in particular even. Therefore $a = b$.

By Lemma 3, the representation π of K^*/k^* with values in $SO(V \oplus \mathbf{C})$ lifts to the spin group $\mathrm{Spin}(V \oplus \mathbf{C})$ if and only if

$$\left(\nu^a \cdot \prod_{\mu \in X} \mu\right)(\pi_K) = 1.$$

As π_K has order 2 in K^*/k^*, all the characters of K^*/k^* take the value ± 1 on π_K. Let r be the number of characters μ from X such that $\mu(\pi_K) = 1$, and let s be the number of

characters μ from X such that $\mu(\pi_K) = -1$. Therefore from the decomposition of π as in (i) we get,

$$\dim(\pi) = 2(r+s) + 2a \tag{ii}$$

$$\Theta_\pi(\pi_K) = 2(r-s), \tag{iii}$$

$$\left(\nu^a \cdot \prod_{\mu \in X} \mu\right)(\pi_K) = (-1)^{s+a}. \tag{iv}$$

From (ii) and (iii),

$$\dim(\pi) - \Theta_\pi(\pi_K) = 4s + 2a. \tag{v}$$

Using Lemma 5 for the character of π at π_K we get

$$\Theta_\pi(\pi_K) = -2G_\chi \cdot \omega(2)\chi(\pi_K),$$

and as $\dim(\pi) = (q+1)q^{f-1}$, we get from (v) that

$$(q+1)q^{f-1} + 2G_\chi \cdot \omega(2)\chi(\pi_K) = 4s + 2a. \tag{vi}$$

We next calculate the epsilon factor $\varepsilon(\pi)$. As the associated representation of the Weil group is induced from the character χ of K^*.

$$\begin{aligned}\varepsilon(\pi) &= \varepsilon(\mathrm{Ind}_{K^*}^{W_k}\chi, \psi_k) \\ &= \varepsilon(\mathrm{Ind}_{K^*}^{W_k}(\chi - 1), \psi_k) \cdot \varepsilon(\mathrm{Ind}_{K^*}^{W_k} 1, \psi_k) \\ &= \varepsilon(\chi, \psi_K)\varepsilon(\omega_{K/k}, \psi_k)\end{aligned}$$

Here ψ_k is any additive character of k, and ψ_K is the additive character of K obtained from ψ_k using the trace map from K to k.

We now use the theorem of Frohlich and Queyrut to calculate $\varepsilon(\chi, \psi_K)$. As the restriction of χ to k^* is $\omega_{K/k}$ and not the trivial character, we cannot directly apply this theorem. However, a slight modification works. For this observe that as k has odd residue characteristic, the quadratic character $\omega_{K/k}$ of k^* is trivial on $1 + \pi_k \mathcal{O}_k$ where $\mathcal{O}_k$ (respectively $\mathcal{O}_K$) is the maximal compact subring of k (respectively K). Also, since K is a ramified extension,

$$\mathcal{O}_K^*/(1 + \pi_K \mathcal{O}_K) \cong \mathcal{O}_k^*/(1 + \pi_k \mathcal{O}_k).$$

Use this isomorphism to extend $\omega_{K/k}$ from $\mathcal{O}_k^*$ to $\mathcal{O}_K^*$ and then extend this characterof $\mathcal{O}_K^* \cdot k^*$ to K^* in one of the two possible ways. Denote this extension of $\omega_{K/k}$ to K^* by $\tilde{\omega}$. As the conductor of $\tilde{\omega}$ is 1, by Lemma 6,

$$\begin{aligned}\varepsilon(\pi) &= \varepsilon(\chi \cdot \tilde{\omega} \cdot \tilde{\omega}^{-1}, \psi_K) \cdot \varepsilon(\omega_{K/k}, \psi_k) \\ &= \varepsilon(\chi \cdot \tilde{\omega}, \psi_K) \cdot \tilde{\omega}(y) \cdot \varepsilon(\omega_{K/k}, \psi_k) \\ &= (\chi \cdot \tilde{\omega})(\pi_K) \cdot \tilde{\omega}(y) \cdot \varepsilon(\omega_{K/k}, \psi_k)\end{aligned} \tag{vii}$$

where y is the element of K^* with the property that

$$\chi \cdot \tilde{\omega}(1 + x) = \psi(xy) \quad \text{for all } x \text{ with } \mathrm{val}(x) \geqslant \tfrac{1}{2}\mathrm{cond}\,\chi,$$

therefore $y = \pi_K^{-(2f+1)} a_0(\chi) +$ higher order terms. It follows that

$$\chi(1 + \pi_K^{2f-1} x) = \psi(\pi_k^{-1} a_0(\chi) \cdot x).$$

From the definition of epsilon factors,

$$\sum_{x\in(C_k/\pi_k)^*} \omega(x)\psi(\pi_k^{-1}x) = \sqrt{q}\omega_{K/k}(\pi_k)\varepsilon(\omega_{K/k},\psi_k),$$

and therefore,

$$\sum_{x\in(C_k/\pi_k)^*} \omega(x)\chi(1+\pi_K^{2f-1}x) = \sqrt{q}\omega_{K/k}(a_0(\chi)\cdot\pi_k)\varepsilon(\omega_{K/k},\psi_k).$$

Comparing with the definition of G_χ, we get

$$G_\chi = \omega_{K/k}(a_0(\chi)\cdot\pi_k)\cdot\varepsilon(\omega_{K/k},\psi_k).$$

Using (vii),

$$\begin{aligned}\varepsilon(\pi) &= (\chi\cdot\tilde{\omega})(\pi_K)\tilde{\omega}(\pi_K^{-(2f+1)}a_0(\chi))\cdot\varepsilon(\omega_{K/k},\psi_k)\\ &= \chi(\pi_K)\cdot\omega(-1)^{f+1}G_\chi.\end{aligned}$$

Finally, we can use (vi) to give the value of s as follows:

$$4s+4a = (q+1)q^{f-1}+2a+2\varepsilon(\pi).$$

We note that by Tunnell's theorem, the trivial character of K^* appears in π if and only if

$$\varepsilon(\pi)\cdot\varepsilon(\pi\otimes\omega_{K/k}) = -\omega_{K/k}(-1).$$

But since $\pi\cong\pi\otimes\omega_{K/k}$, and $\varepsilon(\pi)=\pm 1$, the trivial character of K^* appears in π, i.e. $a=1$, if and only if $\omega_{K/k}(-1)=-1$. Now the proposition can be deduced by a case-by-case analysis depending on the values of $\omega(2)$ and $\omega(-1)$.

Propositions 3 and 4 can now be combined using Lemma 4 to give the following theorem.

Theorem 1. *Let π be an irreducible representation of D_k^*/k^* with values in $O(V)$ associated to a character χ of K^* for a quadratic extension K of k. Then the associated representation with values in $SO(V\oplus\mathbf{C})$ lifts to the spin group, $Spin(V\oplus\mathbf{C})$, if and only if $\omega(-2)=-1$ and $\varepsilon(\pi)=\omega(-1)$ if K is ramified, and $\omega(-1)^{f-1}=-1$ and $\varepsilon(\pi)=-1$ if K is unramified and the conductor of π is $2f$.*

Remark 2. We do not know when an orthogonal representation of a connected compact Lie group can be lifted to the spin group, say in terms of the highest weight of the representation. The question is interesting for finite groups too, for instance the symmetric group all whose representations are known to be orthogonal, or for finite groups of Lie type.

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Irrationality of linear combinations of eigenvectors

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Abstract. A given $n \times n$ matrix of rational numbers acts on $\mathbf{C}^n$ and on $\mathbf{Q}^n$. We assume that its characteristic polynomial is irreducible and compare a basis of eigenvectors for $\mathbf{C}^n$ with the standard basis for $\mathbf{Q}^n$. Subject to a hypothesis on the Galois group we prove that vectors from these two bases are as independent of each other as possible.

Keywords. Irrationality; Galois group; eigenvectors.

A square matrix $A \in GL(n, \mathbf{Q})$ can be considered as acting on $\mathbf{Q}^n$ and on $\mathbf{Q}^n \otimes \mathbf{C} = \mathbf{C}^n$. The action on $\mathbf{C}^n$ is best understood in terms of eigenvectors and that on $\mathbf{Q}^n$ in terms of the standard basis $e_1, \ldots, e_n$ where $e_i = (\delta_{ij})_{j=1}^n$. We shall study the possibility of linear dependence (over $\mathbf{C}$) between vectors from these two bases.

An eigenvector corresponding to an irrational eigenvalue clearly cannot lie in $\mathbf{Q}^n$. But can it lie in $V \otimes \mathbf{C}$ where V is some codimension one subspace of $\mathbf{Q}^n$? How many of the coordinates of an eigenvector can be rational? And could a non-zero $\mathbf{C}$-linear combination of r eigenvectors lie in $V \otimes \mathbf{C}$ where V is some codimension r subspace of $\mathbf{Q}^n$? Because we can work with A conjugated by a change of basis matrix in $GL(n, \mathbf{Q})$ it is sufficient to consider these questions for subspaces spanned by vectors of the standard basis of $\mathbf{Q}^n$.

To avoid rational eigenvalues let us assume that the characteristic polynomial $\chi(A)$ is irreducible over $\mathbf{Q}$. Since $\chi(A)$ is separable there are n distinct eigenvalues and A is diagonalizable. Moreover, there is no A-invariant subspace of $\mathbf{Q}^n$. Now if there was an A-invariant subspace U then eigenvectors in $U \otimes \mathbf{C}$ would be linearly dependent over $\mathbf{C}$ on vectors that form a $\mathbf{Q}$-basis for U, and we have avoided this type of possibility of linear dependence by the hypothesis that $\chi(A)$ is irreducible.

Let F denote the splitting field extension of $\chi(A)$ over $\mathbf{Q}$ and Γ denote the Galois group of this extension. Then Γ acts on the set of roots of $\chi(A)$. This action is transitive [1, p. 66]. When $1 \leqslant r < n$ we call the action *r-homogeneous* if, for any two subsets consisting of r roots, there is an element of Γ that takes the elements of the first set to those of the second. Certainly the action of Γ is 1-homogeneous. It is r-homogeneous if and only if it is $(n-r)$-homogeneous. If Γ is the symmetric or alternating group on n symbols then the action is r-homogeneous for each r.

Theorem. *Suppose that the characteristic polynomial $\chi(A)$ of $A \in GL(n, \mathbf{Q})$ is irreducible, so that the eigenvalues $\alpha_1, \ldots, \alpha_n$ of A are distinct. Consider a matrix $B \in GL(n, \mathbf{C})$ whose i-th row is a left eigenvector of A corresponding to the eigenvalue α_i, $1 \leqslant i \leqslant n$. Let Γ denote the Galois group of the splitting field extension $F:\mathbf{Q}$ of $\chi(A)$, and fix $1 \leqslant r < n$. If, for each q with $1 \leqslant q \leqslant r$, the action of Γ on $\{\alpha_1, \ldots, \alpha_n\}$ is q-homogeneous then every $r \times r$ minor of B has non-zero determinant.*

Remark 1. The determinant of an $r \times r$ minor of B is equal (at least up to sign) to the determinant of a matrix obtained from B by replacing the $n-r$ rows not in that minor by the $n-r$ vectors of the standard basis that do not correspond to any of the r columns in the minor. Thus the theorem asserts that any set of r eigenvectors and $n-r$ vectors from the standard basis is independent. (Independence over F is equivalent to independence over $\mathbf{C}$ since both are equivalent to the vanishing of the determinant.) Two corollaries follow immediately.

COROLLARY 1

If Γ is the symmetric or alternating group on n symbols then any set of n vectors taken from among the standard basis vectors and eigenvectors corresponding to different eigenvalues is independent over $\mathbf{C}$.

COROLLARY 2

An eigenvector of A cannot lie in $V \otimes \mathbf{C}$ when V is a codimension one subspace of $\mathbf{Q}^n$.

Remark 2. If $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$ has $x_1, \ldots, x_{r+1} \in \mathbf{Q}$ then $x = f_1 + \Sigma_{j=r+2}^{n} x_j e_j \in V \otimes \mathbf{C}$ for the codimension r subspace V of $\mathbf{Q}^n$ spanned by $f_1, e_{r+2}, \ldots, e_n$ where $f_1 := \Sigma_{j=1}^{r+1} x_j e_j \in \mathbf{Q}^n$. Thus Corollary 2 implies that no eigenvector can have two coordinates rational; the conclusion of the Theorem implies that a $\mathbf{C}$-linear combination of r eigenvectors can never have $r+1$ coordinates rational. However, the precise number of coordinates that are rational can change if we change the basis of $\mathbf{Q}^n$.

Remark 3. If

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

then $\chi(A) = x^4 - x^2 + 1$ is irreducible and A has eigenvalues $\pm\alpha$, $\pm\alpha^{-1}$ for α a square root of $(1 + i\sqrt{3})/2$. Then Galois group of $\chi(A)$ is the Klein four group, which is not 2-homogeneous. So this case does not satisfy the *hypotheses* of the theorem if $r = 2$. A has left eigenvectors $v_1 = (1, \alpha, \alpha^2, \alpha^3)$ and $v_2 = (1, -\alpha, \alpha^2, -\alpha^3)$ corresponding to the eigenvalues $\pm\alpha$. But then $v_1 + v_2 = 2e_1 + 2\alpha^2 e_3$ so that these four vectors are linearly dependent and the *conclusion* of the theorem is not satisfied. Thus some hypothesis on the Galois group *is* needed.

Remark 4. The theorem arose from work on hyperbolic total automorphisms. Here A is assumed to have *only integer* entries and $\det(A) = \pm 1$. Then A induces an automorphism $\tilde{A}$ of the quotient group $\mathbf{R}^n/\mathbf{Z}^n$, which is the n-dimensional torus T^n. A vector subspace of $\mathbf{R}^n$ that has a basis in $\mathbf{Q}^n$ or $\mathbf{Z}^n$ corresponds to a lower-dimensional torus in T^n. If A has no eigenvalue of modulus 1 the toral automorphism $\tilde{A}$ is called *hyperbolic*. A hyperbolic $\tilde{A}$ has elaborate dynamical properties: on the one hand, for some $x \in T^n$ the orbit $\{\tilde{A}^k x : k \in Z\}$ is dense in T^n, on the other hand the periodic (i.e. finite) orbits are the orbits of rational points (i.e. points of $\mathbf{Q}^n/\mathbf{Z}^n$) and these form a dense subset of T^n. (See Theorems 3.3 and 6.2 of [2] or 1.11 of [3].) Study of these dynamical

properties uses $\mathbf{R}^n = E^s \oplus E^u$ where $E^s = \{v \in \mathbf{R}^n : A^k v \to 0 \text{ as } k \to \infty\}$ and $E^u = \{v \in \mathbf{R}^n : A^k v \to 0 \text{ as } k \to -\infty\}$. Our theorem gives algebraic conditions under which the projections of E^s and E^u to T^n are in general position with respect to the lower-dimensional tori. Let χ_r denote the characteristic polynomial of the automorphism induced by $\tilde{A}$ on the homology group $H_r(T^n)$. Then the roots of χ_r are products of r distinct roots of $\chi(A)$. If χ_r is irreducible then the action of Γ is r-homogeneous, which helps in checking the hypothesis of the theorem.

Proof of Theorem. We work in F^n where $(A - \alpha_1 I)\ldots(A - \alpha_n I) = 0$ and each $A - \alpha_i I$ has nullity one. For each j, choose a left eigenvector $v_j \in F^n$ corresponding to α_j.

Any element σ of the Galois group Γ is a field isomorphism $\sigma: F \to F$ that leaves $\mathbf{Q}$ fixed pointwise. σ induces a permutation of $\{\alpha_1, \ldots, \alpha_n\}$ and we shall write $\sigma(\alpha_j) = \alpha_{\pi(\sigma)(j)}$.

Now σ induces a $\mathbf{Q}$-linear map $\tilde{\sigma}: F^n \to F^n$. Up to multiplication by constants, $\tilde{\sigma}$ permutes the eigenvectors of A because $\tilde{\sigma}(v_j)A = \tilde{\sigma}(v_j A) = \tilde{\sigma}(\alpha_j v_j) = \sigma(\alpha_j)\tilde{\sigma}(v_j)$ so that $\tilde{\sigma}(v_j) = c(\sigma, j) v_{\pi(\sigma)(j)}$ for some non-zero $c(\sigma, j) \in F$.

Now suppose, if possible, that r vectors from $\{v_1, \ldots, v_n\}$ and $n - r$ from $\{e_1, \ldots, e_n\}$ in F^n are linearly dependent over F and that r is the least number for which this is true. By Remark 1 it suffices to find a contradiction to the existence of such vectors. By renumbering if necessary we can assume that the vectors are $v_1, \ldots, v_r, e_{r+1}, \ldots, e_n$. By the dependence there are $\beta_1, \ldots, \beta_n \in F$, not all zero, with

$$\sum_{j=1}^{r} \beta_j v_j = \sum_{j=r+1}^{n} \beta_j e_j. \tag{1}$$

Since r is the least possible, $\beta_j \neq 0$ for $1 \leqslant j \leqslant r$.

Since the Galois group Γ is r-homogeneous we can, for $k = 0, 1, \ldots, n - r$, find $\sigma_k \in \Gamma$ for which the permutation $\pi(\sigma_k)$ maps $\{1, \ldots, r\}$ to $\{k + 1, \ldots, k + r\}$. Apply each σ_k to (1). This gives

$$\sum_{j=1}^{r} \sigma_k(\beta_j) c(\sigma_k, j) v_{\pi(\sigma_k)(j)} = \sum_{j=r+1}^{n} \sigma_k(\beta_j) e_j, \quad 0 \leqslant k \leqslant n - r. \tag{2}$$

The $n - r + 1$ vectors on the left hand sides of (2) all lie in the $(n - r)$-dimensional subspace of F^n spanned by $e_{r+1}, \ldots, e_n$ so they are linearly dependent over F. Thus, for some $k \leqslant n - r$, $\Sigma_{j=1}^{r} \sigma_k(\beta_j) c(\sigma_k, j) v_{\pi(\sigma_k)(j)}$ is a linear combination of $\Sigma_{j=1}^{r} \sigma_m(\beta_j) c(\sigma_m, j) v_{\pi(\sigma_m)(j)}$, $0 \leqslant m < k$. Now $\sigma_k(v_j) = c(\sigma_k, j) v_{\pi(\sigma_k)(j)} = c(\sigma_k, j) v_{r+k}$ when $j = (\pi(\sigma_k))^{-1}(r + k)$. For this value of j it is $\sigma_k(\beta_j) c(\sigma_k, j)$ that is the (non-zero) coefficient of v_{r+k} and so v_{r+k} is a linear combination of $v_1, \ldots, v_{r+k-1}$, which contradicts the independence of the eigenvectors, and so completes the proof.

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On the zeros of $\zeta^{(l)}(s) - a$ (on the zeros of a class of a generalized Dirichlet series – XVII)*

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Abstract. Some very precise results (see Theorems 4 and 5) are proved about the a-values of the lth derivative of a class of generalized Dirichlet series, for $l \geqslant l_0 = l_0(a)$ (l_0 being a large constant). In particular for the precise results on the zeros of $\zeta^{(l)}(s) - a$ (a any complex constant and $l \geqslant l_0$) see Theorems 1 and 2 of the introduction.

Keywords. Riemann zeta function; generalized Dirichlet series; derivatives; distribution of zeros.

1. Introduction

The object of this paper is to prove the following two theorems.

Theorem 1. *Let* $\delta = \left(\log\left(\frac{\log 3}{\log 2}\right)\right)\left(\log\frac{3}{2}\right)^{-1}$. *There exists an effective constant* $\varepsilon_0 > 0$ *such that if* ε *is any constant satisfying* $0 < \varepsilon \leqslant \varepsilon_0$, *then the rectangle*

$$\left\{\sigma \geqslant l(\delta - \varepsilon), 2k\pi\left(\log\frac{3}{2}\right)^{-1} \leqslant t \leqslant (2k+2)\pi\left(\log\frac{3}{2}\right)^{-1}\right\}$$

contains precisely one zero of $\zeta^{(l)}(s)$, *provided* l *exceeds a constant* $l_0 = l_0(\varepsilon)$ *depending only on* ε. *This zero is a simple zero. Moreover this zero does not lie on the boundary of this rectangle and further lies in*

$$\sigma \leqslant l(\delta + \varepsilon).$$

Here as usual $s = \sigma + it$ *and* k *is any integer, positive negative or zero.*

Theorem 2. *Let* $\delta = (\log\log 15)(\log 15)^{-1}$ *and* a *any non-zero complex constant. There exists an effective constant* $\varepsilon_0 > 0$ *such that if* ε *is any constant satisfying* $0 < \varepsilon \leqslant \varepsilon_0$, *then the rectangle*

$$\{\sigma \geqslant l(\delta - \varepsilon),\ T_0 - \pi(\log 15)^{-1} \leqslant t \leqslant T_0 + \pi(\log 15)^{-1}\}$$

where $T_0 = (\mathrm{Im}\ \log\frac{1}{a} + \pi l + 2k\pi)(\log 15)^{-1}$, *contains precisely one zero of* $\zeta^{(l)}(s) - a$, *provided* l *exceeds an effective constant* $l_0 = l_0(a, \varepsilon)$ *depending only on* a *and* ε. *This zero is a simple zero. Moreover this zero does not lie on the boundary of this rectangle and further lies in*

$$\sigma \leqslant l(\delta + \varepsilon).$$

*Dedicated to Prof. Paul Erdös on his eighty-first birthday

Here k is any integer, positive negative or zero.

Remark. In [1] we dealt with slightly different questions on the zeros in $\sigma > \frac{1}{2}$ of $\zeta^{(l)}(s) - a$ where a is any complex constant and l is any fixed positive integer. Interested reader may consult this paper. However the results of the present paper deal with large l and are more precise.

The main ingredient of the proof of Theorems 1 and 2 (and the more general results to be stated and proved in § 3 and § 4) is the following theorem (see Theorem 3.42 on page 116 on [2]).

Theorem 3. (Rouché's Theorem). *If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C, and $|g(z)| < |f(z)|$ on C then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.*

Remark 1. In what follows we use s in place of z.

Remark 2. It is somewhat surprising that we can prove (with the help of Theorem 3) Theorems 4 and 5, which are much more general than Theorems 1 and 2. These will be stated in § 3 and § 4 respectively.

Remark 3. Theorems 4 and 5 can be generalized to include derivatives of ζ and L functions and also of ζ function of ray classes of any algebraic number field and so on. But we have not done so.

2. Notation

$\{\lambda_n\}(n = 1, 2, 3, \ldots)$ will denote any sequence of real numbers with $\lambda_1 = 1$ and $\frac{1}{A} \leqslant \lambda_{n+1} - \lambda_n \leqslant A$ where $A(\geqslant 1)$ is any fixed constant. $\{a_n\}(n = 1, 2, 3, \ldots)$ will denote any sequence of complex numbers with $a_1 = 1$ and $|a_n| \leqslant n^A$. k will be any integer, positive negative or zero. $\delta_n(n \geqslant 2)$ will denote $(\log\log \lambda_n)(\log \lambda_n)^{-1}$

3. A generalization of Theorem 1

Theorem 4. *Let $n_0 > 1$ be any integer, $|a_{n_0}| > A^{-1}$, $|a_{n_0+1}| > A^{-1}$ and $\delta = \left(\log\left(\frac{\log \lambda_{n_0+1}}{\log \lambda_{n_0}}\right)\right) \times \left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}$. Also let $\lambda_{n+1} < \lambda_n^2$ for all $n > 1$. There exists an effective constant ε_0 such that if ε is any constant satisfying $0 < \varepsilon \leqslant \varepsilon_0$, then the rectangle*

$$\left\{\sigma \geqslant l(\delta - \varepsilon),\ T_0 + 2k\pi\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1} \leqslant t \leqslant T_0 + (2k+2)\pi\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}\right\}$$

where $T_0 = \left(\operatorname{Im}\log\left(\frac{a_{n_0+1}}{a_{n_0}}\right)\right)\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}$, contains precisely one zero of the analytic function

$$\sum_{n \geqslant n_0} a_n (\log \lambda_n)^l \lambda_n^{-s}$$

provided l exceeds an effective positive constant $l_0 = l_0(A, \varepsilon, n_0)$ depending only on the parameters indicated. This zero is a simple zero. Moreover this zero does not lie on the boundary of this rectangle and further lies in

$$\sigma \leqslant l(\delta + \varepsilon).$$

Remark. Theorem 1 follows by taking $n_0 = 2, \lambda_n = n$ and $a_n = 1$ for all n.

The following lemma will be used in this section and also while applying Theorem 5 of § 4 to deduce Theorem 2.

Lemma 1. *For any $\delta > 0$ the function $(\log x)x^{-\delta}$ (of x in $x \geqslant 1$) is increasing for $1 \leqslant x \leqslant \exp(\delta^{-1})$ and decreasing for $x \geqslant \exp(\delta^{-1})$. It has precisely one maximum at $x = \exp(\delta^{-1})$.*

Remark. The maximum value is $(e\delta)^{-1}$. The proof of this lemma is trivial and will be left as an exercise.

To prove Theorem 4 we apply Theorem 3 to

$$f(s) = 1 + \left(\frac{a_{n_0+1}}{a_{n_0}}\right)\left(\frac{\log \lambda_{n_0+1}}{\log \lambda_{n_0}}\right)^l \left(\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-s}$$

and

$$g(s) = \sum_{n \geqslant n_0+2} a'_n \left(\frac{\log \lambda_n}{\log \lambda_{n_0}}\right)^l \left(\frac{\lambda_n}{\lambda_{n_0}}\right)^{-s}$$

where $a'_n = a_n(a_{n_0})^{-1}$. It suffices to prove that $f(s) + g(s)$ has its zeros as claimed in Theorem 4.

Lemma 2. *The zeros of $f(s)$ are all simple and are given by $s = s_0$ where*

$$s_0 = \left(\log(-a'_{n_0+1}) + l \log\left(\frac{\log \lambda_{n_0+1}}{\log \lambda_{n_0}}\right)\right)\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1},$$

for all possible values of $\log(-a'_{n_0+1})$. If $s_0 = \sigma_0 + it_0$ then

$$\sigma_0 = \left(\log|a'_{n_0+1}| + l \log\left(\frac{\log \lambda_{n_0+1}}{\log \lambda_{n_0}}\right)\right)\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1},$$

and

$$t_0 = (\operatorname{Im} \log(-a'_{n_0+1}))\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}.$$

Also

$$f(s) = 1 - \left(\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-s+s_0}.$$

Proof. The proof is trivial.

Lemma 3. *For $\sigma \geqslant 200A$, we have*

$$|g(s)| \leqslant \left(\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-\sigma+\sigma_0} S$$

where

$$S = \sum_{n \geqslant n_0+2} |a_n||a_{n_0+1}|^{-1} \left(\frac{\log \lambda_{n_0}}{\log \lambda_{n_0+1}}\right)^l \left(\frac{\lambda_n}{\lambda_{n_0+1}}\right)^{-\sigma}.$$

Proof. The proof follows from

$$|g(s)| \leqslant \sum_{n \geqslant n_0+2} |a_n'| \left(\frac{\log \lambda_n}{\log \lambda_{n_0}}\right)^l \left(\frac{\lambda_n}{\lambda_{n_0}}\right)^{-\sigma}$$

$$= \sum_{n \geqslant n_0+2} |a_n'| \left(\frac{\log \lambda_n}{\log \lambda_{n_0}}\right)^l \left(\frac{\lambda_n}{\lambda_{n_0+1}}\right)^{-\sigma} \left(\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-\sigma}$$

and the fact that

$$\left(\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{\sigma_0} = |a_{n_0+1}'| \left(\frac{\log \lambda_{n_0+1}}{\log \lambda_{n_0}}\right)^l.$$

Remark. Hereafter we write $\sigma_0 = \delta_0 l$ and

$$\delta_0 = l^{-1}(\log |a_{n_0+1}'|) \left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1} + \delta.$$

Also we remark that the condition $\sigma \geqslant l(\delta_0 - \varepsilon)$ is the same as $\sigma \geqslant l(\delta - \varepsilon)$ with a change of ε.

Lemma 4. *Let $S = S(\sigma)$. Then for $\sigma \geqslant l(\delta - \varepsilon)$ we have,*

$$S(\sigma) < \frac{1}{1000},$$

provided $l \geqslant l_0 = l_0(A, \varepsilon, n_0)$, which is effective.

To prove this lemma it suffices to prove that

$$S(l(\delta - \varepsilon)) < \frac{1}{1000}.$$

This will be done in two stages. We have (by Lemma 3)

$$S(l(\delta - \varepsilon)) = \sum_{n \geqslant n_0+2} |a_n||a_{n_0+1}|^{-1} \left\{ \left(\frac{\log \lambda_n}{\log \lambda_{n_0+1}}\right) \left(\frac{\lambda_n}{\lambda_{n_0+1}}\right)^{-\delta+\varepsilon} \right\}^l.$$

In Lemma 5 we prove that $\exp(\delta^{-1}) < \lambda_{n_0+1}$ and so by Lemma 1 it follows that $(\log \lambda_n)\lambda_n^{-\delta}$ is decreasing for $n \geqslant n_0 + 2$. Hence it suffices to prove that

$$\left(\frac{\log \lambda_{n_0+2}}{\log \lambda_{n_0+1}}\right) \left(\frac{\lambda_{n_0+2}}{\lambda_{n_0+1}}\right)^{-\delta+\varepsilon} < 1.$$

This will be done in Lemma 6. This would complete the proof of Lemma 4 since for all large n

$$\left(\frac{\log \lambda_n}{\log \lambda_{n_0+1}}\right) \left(\frac{\lambda_n}{\lambda_{n_0+1}}\right)^{-\delta+\varepsilon}$$

is less than a negative constant power of λ_n.

Lemma 5. *We have*

$$\exp(\delta^{-1}) < \lambda_{n_0+1}.$$

Proof. Since for $0 < x < 1$ we have $-\log(1-x) > x$, it follows that

$$\begin{aligned}\delta &= \left(-\log\left(1-\left(1-\frac{\log\lambda_{n_0}}{\log\lambda_{n_0+1}}\right)\right)\right)\left(\log\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1} \\ &> \left(1-\frac{\log\lambda_{n_0}}{\log\lambda_{n_0+1}}\right)\left(\log\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1} \\ &= (\log\lambda_{n_0+1})^{-1}.\end{aligned}$$

This proves the lemma.

Lemma 6. *We have*

$$\left(\frac{\log\lambda_{n_0+2}}{\log\lambda_{n_0+1}}\right)\left(\frac{\lambda_{n_0+1}}{\lambda_{n_0+2}}\right)^{\delta} < 1.$$

Proof. We have $\lambda_{n_0+2} < \lambda_{n_0+1}^2$ and also for $0 < x < 1$ we have $\log(1+x) < x$. Using these we obtain

$$\left(1+\left(\log\frac{\lambda_{n_0+2}}{\lambda_{n_0+1}}\right)(\log\lambda_{n_0+1})^{-1}\right)^{\log\lambda_{n_0+1}} < \frac{\lambda_{n_0+2}}{\lambda_{n_0+1}}$$

and so

$$\left(\frac{\log\lambda_{n_0+2}}{\log\lambda_{n_0+1}}\right)\left(\frac{\lambda_{n_0+1}}{\lambda_{n_0+2}}\right)^{(\log\lambda_{n_0+1})^{-1}} < 1$$

and since $(\log\lambda_{n_0+1})^{-1} < \delta$, we obtain Lemma 6. Lemmas 2 and 4 complete the proof of Theorem 4.

4. A generalization of Theorem 2

Theorem 5. *Let δ_{n_1} be the maximum of δ_n taken over all n for which $a_n \neq 0$ and $n > 1$. Suppose that for all $n \neq 1, n_1$ we have $\delta_{n_1} - \delta_n \geqslant A^{-1}$ and also $\lambda_{n_1} - e \geqslant A^{-1}$. We further suppose that $|a_{n_1}| \geqslant A^{-1}$ and put $\delta_{n_1} = \delta$. There exists an effective constant ε_0 such that for all ε satisfying $0 < \varepsilon \leqslant \varepsilon_0$, the rectangle*

$$\{\sigma \geqslant l(\delta-\varepsilon),\ T_0 - \pi(\log\lambda_{n_1})^{-1} \leqslant t \leqslant T_0 + \pi(\log\lambda_{n_1})^{-1}\}$$

where $T_0 = (\operatorname{Im}\log(-a_{n_1}) + 2k\pi)(\log\lambda_{n_1})^{-1}$, contains precisely one zero of the analytic function

$$1 + \sum_{n=2}^{\infty} a_n(\log\lambda_n)^l \lambda_n^{-s}$$

provided l exceeds an effective constant $l_0 = l_0(A, \varepsilon, n_1)$ depending only on the parameters indicated. This zero is a simple zero. Moreover this zero does not lie on the boundary of

this rectangle and further lies in

$$\sigma \leqslant l(\delta + \varepsilon).$$

Remark. Theorem 2 follows by taking $\lambda_n = n$ and $a_n = (-1)^{l+1} a^{-1}$ for all $n \geqslant 2$. Note that the maximum of δ_n occurs when $n = 15$. It is necessary to check that $\delta_{15} > \delta_{16}$. In fact we have

$$e^e = 15{\cdot}21\ldots, \log_{10}\delta_{15}^{-1} = 0{\cdot}434357\ldots \text{ and } \log_{10}\delta_{16}^{-1} = 0{\cdot}434455\ldots,$$

by using tables.

To prove Theorem 5 we apply Theorem 3 to

$$f(s) = 1 + a_{n_1}(\log \lambda_{n_1})^l \lambda_{n_1}^{-s}$$

and

$$g(s) = \sum^{*} a_n (\log \lambda_n)^l \lambda_n^{-s}$$

where the asterisk denotes the restrictions $n \neq 1, n_1$.

Lemma 1. *The zeros of $f(s)$ are all simple and are given by $s = s_0$ where*

$$s_0 = (\log(-a_{n_1}) + l \log\log \lambda_{n_1})(\log \lambda_{n_1})^{-1}$$

for all possible values of $\log(-a_{n_1})$. *If $s = \sigma_0 + it_0$, then*

$$\sigma_0 = (\log|a_{n_1}| + l \log\log \lambda_{n_1})(\log \lambda_{n_1})^{-1}$$

and

$$t_0 = (\operatorname{Im} \log(-a_{n_1}))(\log \lambda_{n_1})^{-1}.$$

Also

$$f(s) = 1 - \lambda_{n_1}^{-s+s_0}.$$

Remark. We write $\sigma_0 = \delta_0 l$ and $\delta_0 = l^{-1}(\log|a_{n_1}|)(\log \lambda_{n_1})^{-1} + \delta$. The condition $\sigma \geqslant l(\delta_0 - \varepsilon)$ is the same as $\sigma \geqslant l(\delta - \varepsilon)$ with a change of ε.

Proof. The proof is trivial.

Lemma 2. *For $\sigma \geqslant l(\delta - \varepsilon)$, we have*

$$|g(s)| \leqslant \sum^{*} |a_n| (\log \lambda_n)^l \lambda_n^{-l\delta + l\varepsilon}.$$

Proof. LHS is trivially not more than

$$\sum^{*} |a_n| (\log \lambda_n)^l \lambda_n^{-\sigma}$$

for all $\sigma \geqslant 200A$. This proves the lemma.

Lemma 3. *We have for $\sigma \geqslant l(\delta - \varepsilon)$,*

$$|g(s)| \leqslant \frac{1}{1000}.$$

Proof. Using $\log \lambda_n = (\lambda_n)^{\delta_n}$ we obtain, by Lemma 2,

$$|g(s)| \leqslant \sum^{*} |a_n| (\lambda_n^{-(\delta-\delta_n)+\varepsilon})^l.$$

By the hypothesis of Theorem 5 we see that $\delta - \delta_n \geqslant A^{-1}$ (note also that $\lambda_{n_1} - e \geqslant A^{-1}$ so that $\delta \geqslant \dfrac{\text{loglog}(e + A^{-1})}{\log(e + A^{-1})}$ if $\lambda_{n_1} \leqslant e^e$) and so Lemma 3 is proved.

Lemmas 1 and 3 complete the proof of Theorem 5.

Open questions

1) How much can one generalize Theorems 1 and 2?
2) Whatever the integer constant $l \geqslant 1$ and whatever the complex constant a, prove that $\zeta^{(l)}(s) - a$ has infinity of simple zeros in $\sigma > \frac{1}{2}$, (more precisely $\gg T$ simple zeros in $(\sigma \geqslant \frac{1}{2} + \delta,\ T \leqslant t \leqslant 2T)$ for some absolute constant $\delta > 0$).

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A note on the growth of topological Sidon sets

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Abstract. We give an estimate for the number of elements in the intersection of topological Sidon sets in $\mathbf{R}^n$ with compact convex subsets and deduce a necessary and sufficient conditions for an orbit of a linear transformation of $\mathbf{R}^n$ to be a topological Sidon set.

Keywords. Topological Sidon sets; growth of sets.

Given a locally compact abelian group G, a subset Λ of the dual group X is called a topological Sidon set if any $b \in l^\infty(\Lambda$, namely any bounded complex-valued functions on Λ, is the restriction to Λ of the Fourier transform of a complex bounded Radon measure on G. These sets play an important role in harmonic analysis ([LR], [M]). When G is compact, X is discrete and the notion of topological Sidon sets coincides with that of Sidon sets. ([LR], [M].)

For any topological Sidon set Λ as above there exist $c \geqslant 1$ and a compact subset K of G such that any $b \in l^\infty(\Lambda)$ is the Fourier transform of a measure which is supported on K and has norm at most $c\|b\|_\infty$. When this condition holds for a $c \geqslant 1$ and a compact subset K, Λ is called a (c, K) topological Sidon set.

Sidon sets are known to be 'thin' set ([LR], [M], [P]). Further, estimates are known for the number of elements in intersections of Sidon sets with finite subsets (see Theorem 3). The purpose of this note is to give the similar estimate for the number of elements in intersections of topological Sidon sets in $\mathbf{R}^m$ with compact convex subsets. Let l denote the Lebesgue measure on $\mathbf{R}^m$. For a set E we denote by $|E|$ the cardinality of E. Then our result shows in particular the following.

Theorem 1. *Let $m \in \mathbf{N}$. Then for any compact set $K \subset \mathbf{R}^m$ and $c \geqslant 1$, there exist a $d > 0$ and a neighbourhood U of $0 \in \mathbf{R}^m$ such that for any (c, K) topological Sidon set Λ of $\mathbf{R}^m$ and any convex subset A of $\mathbf{R}^m$ we have*

$$|\Lambda \cap A| \leqslant d \log(l(A + U)/l(U)).$$

We deduce from the theorem the following criterion for orbits of linear transformations to be topological Sidon sets.

COROLLARY

Let $A: \mathbf{R}^m \mapsto \mathbf{R}^m$ be a linear transformation and $v \in \mathbf{R}^m$. Then $\{A^n(v) | n \in \mathbf{N}\}$ is an infinite topological Sidon set if and only if v is not contained in any A-invariant subspace of $\mathbf{R}^m$ on which all the eigenvalues are of absolute value at most 1.

While the estimate as in the theorem is adequate for the above corollary, it seems worthwhile to note that our argument below gives not just existence of a neighbourhood U, but a concrete way of choosing such a neighbourhood. This is of some interest since the right hand side would typically be big when U is small and so for getting a better estimate one would be interested in choosing U as big as may be allowable. We shall prove the following stronger version of theorem 1.

Theorem 2. *Let $m \in \mathbf{N}$. Then for any $c \geqslant 1$ there exists a $d > 0$ such that the following holds: for any compact set K of $\mathbf{R}^m$, any (c, K) topological Sidon set Λ of $\mathbf{R}^m$ and any convex subset A of $\mathbf{R}^m$ we have $|\Lambda \cap A| \leqslant d \log(l(A+3U)/l(U))$, where $U = \{\lambda \in \mathbf{R}^m | \sup_{x \in K \cup B} |\Sigma_{i=1}^m \lambda_i x_i| \leqslant 1/4\pi c\}$, B being any basis of $\mathbf{R}^n$.*

We shall now recall a result from [LR], on which our proof of Theorem 2 is based, prove some preparatory results and then proceed to prove the theorem.

A finite subset A of a discrete topological group X is said to be a *test set of order M*, where $M \geqslant 1$, if $|A^2 A^{-1}| \leqslant M|A|$.

Theorem 3 [LR]. *If $E \subset X$ is a Sidon set with Sidon constant $\kappa \geqslant 1$, then $|A \cap E| \leqslant 2\kappa^2 eM \log|A|$ for test sets of order M such that $|A| \geqslant 2$.*

The following proposition signifies that any countable set close to a topological Sidon set is again a topological Sidon set. It is just a higher dimensional version of Lemma 3 of Ch. VI of [M] and is deduced analogously, as indicated below.

PROPOSITION 1

Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ be a (c, K) topological Sidon set in $\mathbf{R}^m$ and $\rho > 1$ be given. Let ε be such that $(1-\varepsilon c)^{-1} = \rho$ and let $0 < \theta < \varepsilon$ and $W = \{\lambda \in \mathbf{R}^m | \sup_{x \in K} |\sum_{i=1}^m \lambda_i x_i| \leqslant \theta/4\pi\}$. For each n let $\lambda_n' \in \lambda_n + W$. Then $\Lambda' = \{\lambda_n'\}_{n=1}^\infty$ is a topological Sidon set and further any function b in $l^\infty(\Lambda')$ is the restriction to Λ' of the Fourier transform of a measure $\mu \in M(\mathbf{R}^m)$ with $\|\mu\| < \rho c \|b\|_\infty$.

Proof. Since Λ is a topological Sidon set, Λ is a coherent set of frequencies. (cf: [M], Theorem I of Ch. VI for a proof in the case $m = 1$. The proof actually holds in general.) We now argue as in the proof of the assertion (a)$\Rightarrow$(c) in Theorem X of Ch. IV of [M]: The argument there shows that for the set W as above $\{\lambda_n + W\}_{n=1}^\infty$ are mutually disjoint and if $H: \Lambda + W \to \Lambda \times W$ is the (well-defined) map such that $H(\lambda_n + u) = (\lambda_n, u)$, for all $n \in \mathbf{N}$, $u \in W$, then for each $g \in B(\Lambda \times W)$, $g \circ H \in B(\Lambda + W)$ and $\|g \circ H\|_{B(\Lambda + W)} < \rho \|g\|_{B(\Lambda \times W)}$.

Let $b \in l^\infty(\Lambda')$ be given. Let $\tilde{f}(\lambda_n + u) = b(\lambda_n'), \forall n, \forall u \in W$ and let f be the restriction of $\tilde{f}$ to Λ. Since Λ is a (c, K) topological Sidon set, there exists a measure $\mu \in M(\mathbf{R}^m)$ such that $\hat{u} = f$ on Λ and $\|\mu\| \leqslant c\|f\|_\infty$. Then $\mu \times \delta_0$ yields an element of $B(\Lambda \times W)$; we denote it by g and put $\bar{v} = g \circ H \in B(\Lambda + W)$. Then

$$\|\bar{v}\|_{B(\Lambda+W)} < \rho \|g\|_{B(\Lambda \times W)} \leqslant \rho \|\mu \times \delta_0\| \leqslant \rho c \|f\|_0 = \rho c \|\tilde{f}\|_\infty.$$

Hence there exists a measure $\nu \in M(\mathbf{R}^m)$ such that $\|\nu\| < \rho c \|\tilde{f}\|_\infty$ and $\bar{v} = \tilde{f}$ on $\Lambda + W$; in particular $\|\nu\| < \rho c \|b\|_\infty$ and $\hat{\nu} = b$ on Λ'.

Let $\{x_1, \ldots, x_m\}$ be any linearly independent set in $\mathbf{R}^m$ with m elements. Then any translate of the set $\{\Sigma_{i=1}^m t_i x_i | 0 \leqslant t_i \leqslant 1\}$ is called a *parallelopiped* in $\mathbf{R}^m$; further if $\{x_1, \ldots, x_m\}$ is an orthogonal set, then such a parallelopiped is called a *box*.

PROPOSITION 2

Let A be a compact, convex subset of $\mathbf{R}^m$ with nonempty interior. Then A contains a parallelopiped P such that $l(A) \leqslant (2m)^m l(P)$.

Proof. By a suitable translation, we can assume that $0 \in A$. We define an orthogonal set $\{x_1, \ldots, x_m\}$ in $\mathbf{R}^m$ and a linearly independent subset $\{y_1, \ldots, y_m\}$ of A by induction as follows. Let $x_1 = y_1 \in A$ be an element of maximum norm. Assume that for some $k \leqslant m-1$, an orthogonal set $\{x_1, \ldots, x_k\}$ and a linearly independent subset $\{y_1, \ldots, y_k\}$ are chosen. Let $P_k : \mathbf{R}^m \to \langle x_1, \ldots, x_k \rangle^{\perp}$ be the orthogonal projection map onto the subspace of $\mathbf{R}^m$ orthogonal to $\{x_1, \ldots, x_k\}$. Choose x_{k+1} to be an element of maximum norm in $P_k(A)$; since A has nonempty interior $x_{k+1} \neq 0$. Let $y_{k+1} \in A$ be such that $P_k(y_{k+1}) = x_{k+1}$. Clearly $\{x_1, \ldots, x_{k+1}\}$ is an orthogonal set and $\{y_1, \ldots, y_{k+1}\}$ are linearly independent. By induction this yields the sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ as desired.

Let I be the box generated by $\{x_1, \ldots, x_m\}$, i.e. $I = \{\Sigma_{i=1}^m t_i x_i | 0 \leqslant t_i \leqslant 1\}$. Let $J = \{\Sigma_{i=1}^m t_i x_i | -1 \leqslant t_i \leqslant 1\}$. Then $l(J) = 2^m l(I)$. If $a \in A$ and $(a_1, \ldots, a_m)$ are the coordinates of a with respect to the vectors $x_1, \ldots, x_m$, then $|a_i| \leqslant \|x_i\| \; \forall i$ and hence $a \in J$. Therefore $A \subseteq J$ and consequently $l(A) \leqslant l(J) = 2^m l(I)$. Let P be the parallelopiped generated by $\{y_1/m, \ldots, y_m/m\}$, i.e. $P = \{\Sigma_{i=1}^m t_i y_i/m | 0 \leqslant t_i \leqslant 1\}$. Since A is convex and $0 \in A$ it follows that $P \subseteq A$. The matrix of the transformation $x_i \leftrightarrow y_i$ is lower triangular with diagonal entries equal to 1. Therefore $l(P) = m^m l(I)$. Hence we get $l(A) \leqslant (2m)^m l(P)$.

Proof of Theorem 2. Write $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$. Let B be any basis of $\mathbf{R}^m$. Let $\theta \in (0, 1/c)$ be arbitrary and let $\varepsilon \in (\theta, 1/c)$. Let $U_\theta = \{\lambda \in \mathbf{R}^m | \sup_{x \in K \cup B} |\Sigma_{i=1}^m \lambda_i x_i| \leqslant \theta/4\pi\}$. We apply Proposition 1 to $\rho = (1 - \varepsilon c)^{-1}$ and ε, θ and U_θ as above. Clearly, U_θ is a convex, compact and symmetric neighbourhood of 0. Applying Proposition 2 to U_θ, we get a parallelopiped $P \subseteq U_\theta$ such that $l(U_\theta) \leqslant (2m)^m l(P)$. Let $\{z_1, \ldots, z_m\}$ be such that P is a translate of $\{\Sigma t_i z_i | 0 \leqslant t_i \leqslant 1\}$. Let L be the lattice generated by $\{z_1, \ldots, z_m\}$. If we choose $\lambda_n' \in (\lambda_n + U_\theta) \cap L$, then $\Lambda' = \{\lambda_n'\}_{n=1}^{\infty}$ is a coherent set of frequencies with respect to $(1, F)$, where F is a fundamental domain of the annihilator L° of L. By Proposition 1, Λ' is a topological Sidon set and any $b \in L^\infty(\Lambda')$ is the restriction to Λ' of the Fourier transform of a measure $\mu \in M(\mathbf{R}^m)$ with $\|\mu\| \leqslant \rho c \|b\|_\infty$. This implies that Λ' is a $(\rho c, F)$ topological Sidon set ([M]). Since $\hat{L} = \mathbf{R}^m / L^\circ$ and F is a fundamental domain for L° in $\mathbf{R}^m$, this is equivalent to saying that Λ' is a Sidon set in L with Sidon constant ρc.

Now let A be a compact, convex subset of $\mathbf{R}^m$. Put $A + U_\theta = B$ and $B + U_\theta = C$. We shall prove that $C \cap L$ is a test set with associated constant $(18m)^m$. We have

$$|(C \cap L) + (C \cap L) - (C \cap L)| \leqslant l(C + C - C + U_\theta)/l(P)$$
$$\leqslant l((C + U_\theta) + (C + U_\theta) - (C + U_\theta))/l(P).$$

$C + U_\theta$ is a convex, compact subset of $\mathbf{R}^m$ with nonempty interior. Applying Proposition 2 we get a parallelopiped $P_1 \subset C + U_\theta$ such that

$$l((C + U_\theta) + (C + U_\theta) - (C + U_\theta)) \leqslant (6m)^m l(P_1).$$

Then

$$(6m)^m l(P_1) \leqslant (6m)^m l(C + U_\theta) = (6m)^m l(B + U_\theta + U_\theta) \leqslant (6m)^m 3^m l(B),$$

because B contains a translate of U_θ. These inequalities and the fact that U_θ contains

P yields that

$$|(C\cap L)+(C\cap L)-(C\cap L)| \leqslant (18m)^m l(B)/l(P)$$
$$\leqslant (18m)^m |L\cap(B+U_\theta)| = (18m)^m |C\cap L|.$$

This proves that $C\cap L$ is a test set as claimed. By applying Theorem 3 to $C\cap L$ we now get that

$$|\Lambda\cap A| \leqslant |\Lambda\cap(A+U_\theta)| \leqslant |\Lambda'\cap(A+2U_\theta)| \leqslant d_1 \log|L\cap(A+2U_\theta)|,$$

where $d_1 = 2e(\rho c)^2(18m)^m$. Then

$$|\Lambda\cap A| \leqslant d_1 \log(l(A+2U_\theta+U_\theta)/l(P)) \leqslant d\log(l(A+3U_\theta)/l(U_\theta))$$

where d is a constant depending on c, ε and m. By letting $\theta\to 1/c$ we get the required result.

The following theorem is analogous to Theorem II in Ch. VI of [M].

Theorem 4. *If $\{\lambda_n\}_{n=1}^\infty$ is a sequence in $\mathbf{R}^m$ such that for some $\alpha>1$ we have for all large n, $\|\lambda_{n+1}\| \geqslant \alpha\|\lambda_n\|$ then $\{\lambda_n\}_{n=1}^\infty$ is a topological Sidon set.*

This can be deduced from the following lemma in the same way as Theorem II in Ch. VI of [M] from the analogous lemma there.

Lemma. If $\{\lambda_n\}_{n=1}^\infty$ is a sequence in $\mathbf{R}^m$ such that $\|\lambda_{n+1}\| \geqslant 6\|\lambda_n\|$, $\forall n$, and if $\{b_n\}_{n=1}^\infty$ is any sequence in $\mathbf{T}$, then there exists a point $s\in\mathbf{R}^m$ such that $\|s\| \leqslant 1/\|\lambda_1\|$ and $|\langle s,\lambda_n\rangle - b_n| \leqslant 1, \forall n$.

Proof. Let $\lambda=(a_1,\ldots,a_m)$ be a nonzero element in $\mathbf{R}^m$. Let B be a ball in $\mathbf{R}^m$ with radius $1/\|\lambda\|$ and centre at x_0. Let $\beta=(a_1/\|\lambda\|^2,\ldots,a_m/\|\lambda\|^2)$. Then the points $x_0\pm\beta$ are contained in the boundary of B and each of the two line segments joining x_0 to $x_0\pm\beta$ is mapped onto $\mathbf{T}$ by the map $x\to\langle x,\lambda\rangle$. Therefore given any $b\in\mathbf{T}$ we can find a point $y\in B$ such that $\langle y,\lambda\rangle = b$ and $B(y,1/2\|\lambda\|)\subseteq B$. By induction we choose balls B_n and points $y_n\in B_n$ such that $B_{n+1}\subset B(y_n,1/6\|\lambda_n\|)\subset B_n$, $\forall n$ as follows: Let B_1 be the ball with centre at 0 and radius $=1/\|\lambda_1\|$. Let $y_1\in B_1$ be such that $\langle y_1,\lambda_1\rangle = b_1$ and $B(y_1,1/6\|\lambda_1\|)\subset B_1$. Suppose B_n and y_n have been chosen satisfying the above conditions. Let B_{n+1} be the ball with centre at y_n and radius $=1/\|\lambda_{n+1}\|$. Then $B_{n+1}\subset B(y_n,1/6\|\lambda_n\|)\subseteq B_n$. Choose $y_{n+1}\in B_{n+1}$ such that $\langle y_{n+1},\lambda_{n+1}\rangle = b_{n+1}$ and $B(y_{n+1},1/6\|\lambda_{n+1}\|)\subset B_{n+1}$. Let s be the point of intersection of $\{B_n\}$. Then $s\in\cap_1^\infty B(y_n,1/6\|\lambda_n\|)$ also. Then for all n, $\|s-y_n\| \leqslant 1/6\|\lambda_n\|$ and hence $|\langle s,\lambda_n\rangle - b_n| = |\langle s,\lambda_n\rangle - \langle y_n,\lambda_n\rangle| \leqslant 1$; which proves the lemma.

Proof of the Corollary. There exists a unique largest A-invariant subspace V of $\mathbf{R}^m$ such that all eigenvalues of A on V are of absolute value at most 1. Suppose $v\in V$. Using Jordan decomposition it is easy to see that there exists a $c>0$ such that $\|A^n(v)\| \leqslant cn^{m-1}$ for all n. Let $r_n = cn^{m-1}$ and B_n the ball with centre at 0 and radius r_n. If $\{A^n(v)\}_{n=1}^\infty = \Lambda$ is an infinite topological Sidon set then $A^n(v)$, $n\in\mathbf{N}$, are all distinct and hence by Theorem 2 above, we have, $n \leqslant |B_n\cap\Lambda| \leqslant d\log(l(B_n+3U)/l(U))$ for some compact neighbourhood U of 0. Therefore there exists a constant D such that $n \leqslant D\log r_n$ for all n. Since $r_n = cn^{m-1}$ this implies that $n/\log n$ is bounded which is a contradiction.

Now suppose that $v \notin V$. Using Jordan decomposition one can see that there exists a $c > 1$ and an integer $k \geqslant 1$ such that $\| A^{n+k}(v) \| \geqslant c \| A^n(v) \|$, for all large n. It follows from Theorem 4 that Λ is a finite union of topological Sidon sets. Since Λ is uniformly discrete it is a topological Sidon set.

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Characterization of polynomials and divided difference

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Abstract. For distinct points $x_1, x_2, \ldots, x_n$ in $\mathscr{R}$ (the reals), let $f[x_1, x_2, \ldots, x_n]$ denote the divided difference of f. In this paper, we determine the general solution $f, g: \mathscr{R} \to \mathscr{R}$ of the functional equation

$$f[x_1, x_2, \ldots, x_n] = g(x_1 + x_2 + \cdots + x_n)$$

for distinct $x_1, x_2, \ldots, x_n$ in $\mathscr{R}$ without any regularity assumptions on the unknown functions.

Keywords. Characterization of polynomials; divided difference; distinct points; unknown functions.

Let $\mathscr{R}$ be the set of all real numbers. It is well-known that for quadratic polynomials the Mean Value Theorem takes the form

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x + y}{2}\right).$$

Conversely, if f satisfies the above functional-differential equation, then $f(x) = ax^2 + bx + c$ (see [1] and [4]). A cubic polynomial satisfies the following functional equation

$$f[x, y, z] = \frac{1}{2} f''\left(\frac{x + y + z}{3}\right), \tag{1}$$

where $f[x, y, z]$ denotes the divided difference of f. Recently, Bailey [2] has shown that if the above functional-differential equation holds, then f is a cubic polynomial. For distinct points $x_1, x_2, \ldots, x_n$ in $\mathscr{R}$, the *divided difference* of f is defined as

$$f[x_1, \ldots, x_n] := \frac{[x_1, x_2, \ldots, x_n; f]}{[x_1, x_2, \ldots, x_n]},$$

where

$$[x_1, x_2, \ldots, x_n; f] = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & f(x_1) \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & f(x_2) \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 1 & x_n & x_n^2 & \cdots & x_2^{n-2} & f(x_n) \end{bmatrix}$$

and

$$[x_1, x_2, \ldots, x_n] = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} = \prod_{j>i} (x_j - x_i),$$

$$i, j = 1, 2, \ldots, n.$$

This definition of the divided difference is the same as the one given in [2]. In explicit form, $f[x_1,\ldots,x_n]$ can be written as

$$f[x_1,\ldots,x_n]=\sum_{i=1}^{n}\frac{f(x_i)}{\Pi_{j\neq i}(x_j-x_i)},\quad j=1,2,\ldots,n.$$

Bailey, generalizing a result of Aczél [1], has shown in [2] that if $f\colon \mathfrak{R}\to\mathfrak{R}$ is a differentiable function satisfying the functional equation

$$f[x,y,z]=h(x+y+z), \tag{2}$$

(which is a generalization of functional-differential eq. (1)), then f is a polynomial of degree at most three. In Bailey's proof the differentiability of f plays a crucial role. In [2], Bailey wrote "*One is also led to wonder if* $f[x_1,x_2,\ldots,x_n]=h(x_1+x_2+\cdots+x_n)$ *and* f *continuous (or perhaps differentiable) will imply that* f *is a polynomial of degree no more than* n. *At this point we have no answer.*" In this paper, we provide an answer to this problem. Our method is simple and direct. Further, we do not impose any regularity conditions like continuity, differentiability or boundedness on f etc. For characterization of polynomials with mean value property, the interested reader should refer to [1], [2], [3], [4] and [5] and references therein.

Lemma. Let S be a finite subset of $\mathfrak{R}$ symmetric about zero (that is, $-S=S$) and let $f,g\colon\mathfrak{R}\to\mathfrak{R}$ be functions satisfying the functional equation

$$f(x)-f(y)=(x-y)g(x+y)\quad\text{for all } x,y\in\mathfrak{R}\backslash S. \tag{3}$$

Then

$$f(x)=ax^2+bx+c\quad\text{and}\quad g(y)=ay+b \tag{4}$$

for $x\in\mathfrak{R}\backslash S$ and $y\in\mathfrak{R}$, where a,b,c are some constants.

Proof. Putting $y=-x$ in (3), we obtain

$$f(x)-f(-x)=2xg(0),\quad\text{for } x\in\mathfrak{R}\backslash S. \tag{5}$$

Changing y into $-y$ in (3), we get

$$f(x)-f(-y)=(x+y)g(x-y),\quad\text{for } x,y\in\mathfrak{R}\backslash S$$

which after subtracting (3) from it and using (5) gives

$$(x+y)(g(x-y)-g(0))=(x-y)(g(x+y)-g(0)),\quad\text{for } x,y\in\mathfrak{R}\backslash S. \tag{6}$$

Fix a nonzero $u\in\mathfrak{R}$. Let $v\in\mathfrak{R}$ such that $(u\pm v)/2\notin S$ and put $x=(u+v)/2$ and $y=(u-v)/2$. Then $x+y=u$ and $x-y=v$ and by (6) to get

$$u(g(v)-g(0))=v(g(u)-g(0)),\quad\text{for } v\in\mathfrak{R}\backslash(2S\pm u), \tag{7}$$

where $2S\pm u$ denotes the set $\{2s+u|s\in S\}\cup\{2s-u|s\in S\}$.

For each fixed u, (7) shows that g is linear in v, that is of the form $av+b$, except on the finite set $2S\pm u$. To conclude that g is linear on $\mathfrak{R}$ (reals), one has to note that, if one takes two suitable different values of u, which is now treated as a parameter, the exceptional sets involved are disjoint and so $g(v)=av+b$ for all real v with the same constants everywhere.

Substituting this for g in (3) yields

$$f(x) - ax^2 - bx = f(y) - ay^2 - by, \quad \text{for } x, y \in \mathfrak{R} \setminus S. \tag{8}$$

Choosing any $y \in \mathfrak{R} \setminus S$ in (8) yields that $f(x) = ax^2 + bx + c$ for $x \in \mathfrak{R} \setminus F$, for some constant c, which is the required form of f in (4). This completes the proof of the lemma.

Theorem. *Let $f, g: \mathfrak{R} \to \mathfrak{R}$ satisfy the functional equation*

$$f[x_1, x_2, \ldots, x_n] = g(x_1 + x_2 + \cdots + x_n), \tag{FE}$$

for distinct $x_1, x_2, \ldots, x_n$, that is, for $x_i \neq x_j (i \neq j, i, j = 1, 2, \ldots, n)$. Then f is a polynomial of degree at most n and g is linear, that is, a polynomial of first degree.

Proof. It is easy to see that if f is a solution of (FE), so also $f(x) + \Sigma_{k=0}^{n-2} a_k x^k$. So, we can assume without loss of generality that $f(0) = 0 = f(y_1) = \cdots = f(y_{n-2})$ for $y_1, y_2, \ldots, y_{n-2}$ distinct and different from zero. Obviously there are plenty of choices for $0, y_1, \ldots, y_{n-2}$.

Putting in (FE), $(x, 0, y_1, \ldots, y_{n-2})$ and $(x, 0, y, y_1, \ldots, y_{n-3})$ for $(x_1, x_2, \ldots, x_n)$, we get

$$f(x) = -x(y_1 - x)\cdots(y_{n-2} - x) g\left(x + \sum_{k=1}^{n-2} y_k\right) \tag{9}$$

and

$$\left.\begin{aligned} \frac{f(x)}{x(x-y)(y_1 - x)\cdots(y_{n-3} - x)} - \frac{f(y)}{y(x-y)(y_1 - y)\cdots(y_{n-3} - y)} \\ = g(x + y + \Sigma_{k=1}^{n-3} y_k) \end{aligned}\right\} \tag{10}$$

respectively for $x \neq 0, y_1 \cdots y_{n-2}$ and $y \neq x$.

Now (10) can be rewritten as

$$l(x) - l(y) = (x - y) g\left(x + y + \sum_{k=1}^{n-3} y_k\right),$$

where $l(x) := \dfrac{f(x)}{x(y_1 - x)\cdots(y_{n-3} - x)}$ for $x, y \neq 0, y_1, \ldots, y_{n-3}$. Then by Lemma and the arbitrary choice of $0, y_1, \ldots, y_{n-3}$ we get that g is linear and $l(x)$ is quadratic. Hence by (9) f is a polynomial of degree at most n. This proves the theorem.

Remark. The same conclusion can be obtained without using the Lemma as follows. Subtracting (10) from (9), we have

$$L(x) - L(y) = (x - y) \frac{g(x + \Sigma_{k=1}^{n-2} y_k) - g(x + y + \Sigma_{k=1}^{n-3} y_k)}{y - y_{n-2}} \tag{11}$$

where $L(x) = \dfrac{f(x)}{x(y_1 - x)\cdots(y_{n-2} - x)}$, for $x, y \neq 0, y_1, y_2, \ldots, y_{n-2}$. Interchanging x and y in (11) and adding the resulting equation to (11), we get

$$(x - y) g\left(x + y + \sum_{k=1}^{n-3} y_k\right) = (x - y_{n-2}) g\left(x + \sum_{k=1}^{n-2} y_k\right) - (y - y_{n-2}) g\left(y + \sum_{k=1}^{n-2} y_k\right)$$

for $x, y \neq 0, y_1, y_2, \ldots, y_{n-2}$. Replacing x by $x + y_{n-2}$ and y by $y + y_{n-2}$ in the above, we obtain

$$(x - y)G(x + y) = xG(x) - yG(y) \tag{12}$$

where

$$G(x) = g(x + y_1 + \cdots + y_{n-3} + 2y_{n-2})$$

for $x, y \neq 0, -y_{n-2}, (y_1 - y_{n-2}), \ldots, (y_{n-3} - y_{n-2})$. With $y = -x$, (12) becomes

$$2xG(0) = x(G(x) + G(-x)) \tag{13}$$

for $x \neq 0, \pm y_{n-2}, \pm(y_1 - y_{n-2}), \ldots, \pm(y_{n-3} - y_{n-2})$. Replace y by $-y$ in (12) and subtract the resultant equation from (12) and use (13) to get

$$(x + y)(G(x - y) - G(0)) = (x - y)(G(x + y) - G(0)) \tag{14}$$

for $y \neq 0, \pm y_{n-2}, \pm(y_1 - y_{n-2}), \ldots, \pm(y_{n-3} - y_{n-2})$, and $x \neq 0, -y_{n-2}, (y_1 - y_{n-2}), \ldots, (y_{n-3} - y_{n-2})$. As in the Lemma, it can be shown that G is linear so that g is also linear, $g(x) = cx + b$. This g in (9) shows that f is a polynomial of degree at most n.

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A theorem concerning a product of a general class of polynomials and the *H*-function of several complex variables

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Abstract. A theorem concerning a product of a general class of polynomials and the *H*-function of several complex variables is given. Using this theorem certain integrals and expansion formula have been obtained. This general theorem is capable of giving a number of new, interesting and useful integrals, expansion formulae as its special cases.

Keywords. *H*-function of several complex variables; general class of polynomials; expansion formulae; integrals.

1. Introduction and the main result

Srivastava [3, p. 1, eq. (1)] introduced the general class of polynomials

$$S_n^m[x] = \sum_{\alpha'=0}^{[n/m]} \frac{(-n)_{m\alpha'}}{\alpha'!} A_{n,\alpha'} x^{\alpha'}, \quad n = 0, 1, 2, \ldots \tag{1}$$

where m is an arbitrary positive integer and the coefficients $A_{n,\alpha'}(n, \alpha' \geqslant 0)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $A_{n,\alpha'}$, the polynomials $S_n^m[x]$ can be reduced to the well-known classical orthogonal polynomials such as Jacobi, Hermite, Legendre, Laguerre polynomials, etc.

For the *H*-function of several complex variables defined by Srivastava and Panda [4; see also 6, p. 251], we derive the following theorem:

The main theorem

If

$$(1-y)^{\alpha+\beta-\gamma}\,{}_2F_1(2\alpha, 2\beta; 2\gamma; y) = \sum_{k=0}^{\infty} a_k y^k \tag{2}$$

then

$$\int_0^1 {}_2F_1\left(\alpha, \beta; \gamma + \frac{1}{2}; y\right) {}_2F_1\left(\gamma - \alpha, \gamma - \beta; \gamma + \frac{1}{2}; y\right) \times S_n^m[y^h] H(z_1 y^{h_1}, \ldots, z_r y^{h_r}) \mathrm{d}y$$

$$= \sum_{\alpha'=0}^{[n/m]} \sum_{k=0}^{\infty} \frac{(-n)_{m\alpha'}}{\alpha'!} A_{n,\alpha'} \frac{(\gamma)_k}{(\gamma + \frac{1}{2})_k} a_k$$

$$H_{A+1, C+1; [B', D']; \ldots; [B^{(r)}, D^{(r)}]}^{0, \lambda+1 : (u', v'); \ldots; (u^{(r)}, v^{(r)})} \left(\begin{matrix} [-k-h\alpha' : h_1, \ldots, h_r], \\ [-k-h\alpha'-1 : h_1, \ldots, h_r], \end{matrix} \right.$$

$$\left. \times \begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi']; \ldots; [(b^{(r)}) : \phi^{(r)}]; \\ [(c) : \psi', \ldots, \psi^{(r)}] : [(d') : \delta']; \ldots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right), \tag{3}$$

where

$$h_i > 0,\ \mathrm{Re}\left(1 + \sum_{i=1}^{r} h_i d_j^{(i)}/\delta_j^{(i)}\right) > 0,$$

$-1/2 < (\gamma - \alpha - \beta) < 1/2$, $|\arg(z_i)| < T_i\pi/2$, $T_i > 0, i = 1, \ldots, r; j = 1, \ldots, u^{(i)}$ *and* m *is an arbitrary positive integer and the coefficients* $A_{n,\alpha'}(n, \alpha' \geqslant 0)$ *are arbitrary constants, real or complex.*

2. Proof of the main theorem

To prove the main theorem, we have (2, p. 75)

$$ {}_2F_1(\alpha, \beta; \gamma + \tfrac{1}{2}; y)\,{}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma + \tfrac{1}{2}; y) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{(\gamma + \frac{1}{2})_k} a_k y^k \tag{4}$$

where a_k is given by (2).

Now, multiply both sides of (4) by $S_n^m[y^h]H(z_1 y^{h_1}, \ldots, z_r y^{h_r})$ and integrate with respect to y between the limits 0 and 1, we have

$$\int_0^1 {}_2F_1\left(\alpha, \beta; \gamma + \frac{1}{2}; y\right){}_2F_1\left(\gamma - \alpha, \gamma - \beta; \gamma + \frac{1}{2}; y\right) \times S_n^m[y^h]H(z_1 y^{h_1}, \ldots, z_r y^{h_r})\,dy = \int_0^1 \sum_{k=0}^{\infty} \frac{(\gamma)_k}{(\gamma + \frac{1}{2})_k} a_k y^k S_n^m[y^h]H(z_1 y^{h_1}, \ldots, z_r y^{h_r})\,dy. \tag{5}$$

Express the H-function of several complex variables using [6, p. 251] and a general class of polynomials by [3, p. 1, eq. (1)] on the right of (5), then interchange the order of integration and summation which is permissible under the conditions mentioned in (3) and evaluating with the following result

$$\int_0^1 y^t S_n^m[y^h]H(z_1 y^{h_1}, \ldots, z_r y^{h_r})\,dy = \sum_{\alpha'=0}^{[n/m]} \frac{(-n)_{m\alpha'}}{\alpha'!} A_{n,\alpha'} H_{A+1, C+1:[B', D'];\ldots;[B^{(r)}, D^{(r)}]}^{0, \lambda+1:(u', v');\ldots;(u^{(r)}, v^{(r)})} \times \left(\begin{matrix} [-t - h\alpha' : h_1, \ldots, h_r], & [(a):\theta', \ldots, \theta^{(r)}]: \\ [-t - h\alpha' - 1 : h_1, \ldots, h_r], & [(c):\psi', \ldots, \psi^{(r)}]: \end{matrix} \right. \times \left. \begin{matrix} [(b):\phi']; \ldots; [(b^{(r)}):\phi^{(r)}]; & z_1 \\ [(d'):\delta'] \ldots; [(d^{(r)}):\delta^{(r)}]; & \vdots \\ & z_r \end{matrix} \right), \tag{6}$$

where $h_i > 0$, $\mathrm{Re}(t + 1 + \Sigma_{i=1}^{r} h_i d_j^{(i)}/\delta_j^{(i)}) > 0$, $|\arg(z_i)| < T_i\pi/2$, $T_i > 0, i = 1, \ldots, r$; $j = 1, \ldots, u^{(i)}$ and m is an arbitrary positive integer and the coefficients $A_{n,\alpha'}(n, \alpha' \geqslant 0)$ are arbitrary constants, real or complex. We arrive at the required result.

3. Applications

If we put $\alpha = \gamma$ in the main theorem, the value of a_k in (2) comes out to be equal to $(\beta)_k$ and the result (3) yields the following interesting integral

$$\int_0^1 {}_2F_1\left(\alpha, \beta; \alpha + \frac{1}{2}; y\right) S_n^m[y^h] H(z_1 y^{h_1}, \ldots, z_r y^{h_r}) \mathrm{d}y$$

$$= \sum_{\alpha'=0}^{[n/m]} \sum_{k=0}^{\infty} \frac{(-n)_{m\alpha'}}{\alpha'!} A_{n,\alpha'} \frac{(\alpha)_k (\beta)_k}{(\alpha + \frac{1}{2})_k k!}$$

$$\times H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [-k-h\alpha' : h_1, \ldots, h_r], \\ [-k-h\alpha'-1 : h_1, \ldots, h_r], \end{matrix} \right.$$

$$\left. \times \begin{matrix} [(a):\theta', \ldots, \theta^{(r)}] : [(b'):\phi'] ; \ldots ; [(b^{(r)}):\phi^{(r)}] ; \\ [(c):\psi', \ldots, \psi^{(r)}] : [(d'):\delta'] ; \ldots ; [(d^{(r)}):\delta^{(r)}] ; \end{matrix} \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right), \tag{7}$$

where $h_i > 0$, $\mathrm{Re}(\beta) < 1/2$, $\mathrm{Re}(1 + \Sigma_{i=1}^r h_i d_j^{(i)}/\delta_j^{(i)}) > 0$, $|\arg(z_i)| < T_i \pi/2$, $T_i > 0$, $i = 1, \ldots, r$; $j = 1, \ldots, u^{(i)}$ and m is an arbitrary positive integer and the coefficients $A_{n,\alpha'}(n, \alpha' \geqslant 0)$ are arbitrary constants, real or complex.

Take $\beta = \alpha + 1/2$ and $\alpha = -e$ (e is a non-negative integer) in (7), we have

$$\int_0^1 {}_1F_0(-e; y) S_n^m[y^h] H(z_1 y^{h_1}, \ldots, z_r y^{h_r}) \mathrm{d}y$$

$$= \sum_{\alpha'=0}^{[n/m]} \sum_{k=0}^{e} \frac{(-n)_{m\alpha'}}{\alpha'!} A_{n,\alpha'} \frac{(-e)_k}{k!}$$

$$\times H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [-k-h\alpha' : h_1, \ldots, h_r], \\ [-k-h\alpha'-1 : h_1, \ldots, h_r], \end{matrix} \right.$$

$$\left. \times \begin{matrix} [(a):\theta', \ldots, \theta^{(r)}] : [(b'):\phi'] ; \ldots ; [(b^{(r)}):\phi^{(r)}] ; \\ [(c):\psi', \ldots, \psi^{(r)}] : [(d'):\delta'] ; \ldots ; [(d^{(r)}):\delta^{(r)}] ; \end{matrix} \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right), \tag{8}$$

where $h_i > 0$, $\mathrm{Re}(1 + \Sigma_{i=1}^r h_i d_j^{(i)}/\delta_j^{(i)}) > 0$, $|\arg(z_i)| < T_i \pi/2$, $T_i > 0$, $i = 1, \ldots, r$; $j = 1, \ldots, u^{(i)}$ and m is an arbitrary positive integer and the coefficients $A_{n,\alpha'}(n, \alpha' \geqslant 0)$ are arbitrary constants, real or complex.

Now evaluating the integral on the left of (8) with the help of (6), we establish the following interesting expansion formula

$$\sum_{\alpha'=0}^{[n/m]} \sum_{k=0}^{e} \frac{(-n)_{m\alpha'}}{\alpha'!} A_{n,\alpha'} \frac{(-e)_k}{k!}$$

$$\times H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [-k-h\alpha' : h_1, \ldots, h_r], \\ [-k-h\alpha'-1 : h_1, \ldots, h_r], \end{matrix} \right.$$

$$\times \begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\Bigg)$$

$$= \Gamma(e+h\alpha'+1)\sum_{\alpha'=0}^{[n/m]}\frac{(-n)_{m\alpha'}}{\alpha'!}A_{n,\alpha'}H_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}$$

$$\times\left(\begin{matrix}[0;h_1,\ldots,h_r],[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b)^{(r)}):\phi^{(r)}];\\ [-e-h\alpha'-1:h_1,\ldots,h_r],[(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\right.$$

$$\left.z_1,\ldots,z_r\right), \tag{9}$$

provided that both sides exist.

4. Special cases

(i) On taking $m=2$ and $A_{n,\alpha'}=(-1)^{\alpha'}$ in (3), we have

Theorem 1(a).

If

$$(1-y)^{\alpha+\beta-\gamma}{}_2F_1(2\alpha,2\beta;2\gamma;y)=\sum_{k=0}^{\infty}a_k y^k$$

then

$$\int_0^1 {}_2F_1(\alpha,\beta;\gamma+\tfrac{1}{2};y)\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+\tfrac{1}{2};y)$$

$$\times y^{hn/2}H_n\left(\frac{1}{2\sqrt{y^h}}\right)H\left(z_1y^{h_1},\ldots,z_ry^{h_r}\right)\mathrm{d}y$$

$$=\sum_{\alpha'=0}^{n/2}\sum_{k=0}^{\infty}\frac{(-n)_{2\alpha'}(-1)^{\alpha'}}{\alpha'!}\frac{(\gamma)_k a_k}{(\gamma+\frac{1}{2})_k}$$

$$\times H_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}\left(\begin{matrix}[-k-h\alpha':h_1,\ldots,h_r],\\ [-k-h\alpha'-1:h_1,\ldots,h_r],\end{matrix}\right.$$

$$\times\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\Bigg), \tag{10}$$

valid under the same conditions as obtainable from (3).

(ii) When $m=1$ and $A_{n,\alpha'}=\binom{n+u}{n}\dfrac{(u+v+n+1)_{\alpha'}}{(u+1)_{\alpha'}}$ in (3), we have

Theorem 1(b).

If

$$(1-y)^{\alpha+\beta-\gamma}{}_2F_1(2\alpha,2\beta;2\gamma;y)=\sum_{k=0}^{\infty}a_k y^k$$

then

$$\int_0^1 {}_2F_1(\alpha,\beta;\gamma+\tfrac{1}{2};y)\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+\tfrac{1}{2};y)$$
$$\times P_n^{(u,v)}(1-2y^h)H(z_1y^{h_1},\ldots,z_ry^{h_r})\,dy$$
$$=\sum_{\alpha'=0}^{n}\sum_{k=0}^{\infty}(-1)^{\alpha'}\binom{n+u}{n-\alpha'}\binom{n+u+v+\alpha'}{\alpha'}\frac{(\gamma)_k a_k}{(\gamma+\frac{1}{2})_k}$$
$$\times H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[-k-h\alpha':h_1,\ldots,h_r],\\ [-k-h\alpha'-1:h_1,\ldots,h_r],\end{matrix}\right.$$
$$\left.\times\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]\ :[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\right), \tag{11}$$

valid under the same conditions as obtainable from (3).

(iii) Letting $m=1$ and $A_{n,\alpha'}=\binom{n+u}{n}\frac{1}{(u+1)_{\alpha'}}$ in (3), we get

Theorem 1(c).

If

$$(1-y)^{\alpha+\beta-\gamma}\,{}_2F_1(2\alpha,2\beta;2\gamma;y)=\sum_{k=0}^{\infty}a_ky^k$$

then

$$\int_0^1 {}_2F_1(\alpha,\beta;\gamma+\tfrac{1}{2};y)\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+\tfrac{1}{2};y)$$
$$\times L_n^{(u)}(y^h)H(z_1y^{h_1},\ldots,z_ry^{h_r})\,dy$$
$$=\sum_{\alpha'=0}^{n}\sum_{k=0}^{\infty}\frac{(-1)^{\alpha'}}{\alpha'!}\binom{n+u}{n-\alpha'}\frac{(\gamma)_k a_k}{(\gamma+\frac{1}{2})_k}$$
$$\times H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[-k-h\alpha':h_1,\ldots,h_r],\\ [-k-h\alpha'-1:h_1,\ldots,h_r],\end{matrix}\right.$$
$$\left.\times\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]\ :[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\right), \tag{12}$$

valid under the same conditions as obtainable from (3).

(iv) Letting $n\to 0$, the theorem given by (3) reduces to a known theorem recently obtained by Chaurasia [1, eq. (1.2), p. 193].

(v) For $n=0$, the results in (6), (7), (8) and (9) reduce to the known results obtained by Chaurasia [1, eqs (2.3), p. 194, (3.1) and (3.2), p. 195 and (3.3), p. 195].

The importance of our results lies in its manifold generality. In view of the generality of the polynomials $S_n^m[x]$, on suitably specializing the coefficients $A_{n,\alpha'}$, and making a free use of the special cases of $S_n^m[x]$ listed by Srivastava and Singh [5], our results can be reduced to a large number of theorems, integrals and expansion formulas etc. involving generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel polynomials, Gould–Hopper polynomials, Brafman polynomials and their various combinations.

Secondly, by specializing the various parameters and variables in the H-function of several complex variables, we can obtain, from our theorems, integrals and expansion formulae etc. involving a remarkably wide variety of useful functions (or products of several such functions) which are expressible in terms of E, F, G and H functions of one and several variables. Thus, the results presented in this paper would at once yield a very large number of results, involving a large variety of polynomials and various special functions occurring in the literature.

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Certain bilateral generating relations for generalized hypergeometric functions

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Abstract. Recently, we introduced a class of generalized hypergeometric functions $I_{n;(b_q)}^{\alpha;(a_p)}(x, w)$ by using a difference operator $\Delta_{x,w}$, where $\Delta_{x,w}f(x) = \dfrac{f(x+w)-f(x)}{w}$. In this paper an attempt has been made to obtain some bilateral generating relations associated with $I_n^{\alpha}(x, w)$. Each result is followed by its applications to the classical orthogonal polynomials.

Keywords. Generalized hypergeometric functions; difference operator; bilateral generating relations; classical orthogonal polynomials.

1. Introduction

In the previous paper [2] we introduced a class of generalized hypergeometric functions $I_{n;(b_q)}^{\alpha;(a_p)}(x, w)$ defined by using a difference operator as follows:

$$I_{n;(b_q)}^{\alpha;(a_p)}(x, w) = \frac{1}{n!(x-w)^{[\alpha w]}}\Delta_{x,w}^{n}\left[(x-w)^{[(\alpha+n)w]} \times {}_{p+1}F_q\left((a_p), -\frac{x}{w}; (b_q); w\right)\right], \tag{1.1}$$

where ${}_{p+1}F_q$ denotes the generalized hypergeometric functions (see, for example, Srivastava and Manocha [8]). We also derived the following relation:

$$I_{n;(b_q)}^{\alpha;(a_p)}(x, w) = \frac{(1+\alpha)_n}{n!}F_{q:1;0}^{p:2;1}\left[\begin{matrix}(a_p): & -n, \dfrac{x}{w}; & -\dfrac{x}{w}; \\ (b_q): & 1+\alpha; & -;\end{matrix}\ \begin{matrix} \\ w, w\end{matrix}\right], \tag{1.2}$$

where $F_{q:s;v}^{p:r;u}(x, y)$ is a double hypergeometric function (see Srivastava and Karlsson [7, p. 27(28)]).

The following definitions and results given by Konhauser [1, p. 303(3)], Srivastava and Manocha [8, p. 243(11)] and Manocha [4, p. 687(1.3)] have been used here in regard to the bilateral generating relations for the generalized hypergeometric function $I_n^{\alpha}(x, w)$:

$$Z_n^{\alpha}(x, k) = \frac{(1+\alpha)_{nk}}{n!}{}_1F_k\left(-n; \Delta(k; 1+\alpha); \left(\frac{x}{k}\right)^k\right), \tag{1.3}$$

where

$$\Delta(k; 1+\alpha) = \frac{\alpha+1}{k}, \frac{\alpha+2}{k}, \ldots, \frac{\alpha+k}{k} \quad (k = 1, 2, 3\ldots);$$

$$L_{m+n}^{(\alpha)}(x) = \frac{(1+\alpha)_m(1+\alpha+m)_n}{(n+m)!}e^x {}_1F_1(\alpha+n+m+1; 1+\alpha; -x); \tag{1.4}$$

$$\sum_{n=0}^{\infty}\binom{m+n}{m}\frac{(\lambda+m)_n}{(-\alpha-\beta+m)_n}P_{m+n}^{(\alpha-m-n,\beta-m-n)}(x)t^n$$

$$=\frac{(l+\alpha+\beta-m)_m}{m!}\left(\frac{1+x}{2}\right)^m\left(1+\frac{1}{2}\left(1+x\right)t\right)^{-\lambda-m}$$

$$\times F_1\left(-\beta;\lambda+m,-m;-\alpha-\beta;\frac{t}{1+\frac{1}{2}(1+x)t},\frac{2}{1+x}\right), \tag{1.5}$$

where F_1 is an Appell function [6]. We also derived the extended linear generating relation [3] as follows:

$$\sum_{n=0}^{\infty}\binom{n+m}{n}\frac{(\lambda)_n}{(1+\alpha+m)_n}I_{n+m;(b_q)}^{\alpha;(a_p)}(x,w)t^n$$

$$=(1-t)^{-\lambda}\binom{\alpha+m}{m}F^{(3)}\left[\begin{matrix}(a_p)::\frac{x}{w}; & -;-: & \lambda; & -m; & -\frac{x}{w};\\ (b_q)::1+\alpha; & -;-:-; & -; & -; & \end{matrix}\ -\frac{wt}{1-t},w,w\right], \tag{1.6}$$

where $F^{(3)}$ is Srivastava's general triple hypergeometric series (see, e.g., Srivastava and Manocha [8, p. 69(39)]).

2. Bilateral generating relations

We have derived the following bilateral generating relations for the generalized hypergeometric function $I_n^{\alpha}(x,w)$:

$$\sum_{n=0}^{\infty}\frac{n!(\eta)_n}{(1+\alpha)_n(1+\beta)_{kn}}Z_n^{\beta}(y,k)I_n^{\alpha}(x,w)t^n$$

$$=(1-t)^{-\eta}F_{q+k+1:0,0,0,0}^{p+2:0,0,0,1}\left[\begin{matrix}[\eta:1,1,1,0],[(a_p):1,0,1,1],\\ [1+\alpha:1,0,1,0],[(b_q):1,0,1,1],\\ \left[\frac{x}{w}:1,0,1,0\right]:-;-;-;\left[-\frac{x}{w}:1\right];\\ [\Delta(k:1+\beta):0,1,1,0]:-;-;-;-;\end{matrix}\ -\frac{wt}{1-t},h,-wh,w\right], \tag{2.1}$$

where F is a generalized Lauricella hypergeometric function of 4 variables and $h=\left\{\left(\frac{y}{k}\right)^k\frac{t}{t-1}\right\}$.

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}\frac{n!(\eta)_n}{(1+\alpha)_n(1+\beta)_{kn}}Z_n^{\alpha}(y,k)I_n^{\alpha}(x,w)t^n$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\eta)_n (-n)_l \left(\frac{y}{k}\right)^{kl}}{(1+\alpha)_n \Delta_l(k:1+\beta) l!} I_n^{\alpha}(x,w) t^n$$

$$= \sum_{l=0}^{\infty} \frac{(\eta)_l \left(\frac{y}{k}\right)^{kl} (-t)^l}{(1+\alpha)_l \Delta_l(k;1+\beta)} \sum_{n=0}^{\infty} \binom{n+l}{n} \frac{(\eta+l)_n}{(1+\alpha+l)_n} I_{n+1}^{\alpha}(x,w) t^n$$

$$= (1-t)^{-\eta} F^{p+2:0,0,0,1}_{q+k+1:0,0,0,0} \left[\begin{matrix} [\eta:1,1,1,0],[(a_p):1,0,1,1], \\ [1+\alpha:1,0,1,0],[(b_q):1,0,1,1], \end{matrix} \right.$$

$$\times \left. \begin{matrix} \left[\frac{x}{w}:1,0,1,0\right]:-;-;-;\left[-\frac{x}{w}:1\right]; \\ [\Delta(k:1+\beta):0,1,1,0]:-;-;-;-; \end{matrix} \quad -\frac{wt}{1-t}, h, -wh, w \right]$$

[using (1.5)].

This completes the proof of (2.1).

Applications

(i) By setting $p = q$ and $a_j = b_j (j = 1, 2, \ldots p)$ in (2.1), we get

$$\sum_{n=0}^{\infty} \frac{n!(\eta)_n}{(1+\alpha)_n (1+\beta)_{kn}} Z_n^{\beta}(y,k) J_n^{\alpha}(x,w) t^n$$

$$= (1-t)^{-\eta} F^{(3)} \left[\begin{matrix} \eta:: -;-;\frac{x}{y} & :-;-;-; \\ -:: -;\Delta(k;1+\beta);1+\alpha:-;-;- & \end{matrix} \quad -\frac{wt}{1-t}, h, -wh \right], \tag{2.2}$$

where $h = \left\{ \left(\frac{y}{k}\right)^k \frac{t}{t-1} \right\}$ and $J_n^{\alpha}(x,w)$ is a modified Jacobi polynomial studied by Parihar and Patel [5].

(ii) On taking $k = 1$, $p = q$, $(a_j) = (b_j)$ and letting $w \to 0$ in (2.1), we get the known result given by Srivastava and Manocha [8, p. 133(9)].

The following results can also be deduced by using the same technique as followed in the previous result.

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{(1+\beta)_n} L_{m+n}^{(\alpha)}(x) I_n^{\beta}(y,w) t^n = (1+\alpha)_m e^x (1-t)^{-1-\alpha-m}$$

$$\times F^{(3)} \left[\begin{matrix} -:: -;1+\alpha+m;(a_p):-\frac{y}{w};-;\frac{y}{w}; \\ -:: -; \qquad -;(b_q):-;1+\alpha;1+\beta; \end{matrix} \quad w, -\frac{x}{1-t}, -\frac{wt}{1-t} \right]$$

[using (1.4)]. (2.3)

Applications

(i) By writing $p = q$ and $(a_j) = (b_j)$ in (2.3), we have

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{(1+\beta)_n} L_{m+n}^{(\alpha)}(x) J_n^{\beta}(y,w) t^n$$
$$= (1+\alpha)_m e^x (1-t)^{-1-\alpha-m}$$
$$\Psi_1\left(1+\alpha+m, \frac{y}{w}; 1+\beta; 1+\alpha; -\frac{wt}{1-t}, -\frac{x}{1-t}\right), \tag{2.4}$$

where Ψ_1 is Humbert's function defined in [7, p. 26(21)] and $J_n^{\alpha}(x,w)$ is a modified Jacobi polynomial studied by Parihar and Patel [5].

(ii) Taking limit as $w \to 0$ in (2.4), we obtain the result given by Srivastava and Manocha [8, p. 160(70)].

$$\sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(y) I_n^{\gamma}(x,w) t^n$$
$$= h^{1+\gamma} F^{p+3:0,1,0}_{q+2:0,0,0} \left[\begin{matrix} [(a_p):1,1,0,1],[1+\gamma:1,0,1,1],\left[\frac{x}{w}1:1,0,0,1\right], \\ [(b_q):1,1,0,1],[1+\gamma:1,0,0,1], \end{matrix} \right.$$
$$\left. \begin{matrix} [-\beta:0,0,1,1]:-;\left[-\frac{x}{w}:1\right];-;-; \\ [-\alpha-\beta:0,0,1,1]:-;-;-; \quad -; \end{matrix} \right.$$
$$\left. \times \frac{(l+y)wth}{2} w, th; -wth \right]$$
[using (1.5)]. (2.5)

where $h = \{1 + \frac{1}{2}(y+1)t\}^{-1}$.

Applications

The following applications are obvious:

$$\sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(y) J_n^{\gamma}(x,w) t^n$$
$$= h^{1+\gamma} F^{(3)} \left[\begin{matrix} 1+\gamma:: -;-\beta; & \frac{x}{w}:-;-;-; \\ -::-;-\alpha-\beta;1+\gamma:-;-;-; & \end{matrix} \right.$$
$$\left. \frac{(y+1)wth}{2}, th, -wth \right], \tag{2.6}$$

where $h = \{1 + \frac{1}{2}(y+1)t\}^{-1}$.

$$\sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(y)\, L_n(x) t^n$$
$$= h^{1+\gamma} F^{(3)} \left[\begin{matrix} 1+\gamma:: -;-\beta;-;-;-; \\ -::-;-\alpha-\beta;1+\gamma:-;-;-; \end{matrix} \right.$$
$$\left. \frac{(y+1)xth}{2}, th, -xth \right], \tag{2.7}$$

where $h = \{1 + \frac{1}{2}(y+1)t\}^{-1}$

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n(\delta)_n}{(1+\alpha)_n} I_n^{\alpha}(x,w) F_4(\gamma+n, \delta+n; \rho_1, \rho_2; Z_1, Z_2) t^n$$

$$= F_{q:1,1,0,1,0}^{p+2:0,0,0,1,1} \left[\begin{matrix} [\gamma:1,1,1,1,0],(\delta:1,1,1,1,0],[(a_p):0,0,0,1,1]: \\ [(b_q):0,0,0,1,1]:\rho_1;\rho_2;-; \end{matrix} \right.$$

$$\left. \begin{matrix} -;-;-;\frac{x}{w};-\frac{x}{w}; \\ 1+\alpha;-; \qquad -; \end{matrix} \quad Z_1, Z_2, t, -wt, w \right]. \tag{2.8}$$

Applications

[As usual, we get]

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n(\delta)_n}{(1+\alpha)_n} J_n^{\alpha}(x,w) F_4(\gamma+n, \delta+n; \rho_1, \rho_2; Z_1, Z_2) t^n$$

$$= F_A^{(4)}(\gamma, \delta; -, -, -, \frac{x}{w}; \rho_1, \rho_2, -, 1+\alpha; Z_1, Z_2, t, -wt), \tag{2.9}$$

where $F_A^{(n)}$ is a Lauricella hypergeometric function of n variables (see [8, p. 60(1)]).

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n(\delta)_n}{(1+\alpha)_n} L_n^{\alpha}(x) F_4(\gamma+n, \delta+n; \rho_1, \rho_2; Z_1, Z_2) t^n$$

$$= F_A^{(4)}(\gamma, \delta: -, -, -, -; \rho_1, \rho_2, -, 1+\alpha; Z_1, Z_2, t, -xt). \tag{2.10}$$

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A localization theorem for Laguerre expansions

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Abstract. Regularity properties of Laguerre means are studied in terms of certain Sobolev spaces defined using Laguerre functions. As an application we prove a localization theorem for Laguerre expansions.

Keywords. Laguerre means, Laguerre series, Sobolev spaces.

1. Introduction

The Laguerre polynomials $L_n^\alpha(x)$, of type $\alpha > -1$ are defined by the generating function identity

$$\sum_0^\infty L_n^\alpha(x)t^n = (1-t)^{-\alpha-1}e^{-(xt)/(1-t)}, \quad |t|<1. \tag{1.1}$$

The associated Laguerre functions are defined by

$$\tilde{\mathscr{L}}_n^\alpha(x) = L_n^\alpha(x)e^{-x/2}x^{\alpha/2} \tag{1.2}$$

and they are the eigenfunctions of the Laguerre differential operator

$$-\frac{d}{dx}\left\{x\frac{d}{dx}\tilde{\mathscr{L}}_n^\alpha(x)\right\} + \left\{\frac{x}{4}+\frac{\alpha^2}{4x}\right\}\tilde{\mathscr{L}}_n^\alpha(x) = \left(n+\frac{\alpha+1}{2}\right)\tilde{\mathscr{L}}_n^\alpha(x) \tag{1.3}$$

Moreover the normalized functions $\mathscr{L}_n^\alpha(x) = \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{1/2}\tilde{\mathscr{L}}_n^\alpha(x)$ form an orthonormal basis for $L^2[(0,\infty), dx]$. Therefore for any $f \in L^2(0,\infty)$ we have the eigenfunction expansion

$$f = \sum_0^\infty a_n \mathscr{L}_n^\alpha(x) \tag{1.4}$$

with

$$a_n = \int_0^\infty f(x)\mathscr{L}_n^\alpha(x)dx$$

Three types of Laguerre expansions have been studied in the literature. The first one is concerned with the Laguerre polynomials $L_n^\alpha(x)$, $\alpha > -1$, which form an orthonormal basis for $L^2[(0,\infty), e^{-x}x^\alpha dx]$. The second type is concerned with the Laguerre functions (1.2) which form an orthogonal family in $L^2[(0,\infty), dx]$. Considering the functions $l_n^\alpha(x) = \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha(x)e^{-x/2}$ as an orthonormal family in $L^2[(0,\infty), x^\alpha dx]$, we get a third type of expansion.

Several authors have studied norm convergence and almost everywhere convergence of Riesz means of such expansions. Some references are Askey-Wainger [2], Muckenhoupt [6], Gorlich-Markett [3], Markett [5], Stempak [7], Thangavelu [10]. Various results can also be seen in [12], [1].

Recently by invoking an equiconvergence theorem of Muckenhoupt for Laguerre expansion, Stempak [8] has proved the following almost everywhere convergence result for expansions with respect to $\mathscr{L}_n^\alpha(x)$ as well as $l_n^\alpha(x)$.

(1) $\Sigma_0^N(g, \mathscr{L}_k^\alpha)_{L^2(dx)} \mathscr{L}_k^\alpha(x) \to g(x)$ for almost every $x \in \mathbb{R}_+$ as $N \to \infty$ for $\frac{4}{3} < p < 4$ if $\alpha > -\frac{1}{2}$, and for $p \in \left(\left(1+\frac{\alpha}{2}\right)^{-1}, 4\right)$ otherwise.

(2) $\Sigma_0^N(g, l_k^\alpha)_{L^2(x^\alpha dx)} l_k^\alpha(x) \to g(x)$ for almost every $x \in \mathbb{R}_+$ as $N \to \infty$ for $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{(2\alpha+1)}$ if $\alpha > -\frac{1}{2}$, and for $1 < p < \infty$ otherwise.

In this paper we study the twisted spherical means associated with the Laguerre expansions which we will call Laguerre means. We consider expansions with respect to the system $\varphi_k^\alpha(x) = L_k^\alpha(x^2)e^{-x^2/2}$. Then the normalized functions

$$\psi_k^\alpha(x) = \left(\frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} \varphi_k^\alpha(x) \tag{1.5}$$

form an orthonormal basis for $L^2[(0,\infty), x^{2\alpha+1}dx]$. We have the mapping $T: L^2[x^{2\alpha+1}dx] \to L^2[x^\alpha dx]$ defined by $Tf(x) = \frac{1}{\sqrt{2}} f(\sqrt{x})$, which is a unitary mapping which takes $\psi_k^\alpha(x)$ to $l_k^\alpha(x)$. Therefore the expansion in ψ_k^α is equivalent to the expansion in l_k^α.

We prove a localization theorem for Laguerre expansion with respect to ψ_k^α without appealing to the equiconvergence theorem. Clearly a localization theorem follows from the almost everywhere convergence result of Stempak given above, but this result only says that if $f \equiv 0$ in a neighbourhood of a point $z \in (0,\infty)$, then $S_N f(w) \to 0$ for almost every w in this neighbourhood. But using the method of Laguerre means we could identify the set on which $S_N f(w) \to 0$.

The twisted spherical mean of a locally integrable function f on $\mathbb{C}^n$ is defined to be

$$f\mu_r(z) = \int_{|w|=r} f(z-w)e^{i/2\,Im(z,\bar{w})} d\mu_r(w), \tag{1.6}$$

where $d\mu_r(w)$ is the normalized surface measure on the sphere $\{|w| = r\}$ in $\mathbb{C}^n$. Such spherical means have been considered by Thangavelu in [11], where its regularity properties are used to prove a localization theorem for the special Hermite expansion of L^2 functions on $\mathbb{C}^n$. The special Hermite expansion of a function f is given by

$$f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} f\varphi_k(z), \tag{1.7}$$

where $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-1/4|z|^2}$. Here $L_k^{n-1}(r)$ stands for the Laguerre polynomial of type $n-1$. Measuring the regularity of $f\mu_r(z)$ using a certain Sobolev space denoted by $W_R^s(\mathbb{R}_+)$, he proved the following localization theorem:

Theorem 1. (*S. Thangavelu*) *Let f be a compactly supported function vanishing in a neighbourhood of a point $z \in \mathbb{C}^n$. Further assume that $f\mu_r(z) \in W_R^{n/2}(\mathbb{R}_+)$ as a function of r. Then $S_N f(z) \to 0$ as $N \to \infty$.*

By assuming certain regularity of $f\mu_r(z)$ as a function of r he could also establish an almost everywhere convergence result for special Hermite expansion. In the study of $f\mu_r(z)$ a crucial role is played by the following series expansion:

$$f\mu_r(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(r) f\varphi_k(z) \tag{1.8}$$

for the twisted spherical means. Here $f\varphi_k$ denotes the twisted convolution of f and φ_k, where twisted convolution of two functions f and g on $\mathbb{C}^n$ is defined by

$$fg(z) = \int_{\mathbb{C}^n} f(z-w)\, g(w) e^{i/2\, Im(z.\bar{w})} dw. \tag{1.9}$$

For a radial function f we have

$$f\varphi_k(z) = (2\pi)^{-n} R_k(f) \varphi_k(z), \tag{1.10}$$

where

$$R_k(f) = \frac{2^{1-n} k!}{(k+n-1)!} \int_0^{\infty} f(s)\, L_k^{n-1}\left(\frac{1}{2}s^2\right) e^{-1/4s^2} s^{2n-1} ds.$$

Therefore from (1.8) it follows that for a radial function f the special Hermite expansion becomes the Laguerre expansion with respect to the family $L_k^{n-1}(\frac{1}{2}|z|^2)e^{-1/4|z|^2}$. The above observation suggests that we can also study the localization problem for Laguerre expansion with respect to the orthogonal family $L_k^{\alpha}(r^2)e^{-1/2r^2}, \alpha > -1$. What we need is something similar to twisted spherical means. Using the local co-ordinates on the sphere $|z| = r$ in $\mathbb{C}^n$ it is easy to see that

$$f\mu_r(z) = c_n \int_0^{\pi} f[(r^2 + |z|^2 + 2r|z|\cos\theta)^{1/2}] \frac{J_{n-3/2}(r|z|\sin\theta)}{(r|z|\sin\theta)^{n-3/2}} \sin^{2n-2}\theta d\theta. \tag{1.11}$$

for a suitable constant c_n.

We define the Laguerre means of order α to be

$$T_r^{\alpha} f(z) = \frac{2^{\alpha}\Gamma(\alpha+1)}{\sqrt{2\pi}} \int_0^{\pi} f[(r^2 + z^2 + 2rz\cos\theta)^{1/2}] \times \frac{J_{\alpha-1/2}(rz\sin\theta)}{(rz\sin\theta)^{\alpha-1/2}} \sin^{2\alpha}\theta d\theta \tag{1.12}$$

Then T_r^{α} is a bounded self adjoint operator on $L^2(\mathbb{R}_+, x^{2\alpha+1}dx)$.

We have the interesting formula, see [12]

$$T_r^{\alpha} \varphi_k^{\alpha}(z) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} \varphi_k^{\alpha}(r) \varphi_k^{\alpha}(z), \tag{1.13}$$

for $\alpha > -\frac{1}{2}, r \geqslant 0, z \geqslant 0$. From the series expansion for $T_r^{\alpha} f(z)$ in terms of $\varphi_k^{\alpha}(z)$ and

using the above formula it is easy to see that $T_r^\alpha f(z)$ has the series expansion

$$T_r^\alpha f(z) = \sum_0^\infty \left(\frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} \right)^2 (f, \varphi_k^\alpha)_\alpha \varphi_k^\alpha(z)\, \varphi_k^\alpha(r), \tag{1.14}$$

$r \geqslant 0, z \geqslant 0, \alpha > -\frac{1}{2}$, where $\varphi_k^\alpha(r) = L_k^\alpha(r^2)e^{-1/2r^2}$. Here $(,)_\alpha$ denotes the inner product in the Hilbert space $L^2[R_+, x^{2\alpha+1}]$. Using this notion of Laguerre means we establish a localization theorem for Laguerre series expansion for $f \in L^2[\mathbb{R}_+, x^{2\alpha+1}dx]$ with respect to the orthogonal family $\varphi_k^\alpha(r)$. Our main result is the following:

Theorem 2. *Let $f \in L^2[\mathbb{R}_+, x^{2\alpha+1}dx], \alpha > -\frac{1}{2}$ be a function vanishing in a neighbourhood B_z of a point $z \in \mathbb{R}_+$. If $w \in B_z$ is such that $T_r^\alpha f(w) \in W_\alpha^{(\alpha+1)/2}(\mathbb{R}_+)$, as a function of r, then $S_N f(w) \to 0$ as $N \to \infty$.*

We use the following notation: $L_\alpha^2(\mathbb{R}_+)$ stands for the space $L^2[\mathbb{R}_+, x^{2\alpha+1}dx]$, and the norm and the inner product in this space are denoted by $\|.\|_\alpha$ and $(.,.)_\alpha$ respectively.

2. The Sobolev space $W_\alpha^s(\mathbb{R}_+)$

The usual Sobolev space $H^s(\mathbb{R}^n)$, for $s \geqslant 0$ is defined to be

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : (-\Delta + 1)^s f \in L^2(\mathbb{R}^n)\}$$

using the operator $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$. Since we are interested in studying the regularity of the function $r \to T_r^\alpha f(z)$, motivated by the expansion (1.14) we define the Sobolev space $W_\alpha^s(\mathbb{R}_+)$ using the operator $L_\alpha = -\left[\dfrac{d^2}{dx^2} + \dfrac{2\alpha+1}{x}\dfrac{d}{dx} - x^2 \right]$, which is a positive definite symmetric operator and the φ_s^α's form the family of eigenfunctions with corresponding eigenvalues $4\left(k + \dfrac{\alpha+1}{2}\right)$. Also we have the normalized functions $\psi_k^\alpha(z)$ forming an orthonormal basis for $L_\alpha^2(\mathbb{R}_+)$. We define for $s \geqslant 0$

$$W_\alpha^s(\mathbb{R}_+) = \{f \in L_\alpha^2(R_+) : L_\alpha^s f \in L_\alpha^2(\mathbb{R}_+)\}. \tag{2.1}$$

where L_α^s is defined using the spectral theorem. In other words

$$f = \sum_{k=0}^\infty (f, \psi_k^\alpha)_\alpha \psi_k^\alpha$$

belongs to W_α^s if and only if,

$$\sum_{k=0}^\infty \left| 4^s \left(k + \frac{\alpha+1}{2}\right)^s (f, \psi_k^\alpha)_\alpha \right|^2 < \infty.$$

We now prove the following useful proposition which is needed for the proof of the main theorem.

PROPOSITION 3

Let $\alpha > -1$ and let φ be a smooth function on $\mathbb{R}_+$ which satisfies the following conditions

(i) $\varphi \equiv 0$ near the origin in $\mathbb{R}_+$

(*ii*) $\left|\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^j \varphi(r)\right| = O\left(\frac{1}{r^{2+j}}\right)$ *as* $r \to \infty$ *for* $j = 0, 1, 2, 3, \ldots, 2m$.

Then the operator $M_\varphi : W^s_\alpha \to W^s_{\alpha+1}$ *defined by* $M_\varphi f = \varphi . f$ *is a bounded operator* $\forall s$ *such that* $s \leqslant m$.

The proof of this proposition needs the following lemmas. Before stating the first lemma we introduce, for each non-negative integer k, the class C_k, consisting of all smooth functions on $\mathbb{R}_+$, vanishing near 0 and which also satisfies the decay condition, $\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^j \varphi = O\left(\frac{1}{r^{2+k+j}}\right)$ as $r \to \infty$. The class C_k satisfies the following properties: (i) $C_{k+1} \subset C_k$, (ii) If $\varphi \in C_k, \frac{1}{r}\varphi \in C_{k+1}, r\varphi \in C_{k-1}$, for $k > 1$, (iii) If $\varphi \in C_k$, $\varphi^{(j)} \in C_{k+j}$.

Lemma 4. *Under the above assumptions on* m, φ *and* α *we have* $L^m_{\alpha+1} \circ M_\varphi \circ L^{-m}_\alpha = \Sigma_{t+k \leqslant m} M_{\varphi_{k,t}} \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L^{t-m}_\alpha$ *with* $\varphi_{k,t} \in C_k$.

Proof. We claim that $L^m_{\alpha+1} \circ M_\varphi$ can be written as a linear combination of the form

$$L^m_{\alpha+1} \circ M_\varphi = \sum_{t+k \leqslant m} M_{\varphi_{k,t}} \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L^t_\alpha \quad \text{with} \quad \varphi_{k,t} \in C_k. \tag{2.2}$$

First we note the following relations

$$L_\alpha M_\varphi = M_\varphi L_\alpha - 2M_{\varphi'} \frac{\mathrm{d}}{\mathrm{d}r} - M_{\left(\varphi'' + \frac{2\alpha+1}{r}\varphi'\right)} \tag{2.3}$$

$$L_{\alpha+1} = L_\alpha - \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} \tag{2.4}$$

Using this relation in the above we get

$$L_{\alpha+1} M_\varphi = M_\varphi L_\alpha - 2M_{(\varphi' + \varphi/r)} \frac{\mathrm{d}}{\mathrm{d}r} - M_{\left(\varphi'' + \frac{2\alpha+1}{r}\varphi'\right)}. \tag{2.5}$$

We also use the relation,

$$L_\alpha \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k = \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha + \sum_{j=0}^{k-1} b_j \left(\frac{1}{r}\right)^j \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-j} + c_1 r \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-1} + c_2 \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-2} \tag{2.6}$$

where b_j, c_1, c_2, are constants. This can be easily proved by induction on k. We prove (2.2) by induction on m. (2.2) is clear for $m = 1$. Assume (2.2) for $m = j$. Now,

$$L^{j+1}_{\alpha+1} \circ M_\varphi = \left(L_\alpha - \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)(L^j_{\alpha+1} M_\varphi)$$

$$= \left(L_\alpha - \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\sum_{t+k \leqslant j} M_{\varphi_{k,t}} \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L^t_\alpha\right)$$

$$= \sum_{t+k\leqslant j} L_\alpha\left(M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^t\right) - 2\sum_{t+k\leqslant j}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^t$$

$$= \sum_{t+k\leqslant j}\left[M_{\varphi_{k,t}}L_\alpha - 2M_{\varphi'_{k,t}}\frac{\mathrm{d}}{\mathrm{d}r} - M_{(\varphi''_{k,t}+\frac{2\alpha+1}{r}\varphi'_{k,t})}\right]\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^t$$

$$-\frac{2}{r}\sum_{t+k\leqslant j}\frac{\mathrm{d}}{\mathrm{d}r}M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^t.$$

$$= \sum_{t+k\leqslant j} M_{\varphi_{k,t}}L_\alpha\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^t - 2\sum_{t+k\leqslant j} M_{\varphi'_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k+1} L_\alpha^t$$

$$-\sum_{t+k\leqslant j} M_{(\varphi''_{k,t}+\frac{2\alpha+1}{r}\varphi'_{k,t})}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^t - \frac{2}{r}\sum_{t+k\leqslant j} M_{\varphi'_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^t$$

$$-\frac{2}{r}\sum_{t+k\leqslant j} M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k+1} L_\alpha^t \tag{2.7}$$

In the above computation we have used (2.3). In view of (2.6), the first term of the above is

$$= \sum_{t+k\leqslant j} M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^{t+1} + \sum_{i=0}^{k-1} b_i\left(\frac{1}{r}\right)^i \sum_{t+k\leqslant j} M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-i} L_\alpha^t$$

$$+ \sum_{t+k\leqslant j} c_1 r M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-1} L_\alpha^t + \sum_{t+k\leqslant j} c_2\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-2}$$

$$= \sum_{t+k\leqslant j+1} M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^t + \sum_{i=0}^{k-1} b_i \sum_{t+k\leqslant j}\left(\frac{1}{r}\right)^i M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-i} L_\alpha^t$$

$$+ \sum_{t+k\leqslant j} c_1 r M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-1} L_\alpha^t + \sum_{t+k\leqslant j} c_2\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-2} L_\alpha^t \tag{2.8}$$

Now by induction hypothesis we have $\varphi_{k,t}\in C_k$. Note that in the second term of the above the coefficient of $\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-i} L_\alpha^t$ is $(1/r)^i\varphi_{k,t}$. We have $(1/r)^i\varphi_{k,t}\in C_{k+i}\subset C_k\subset C_{k-i}$ for $i\geqslant 0$ and also $r\varphi_{k,t}\in C_{k-1}$. Hence the first term in (2.7) is of the required form. The second term of (2.7) can be written as $-2\Sigma_{t+k\leqslant j+1}M_{\varphi'_{k-1,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^t$, and $\varphi_{k,t}\in C_k$ by induction hypothesis. Therefore $\varphi'_{k-1,t}\in C_k$ in view of (iii). Hence the second term of (2.7) is also of the required form. In the third term the coefficient of $\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^t$ is $M_{\varphi''_{k,t}+\frac{2\alpha+1}{r}\varphi'_{k,t}}$ and $\varphi''_{k,t}+\frac{2\alpha+1}{r}\varphi'_{k,t}\in C_{k+2}\subset C_k$ by induction hypothesis and in view of (i), (ii) and (iii). Similarly $\frac{1}{r}\varphi'_{k,t}$ occurring in the fourth term belongs to $C_{k+2}\subset C_k$. Also $\frac{1}{r}\varphi_{k,t}$ occurring in the fifth term $C_{k+1}\subset C_k$. Therefore (2.2) holds for $m=j+1$ also. Thus we have $T^m f = L_{\alpha+1}^m\circ M_\varphi\circ L_\alpha^{-m} f = \Sigma_{t+k\leqslant m}M_{\varphi_{k,t}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L_\alpha^{t-m} f$. Which proves the first lemma.

Lemma 5. $\left(\frac{d}{dr}\right)^i L_\alpha^t: L_\alpha^2(\mathbb{R}_+) \to L_\alpha^2(\mathbb{R}_+)$ *is a bounded operator whenever* i *is a non-negative integer and* $i + t \leqslant 0$

Proof. We prove that $\frac{d}{dr} L_\alpha^t$ is a bounded operator on $L_\alpha^2(\mathbb{R}_+)$ for $1 + t \leqslant 0$. We first note that

$$\frac{d}{dr}\psi_k^\alpha = -r[k^{1/2}\psi_{k-1}^{\alpha+1} + (k+\alpha+1)^{1/2}\psi_k^{\alpha+1}] \tag{2.9}$$

This can be seen as follows. We have

$$\begin{aligned}\frac{d}{dr} L_k^\alpha(r)e^{-1/2r^2} &= \frac{d}{dr} L_k^\alpha(r^2)e^{-r^2/2} - rL_k^\alpha(r^2)e^{-r^2/2} \\ &= -2rL_{k-1}^{\alpha+1}(r^2)e^{-r^2/2} - rL_k^\alpha(r^2)e^{-r^2/2} \\ &= (-r)[L_{k-1}^{\alpha+1}(r^2) + L_{k-1}^{\alpha+1}(r^2) + L_k^\alpha(r^2)]e^{-r^2/2} \\ &= (-r)[L_{k-1}^{\alpha+1}(r^2) + L_k^{\alpha+1}(r^2)]e^{-r^2/2}\end{aligned}$$

Here we have used the relations

(i) $\frac{d}{dr} L_k^\alpha(r) = -L_{k-1}^{\alpha+1}$

and,

(ii) $L_k^{\alpha+1} - L_{k-1}^{\alpha+1} = L_k^\alpha$

Now (2.9) follows from the definition of ψ_k^α. Let $f \in L_\alpha^2(\mathbb{R}_+)$. By definition

$$L_\alpha^t f = 4^t \sum_{k=0}^{\infty} \left(k + \frac{\alpha+1}{2}\right)^t (f, \psi_k^\alpha)_\alpha \psi_k^\alpha$$

$$\frac{d}{dr} L_\alpha^t f(r) = 4^t \sum_{k=0}^{\infty} \left(k + \frac{\alpha+1}{2}\right)^t (f, \psi_k^\alpha)_\alpha \frac{d}{dr}\psi_k^\alpha(r),$$

and using (2.9) we get

$$\begin{aligned}\frac{d}{dr} L_\alpha^t f(r) &= 4^t \cdot \sum_{k=1}^{\infty} \left(k + \frac{\alpha+1}{2}\right)^t k^{1/2}(f, \psi_k^\alpha)_\alpha (-r)\psi_{k-1}^{\alpha+1}(r) \\ &\quad + 4^t \sum_{k=0}^{\infty} \left(k + \frac{\alpha+1}{2}\right)^t (k+\alpha+1)^{1/2}(f, \psi_k^\alpha)_\alpha (-r)\psi_k^{\alpha+1}(r) \\ &= -rTf(r) - rSf(r)\end{aligned} \tag{2.10}$$

where

$$Tf(r) = 4^t \sum_{k=1}^{\infty} \left(k + \frac{\alpha+1}{2}\right)^t k^{1/2}(f, \psi_k^\alpha)_\alpha \psi_{k-1}^{\alpha+1}(r) \tag{2.11}$$

and

$$Sf(r) = 4^t \sum_{k=0}^{\infty} \left(k + \frac{\alpha+1}{2}\right)^t (k+\alpha+1)^{1/2}(f, \psi_k^\alpha)_\alpha \psi_k^{\alpha+1}. \tag{2.12}$$

Therefore,

$$\left\| \frac{\mathrm{d}}{\mathrm{d}r} L_\alpha^t f(r) \right\|_\alpha^2 \leqslant (\| rTf(r) \|_\alpha + \| rSf(r) \|_\alpha)^2$$
$$\leqslant 2(\| rTf(r) \|_\alpha^2 + \| rSf(r) \|_\alpha^2). \tag{2.13}$$

Now using the expansion (2.11) we calculate,

$$\| rTf(r) \|_\alpha^2 = \int_0^\infty r^2 |Tf(r)|^2 r^{2\alpha+1} \mathrm{d}r$$
$$= \int_0^\infty |Tf(r)|^2 r^{2\alpha+3} \mathrm{d}r$$
$$= 4^{2t} \sum_{k=1}^\infty \left(k + \frac{\alpha+1}{2} \right)^{2t} k |(f, \psi_k^\alpha)_\alpha|^2$$
$$\leqslant \sum_{k=1}^\infty 4^{2t} \left(k + \frac{\alpha+1}{2} \right)^{2t+1} |(f, \psi_k^\alpha)_\alpha|^2$$
$$\leqslant \sum_{k=1}^\infty |(f, \psi_k^\alpha)_\alpha|^2$$
$$= \| f \|_\alpha^2 \tag{2.14}$$

since $1 + t \leqslant 0$. Similarly one can see that

$$\| rSf(r) \|_\alpha^2 \leqslant \| f \|_\alpha^2 \tag{2.15}$$

Using (2.14) and (2.15) in (2.13) we see that $\left\| \frac{\mathrm{d}}{\mathrm{d}r} L_\alpha^t f \right\|_\alpha \leqslant 2 \| f \|_\alpha$ for $1 + t \leqslant 0$. Similarly one can show that $\left\| \left(\frac{\mathrm{d}}{\mathrm{d}r} \right)^j L_\alpha^t f \right\|_\alpha \leqslant c \| f \|_\alpha$ for some constant c, whenever $j + t \leqslant 0$, which proves the second lemma.

Proof of proposition 3. We have by definition $W_\alpha^s = L_\alpha^{-s}(L_\alpha^2(\mathbb{R}_+))$. Therefore it is enough to prove that

$$L_{\alpha+1}^s \circ M_\varphi \circ L_\alpha^{-s} : L_\alpha^2(\mathbb{R}_+) \to L_{\alpha+1}^2(\mathbb{R}_+) \tag{2.16}$$

is a bounded operator. Put

$$T^t f = L_{\alpha+1}^t \circ M_\varphi \circ L_\alpha^{-t} f \tag{2.17}$$

where $L_{\alpha+1}^t$ and L_α^{-t} are defined using spectral theorem. Then clearly,

$$\| T^0 f \|_{\alpha+1} = \| \varphi f \|_{\alpha+1}$$
$$\leqslant c_0 \| f \|_\alpha, \tag{2.18}$$

for some constant c_0 independent of f. We will also prove that, for any positive integer m

$$\| T^m f \|_{\alpha+1} \leqslant c_1 \| f \|_\alpha, \tag{2.19}$$

for some constant c_1 independent of f.

Assuming (2.19) for a moment choose $f_1 \in L^2_\alpha(\mathbb{R}_+)$ and $g_1 \in L^2_{\alpha+1}(\mathbb{R}_+)$ to be finite linear combinations of ψ_k^α's and $\psi_k^{\alpha+1}$'s, respectively. Consider the function h which is holomorphic in the region $0 < \mathrm{Re}(z) < m$ and continuous in $0 \leqslant \mathrm{Re}(z) \leqslant m$, defined by:

$$h(z) = (T^z f_1, g_1)_{\alpha+1} = (L^z_{\alpha+1} \circ M_\varphi \circ L^{-z}_\alpha f_1, g_1)_{\alpha+1} \tag{2.20}$$

Then by (2.18) we have,

$$\begin{aligned} |h(iy)| &= |(L^{iy}_{\alpha+1} \circ M_\varphi \circ L^{-iy}_\alpha f_1, g_1)_{\alpha+1}| \\ &= |(\varphi(r)\tilde{f}_1, \tilde{g}_1)_{\alpha+1}| \end{aligned}$$

where $\tilde{f}_1 = L^{-iy}_\alpha f_1$, and $\tilde{g}_1 = L^{-iy}_{\alpha+1} g_1$. Therefore,

$$\begin{aligned} |h(iy)| &\leqslant \| T^0 \tilde{f}_1 \|_{\alpha+1} \| \tilde{g}_1 \|_{\alpha+1} \\ &\leqslant c_0 \| \tilde{f}_1 \|_\alpha \| \tilde{g}_1 \|_{\alpha+1} \end{aligned}$$

and since both L^{-iy}_α and $L^{-iy}_{\alpha+1}$ are unitary operators, we get

$$|h(iy)| \leqslant c_0 \| f_1 \|_\alpha \| g_1 \|_{\alpha+1}$$

Similarly by using (2.19) we get

$$\begin{aligned} |h(m+iy)| &= |(L^{m+iy}_{\alpha+1} \circ M_\varphi \circ L^{-m-iy}_\alpha f_1, g_1)_{\alpha+1}| \\ &= |(L^m_{\alpha+1} \circ M_\varphi \circ L^{-m}_\alpha \tilde{f}_1, \tilde{g}_1)_{\alpha+1}| \\ &\leqslant \| T^m \tilde{f}_1 \|_{\alpha+1} \| \tilde{g}_1 \|_{\alpha+1} \\ &\leqslant c_1 \| f_1 \|_\alpha \| g_1 \|_{\alpha+1} \end{aligned}$$

Thus we have

$$|h(iy)| \leqslant c_0 \| f_1 \|_\alpha \| g_1 \|_{\alpha+1} \tag{2.21}$$

$$|h(m+iy)| \leqslant c_1 \| f_1 \|_\alpha \| g_1 \|_{\alpha+1} \tag{2.22}$$

Since h is a bounded function we have by three lines theorem

$$|h(t+iy)| \leqslant c_0^{1-t/m} c_1^{t/m} \| f_1 \|_\alpha \| g_1 \|_{\alpha+1}$$

for $0 < t < m$. In particular,

$$|h(t)| \leqslant c_0^{1-t/m} c_1^{t/m} \| f_1 \|_\alpha \| g_1 \|_{\alpha+1},$$

that is,

$$|(T^t f_1, g_1)| \leqslant c_0^{1-t/m} c_1^{t/m} \| f_1 \|_\alpha \| g_1 \|_{\alpha+1}. \tag{2.23}$$

Now taking supremum over all such $g_1 \in L^2_{\alpha+1}$ with $\| g_1 \|_{\alpha+1} \leqslant 1$ we get $\| T^t f_1 \|_{\alpha+1} \leqslant c_0^{1-t/m} c_1^{t/m} \| f_1 \|_\alpha$. Therefore T^t is a bounded operator on a dense subset of L^2_α. Therefore it has a norm preserving extension to L^2_α. Thus we have

$$\| T^t f \|_{\alpha+1} \leqslant c_t \| f \|_\alpha \forall f \in L^2_\alpha(\mathbb{R}_+), \quad \text{for } 0 < t < m \tag{2.24}$$

which proves (2.16).

To prove (2.19) we proceed as follows. By Lemma (4) we have $T^m f = \Sigma_{t+k \leqslant m} M_{\varphi_{k,t}} \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L^{t-m}_\alpha$. And by Lemma (5), $\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k L^{t-m}_\alpha$ is a bounded operator on $L^2_\alpha(\mathbb{R}_+)$, whenever $k + (t-m) \leqslant 0$. Also note that since $\varphi_{k,t}$ satisfies the conditions (1) and (2) of

the Proposition 3 for $j=0$, $M_{\varphi_{k+1}}$ maps $L^2_\alpha(\mathbb{R}_+)\to L^2_{\alpha+1}(\mathbb{R}_+)$ boundedly. Thus we get $\|T^m f\|_{\alpha+1}\leqslant c_1\|f\|_\alpha$. This completes the proof of the proposition.

3. Regularity of $T^\alpha_r f(z)$

In this section we prove that the Laguerre means $T^\alpha_r f(z)$ are slightly more regular than f, for $z\neq 0$. To prove this fact we use the series expansion (1.14) for $T^\alpha_r f(z)$. Let $f\in W^s_\alpha$. Then

$$4^s\sum_0^\infty\left(k+\frac{\alpha+1}{2}\right)^s\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}(f,\varphi^\alpha_k)_\alpha\varphi^\alpha_k(r) \tag{3.1}$$

converges in $L^2_\alpha(\mathbb{R}_+)$. We also use the following asymptotic estimates, (see [4] or [9])

$$\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\approx k^{-\alpha} \tag{3.2}$$

$$\psi^\alpha_k(z)\approx k^{-1/4}|z|^{-\alpha-(1/2)}\cos\left(2\sqrt{kz}-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right),\quad z\neq 0 \tag{3.3}$$

$$\psi^\alpha_k(0)\approx k^{\alpha/2}\quad\text{as } k\to\infty \tag{3.4}$$

From (1.14) we have

$$\int_0^\infty|T^\alpha_r f(z)|^2 r^{2\alpha+1}\,dr=\Gamma(\alpha+1)^4\sum_{k=0}^\infty\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}|(f,\psi^\alpha_k)_\alpha|^2|\psi^\alpha_k(z)|^2$$

$$\leqslant c(z)\sum_{k=0}^\infty(1+k)^{-\alpha}(1+k)^{-1/2}|(f,\psi^\alpha_k)_\alpha|^2 \tag{3.5}$$

for $z\neq 0$, in view of (3.2) and (3.3). Also

$$\int_0^\infty|T^\alpha_r f(z)|^2 r^{2\alpha+1}\,dr\approx\sum_{k=0}^\infty|(f,\psi^\alpha_k)_\alpha|^2\quad\text{for } z=0 \tag{3.6}$$

in view of (3.2) and (3.4). Comparing (3.1) and (3.5) we see that $f\in W^s_\alpha\Rightarrow r\to T^\alpha_r f(z)\in W^{s+(\alpha/2)+(1/4)}$. Comparing (3.1) and (3.6) we see that $f\in W^s_\alpha$ if and only if $T^\alpha_r f(z)\in W^s_\alpha$. Thus we have proved the following:

Lemma 6. (*i*) $f\in W^s_\alpha\Rightarrow r\to T^\alpha_r f(z)\in W^{s+(\alpha/2)+(1/4)}_\alpha$, $z\neq 0$.
(*ii*) $f\in W^s_\alpha$ *if and only if* $r\to T^\alpha_r f(0)\in W^s_\alpha$.
Now we prove some properties of Laguerre means $T^\alpha_r f$.

Lemma 7. (*i*) *If* f *is supported in* $z\leqslant b$, *then* $T^\alpha_r f(z)$ *as a function of* r *is supported in* $r\leqslant b+z$.
(*ii*) *If* f *vanishes in a neighbourhood of* z *then* $T^\alpha_r f(z)$ *as a function of* r *vanishes in a neighbourhood of origin in* $\mathbb{R}_+$.

Proof. (i) If f is supported in $z\leqslant b$ then the integral (1.12) vanishes unless $(r^2+z^2+2rz\cos\theta)^{1/2}\leqslant b$. This implies $(r-z)^2\leqslant b^2$. Therefore the integral (1.12) vanishes unless $|r-z|\leqslant b$ or $r\leqslant b+z$.
(ii) Again if f vanishes in a neighbourhood $\{|y-z|<a\}$, $a>0$ of z, the above integral (1.12) is zero if $|(r^2+z^2+2rz\cos\theta)^{1/2}-z|\leqslant a$. Since z is fixed this says that the above

inequality holds for r in a neighbourhood of 0. Now consider the continuous function

$$g(r) = |(r^2 + z^2 + 2rz\cos\theta)^{1/2} - z| - a,$$

defined on $\mathbb{R}_+$. We have $g(0) = -a < 0$. Therefore $g < 0$ in a neighbourhood of 0 as well. This means that for r in some neighbourhood of 0 we have $|(r^2 + z^2 + 2rz\cos\theta)^{1/2} - z| < a$. Thus $T_r^\alpha f(z) \equiv 0$ in that neighbourhood.

4. A localization theorem for Laguerre expansions

Now we are in a position to prove Theorem (2) stated in the Introduction. From (1.14) using the orthogonality of ψ_k^α we get

$$\int_0^\infty T_r^\alpha f(z)\varphi_k^\alpha(r) r^{2\alpha+1}\,dr = \Gamma(\alpha+1)^2 (f, \psi_k^\alpha)_\alpha \psi_k^\alpha(z). \tag{4.1}$$

Again from (1.14) we get,

$$\begin{aligned} S_n^\alpha f(z) &= \sum_{k=0}^{N} (f, \psi_k^\alpha)_\alpha \psi_k^\alpha(z) \\ &= (\Gamma(\alpha+1))^{-2} \int_0^\infty T_r^\alpha f(z) \sum_{k=0}^{N} \varphi_k^\alpha(r) r^{2\alpha+1}\,dr \\ &= (\Gamma(\alpha+1))^{-2} \int_0^\infty T_r^\alpha f(z)\varphi_N^{\alpha+1}(r) r^{2\alpha+1}\,dr. \end{aligned} \tag{4.2}$$

Here we have used the relation $\Sigma_0^N L_k^\alpha(x) = L_N^{\alpha+1}(x)$. We use the above representation for $S_N^\alpha f(z)$ to prove Theorem (2). The proof uses the following fact: If $g \in L_\alpha^2(\mathbb{R}_+)$, then the Fourier–Laguerre coefficients $(g, \psi_k^\alpha)_\alpha \to 0$ as $k \to \infty$. Recalling the definition of ψ_k^α this means that

$$\int_0^\infty g(r)\varphi_k^\alpha(r) r^{2\alpha+1}\,dr = \circ(k^{\alpha/2}) \quad \text{as } k \to \infty. \tag{4.3}$$

Also if $g \in W_\alpha^s(\mathbb{R}_+)$ then,

$$\int_0^\infty g(r)\varphi_k^\alpha(r) r^{2\alpha+1}\,dr = \circ(k^{-s+\alpha/2}) \quad \text{as } k \to \infty. \tag{4.4}$$

From (4.2) we get

$$S_N^\alpha f(z) = (\Gamma(\alpha+1))^{-2} \int_0^\infty \frac{T_r^\alpha f(z)}{r^2} \varphi_N^{\alpha+1}(r) r^{2\alpha+3}\,dr. \tag{4.5}$$

Let $\tilde{h}$ be a smooth function on $(\mathbb{R}_+)$ such that $\tilde{h}(r) \equiv 1$ on the support of $T_r^\alpha f(z)$ and $\tilde{h}(r) \equiv 0$ in a neighbourhood of the origin in $\mathbb{R}_+$. Put $h(r) = \dfrac{\tilde{h}(r)}{r^2}$. Thus we get

$$S_N^\alpha f(z) = (\Gamma(\alpha+1))^{-2} \int_0^\infty h(r)\, T_r^\alpha f(z)\varphi_N^{\alpha+1}(r) r^{2\alpha+3}\,dr \tag{4.6}$$

Now if $T_r^\alpha f(z) \in W_\alpha^{(\alpha+1)/2}$, we have by Proposition 3 $h(r)\, T_r^\alpha f(z) \in W_{\alpha+1}^{(\alpha+1)/2}$. Therefore

by (4.3),

$$S_N^\alpha f(z) = \circ(N^{(-(\alpha+1)/2+(\alpha+1)/2)}) = \circ(1),$$

as $N \to \infty$. Therefore $S_N^\alpha f(z) \to 0$ as $N \to \infty$, which proves the theorem.

In view of Lemma 6, if $f \in W_\alpha^{1/2}$, then $T_r^\alpha f(z) \in W_\alpha^{(\alpha+1)/2}$, for $z \neq 0$. Thus we have the following corollary to the above theorem.

COROLLARY 8

If $f \in W_\alpha^{1/2}$ then the conclusion of Theorem 2 holds at points $z \neq 0$.

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Degree of approximation of functions in the Hölder metric by (e, c) means

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Abstract. Degree of approximation of functions by the (e, c) means of its Fourier series in the Hölder metric is studied.

Keywords. Fourier series; Hölder metric; Banach space.

1. Definitions and notations

Let f be a periodic function with period 2π and integrable in the sense of Lebesgue over $[-\pi, \pi]$. Let the Fourier series of f at $t = x$ be

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{1}$$

Let

$$\Phi_x(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}. \tag{2}$$

Let $S_k(f; x)$ be the kth partial sum of the Fourier series (1). Then it is easily seen that (see [9], p. 50)

$$S_k(f; x) - f(x) = \frac{2}{\pi}\int_0^{\pi} \frac{\Phi_x(t)}{2\sin\frac{1}{2}t} \sin\left(k + \frac{1}{2}\right)t \, dt. \tag{3}$$

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic and continuous functions defined on $[-\pi, \pi]$ under the sup-norm. For $0 < \alpha \leqslant 1$ and some positive constant K, the function space H_α is given by the following:

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leqslant K|x - y|^\alpha\}. \tag{4}$$

The space H_α is a Banach space [7] with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x,y} \Delta^\alpha[f(x, y)] \tag{5}$$

where

$$\|f\|_c = \sup_{-\pi \leqslant x \leqslant \pi} |f(x)| \tag{6}$$

and

$$\Delta^\alpha f(x, y) = \frac{|f(x) - f(y)|}{|x - y|^\alpha} (x \neq y). \tag{7}$$

We shall use the convention that $\Delta^0 f(x, y) = 0$. The metric induced by the norm (5) on H_α is called a Hölder metric. It can be seen that $\|f\|_\beta \leqslant (2\pi)^{\alpha-\beta}\|f\|_\alpha$ for $0 \leqslant \beta < \alpha < 1$.

Thus $(H_\alpha, \|\cdot\|_\alpha)$ is a family of Banach space which decreases as a increases, i.e.

$$C_{2\pi} \supseteq H_\beta \supseteq H_\alpha \quad \text{for } 0 \leqslant \beta < \alpha < 1.$$

DEFINITION

An infinite series $\Sigma_{-\infty}^{\infty} c_n$ with partial sums $\{C_n\}$ is said to be summable $(e, c)(c > 0)$ to sum S, if

$$\lim_{n\to\infty} \sqrt{\frac{c}{\pi n}} \sum_{-\infty}^{\infty} \exp\left(-\frac{ck^2}{n}\right) C_{n+k} = S \tag{8}$$

where it is understood that $C_{n+k} = 0$ when $n + k < 0$.

The (e, c) summability method which is a regular method of summation was introduced by Hardy and Littlewood [4] (cf. also [5]) as an auxiliary method to prove Tauberian theorem for Borel summability.

It is known [6] that, if $c_n = o(1)$ and

$$c = \frac{1}{2}\alpha = \frac{k}{2(1-k)} = \frac{1+q}{2q} \tag{9}$$

then summability of Σc_n by any one of the methods (e, c), Borel exponential method (B, α), Borel integral method $(B', \alpha), \alpha > 0$, Euler method $(E, q)(q > 0)$ and circle method $(\gamma, k)(0 < k < 1)$ implies its summability to the same sum by any of the others.

2. Introduction

Alexits [1] studied the degree of approximation of function of H_α by the Cesàro mean of their Fourier series in the sup-norm. Since $C_{2\pi} \supseteq H_\alpha \supseteq H_\beta$ for $0 \leqslant \beta < \alpha \leqslant 1$, Prosdorff [7] obtained an estimate for $\|\sigma_n(f) - f\|_\beta$ for $f \in H_\alpha$, where $\sigma_n(f)$ is the Fejèr means of the Fourier series of f. Precisely he proved the following:

Theorem A ([7], Theorem 2). *Let $f \in H_\alpha (0 < \alpha \leqslant 1)$ and $0 \leqslant \beta < \alpha$. Then*

$$\|\sigma_n(f) - f\|_\beta = O(1) \begin{cases} n^{\beta-\alpha} & (0 < \alpha < 1) \\ \dfrac{1}{n(\log n)^{\beta-1}} & (\alpha = 1). \end{cases}$$

The case $\beta = 0$ of Theorem A is that of Alexits referred to earlier. Recently Chandra has studied the degree of approximation of functions in Hölder metric by Borel's means [3] and by Euler's means [2]. Precisely, he proved

Theorem B [3]. *Let $0 \leqslant \beta < \alpha \leqslant 1$ and let $f \in H_\alpha$. Then*

$$\|B_n(f) - f\|_\beta = O(n^{\beta-\alpha} \log n)$$

where $B_n(f)$ is the Borel exponential mean of $S_n(f; x)$.

Theorem C [2]. *Let $0 \leqslant \beta < \alpha \leqslant 1$ and let $f \in H_\alpha$. Then*

$$\|E_n^q(f) - f\|_\beta = O(n^{\beta-\alpha} \log n)$$

where $E_n^q(f)$ is the Euler (E, q), $q > 0$ mean of $S_n(f; x)$.

The object of this paper is to find the degree of approximation of functions by the (e, c)-mean of its Fourier series in the Hölder metric. Denoting the (e, c)-mean of $S_n(f; x)$

$$e_n(f, x) = e_n^c(f; x) = \sqrt{\frac{c}{\pi n}} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{ck^2}{n}\right) S_{n+k}(f; x), \tag{10}$$

where $S_{n+x}(f; x) = 0$, $n + k < 0$, we prove the following theorems:

Theorem 1. *Let* $0 < \alpha \leqslant 1$ *and* $0 \leqslant \beta < \alpha$. *Let* $f \in H_\alpha$. *Then*

$$\| e_n(f) - f \|_\beta = O(1) \begin{cases} \dfrac{\log n}{n^{\alpha-\beta}} & (0 < \alpha - \beta \leqslant \tfrac{1}{2}) \\ \dfrac{1}{n^{1/2}} & (\tfrac{1}{2} < \alpha - \beta \leqslant 1). \end{cases}$$

Theorem 2. *Let* $0 < \alpha \leqslant 1$ *and* $0 \leqslant \beta < \alpha$ *and let* $f \in H_\alpha$. *Further, if*

$$\int_{2\pi/2n+1}^{\pi \log n/n^{1/2}} \frac{|\Phi_x(t + (2\pi/2n+1)) - \Phi_x(t)|}{t} \exp(-nt^2/4c)\mathrm{d}t = O\left(\frac{1}{n^\alpha}\right) \tag{11}$$

then

$$\| e_n(f) - f \|_\beta = O(1) \begin{cases} \dfrac{(\log n)^{\beta/\alpha}}{n^{\alpha-\beta}}, & 0 < \alpha - \beta \leqslant \tfrac{1}{2} \\ \dfrac{1}{n^{1/2}}, & \tfrac{1}{2} < \alpha - \beta \leqslant 1. \end{cases}$$

3. Additional notations and estimates

We use the following additional notations:

$$e_n(t) = \sqrt{\frac{c}{\pi n}} \sum_{k=-n}^{\infty} \exp\left(-\frac{ck^2}{n}\right) \sin\left(n + \frac{1}{2}\right) t \tag{12}$$

$$K_n(t) = \sqrt{\frac{c}{\pi n}} \left\{ 1 + 2 \sum_{k=1}^{n} \exp\left(-\frac{ck^2}{n}\right) \cos kt \right\} \tag{13}$$

$$L_n(t) = \sqrt{\frac{c}{\pi n}} \left\{ \sum_{k=n+1}^{\infty} \exp\left(-\frac{ck^2}{n}\right) \sin\left(n + k + \frac{1}{2}\right) t \right\} \tag{14}$$

$$\theta = \theta(n) = \sqrt{\frac{c}{\pi n}} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{ck^2}{n}\right) \tag{15}$$

$$\eta = \eta(n) = \frac{2\pi}{2n+1} \tag{16}$$

$$N = N(n) = \frac{\pi \log n}{n^{1/2}} \tag{17}$$

$$\lambda = \frac{1}{4c} \tag{18}$$

$$F(t) = \Phi_x(t) - \Phi_y(t) \tag{19}$$

Estimates. We need the following estimates:

If $f \in H_\alpha$, $0 < \alpha \leqslant 1$, then

$$F(t) = \begin{cases} O(|t|^\alpha) & (20) \\ O(|x-y|^\alpha) & (21) \end{cases}$$

and

$$F(t) - F(t_1) = O(|t-t_1|^\alpha) \tag{22}$$

$$\exp(-n\lambda(t+\eta)^2) - \exp(-n\lambda t^2) = O(t+\eta)\exp(-n\lambda t^2) \tag{23}$$

$$K_n(t) = \exp(-n\lambda t^2) + \psi(n), \quad \text{where } \psi(n) = O(e^{-\delta n}), \quad c > \delta > 0 \tag{24}$$

$$L_n(t) = O(te^{-\delta n}), \quad (c > \delta > 0) \tag{25}$$

$$\theta(n) - 1 = O(n^{-1/2}). \tag{26}$$

If there is no confusion, we shall write throughout δ as a suitably chosen positive constant not necessarily the same at each occurrence.

Proof of the estimates. Estimates (20) and (21) follow immediately from the definition of $\Phi_x(t)$ and H_α. Now

$$\begin{aligned} 2(F(t) - F(t_1)) &= 2[(\Phi_x(t) - \Phi_y(t)) - (\Phi_x(t_1) - \Phi_y(t_1)] \\ &= 2[(\Phi_x(t) - \Phi_x(t_1)) - (\Phi_y(t) - \Phi_y(t_1))] \end{aligned}$$

and

$$\begin{aligned} 2|\Phi_x(t) - \Phi_x(t_1)| &\leqslant |f(x+t) - f(x+t_1)| + |f(x-t) - f(x-t_1)| \\ &= O(|t_1 - t|^\alpha) \quad \text{as } f \in H_\alpha. \end{aligned}$$

Hence (22) follows at once.

Proof of (23). We put $g(x) = \exp(-n\lambda x^2)$. By mean value theorem for some $0 < \xi < 1$

$$\exp(-n\lambda(t+\eta)^2) - \exp(-n\lambda t^2) = g(t+\eta) - g(t) = \eta g'(t + \xi\eta),$$

from which (23) follows at once.

Proof of (24) is contained in (Siddiqui [8], p. 122), and proof (26) can be found in (Hardy [6], p. 205).

Proof of (25). We have

$$\begin{aligned} L_n(t) &= \sqrt{\frac{c}{n\pi}} \sum_{k=n+1}^{\infty} \exp\left(-\frac{ck^2}{n}\right) \sin\left(n + k + \frac{1}{2}\right)t \\ &= O(n^{-1/2}t) \sum_{k=n+1}^{\infty} k \exp\left(-\frac{ck^2}{n}\right) \\ &= O(n^{-1/2}t) \int_{n+1}^{\infty} x \exp\left(-\frac{cx^2}{n}\right) dx \\ &= O(\sqrt{n}t) \int_{n+1}^{\infty} \frac{x}{n} \exp\left(-\frac{cx^2}{n}\right) dx \end{aligned}$$

$$= O(\sqrt{n}t)\left[\exp\left(-\frac{cx^2}{n}\right)\right]_n^\infty$$

$$= O(\sqrt{n}te^{-cn}) = O(te^{-\delta n}) \quad (0 < \delta < c).$$

Proof of Theorem 1. From (3), (10) and (15), we get taking $S_{n+k}(f;x) = 0$, when $k < -n$

$$e_n(f;x) - f(x) = \sqrt{\frac{c}{\pi n}} \sum_{k=-n}^{\infty} \exp\left(-\frac{ck^2}{n}\right) S_{n+k}(f;x) + (\theta(n)-1)f(x)$$

$$= \frac{2}{\pi}\frac{c}{\pi n}\int_0^\pi \frac{\Phi_x(t)}{2\sin\frac{1}{2}t}\left(\sum_{k=-n}^{\infty} \exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t\right)dt$$

$$+ (\theta(n)-1)f(x).$$

Let

$$l_n(x) - e_n^c(f;x) - f(x).$$

Then using (12) and (19), we obtain

$$l_n(x) = l_n(y) = \frac{2}{\pi}\int_0^\pi \frac{F(t)}{2\sin\frac{1}{2}t} e_n(t)dt + (\theta(n)-1)(f(x)-f(y)). \tag{27}$$

We have

$$e_n(t) = \sqrt{\frac{c}{\pi n}} \sum_{k=-n}^{\infty} \exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t$$

$$= \sqrt{\frac{c}{\pi n}}\left[\sum_{k=-n}^{n} \exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t\right.$$

$$\left. + \sum_{k=n+1}^{\infty} \exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t\right]$$

$$= \sqrt{\frac{c}{\pi n}}\left[\left(1+2\sum_{k=1}^{n} \exp\left(-\frac{ck^2}{n}\right)\cos kt\right)\sin\left(n+\frac{1}{2}\right)t\right.$$

$$\left. + \sum_{k=n+1}^{\infty} \exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t\right]$$

$$= K_n(t)\sin\left(n+\frac{1}{2}\right)t + L_n(t), \tag{28}$$

using (13) and (14).

From (27), (28) and (24), we get

$$l_n(x) - l_n(y) = \frac{2}{\pi}\int_0^\pi \frac{F(t)}{2\sin\frac{1}{2}t} K_n(t)\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$+ \frac{2}{\pi}\int_0^\pi \frac{F(t)}{2\sin\frac{1}{2}t} L_n(t)dt + (\theta(n)-1)(f(x)-f(y)) \tag{29}$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{F(t)}{2\sin\frac{1}{2}t} e^{-n\lambda t^2} \sin\left(n+\frac{1}{2}\right)t\,dt$$
$$+ \frac{2}{\pi}\psi(n)\int_0^{\pi} \frac{F(t)}{2\sin\frac{1}{2}t}\sin\left(n+\frac{1}{2}\right)t\,dt$$
$$+ \frac{2}{\pi}\int_0^{\pi} \frac{F(t)}{2\sin\frac{1}{2}t} L_n(t)dt + (\theta(n)-1)(f(x)-f(y))$$
$$= I + J + K + L, \text{ say.} \tag{30}$$

Using (20), (21) and (24), we get

$$J = O(e^{-\delta n}) \begin{cases} \int_0^{\pi} t^{\alpha-1}\,dt \\ |x-y|^{\alpha}\int_0^{\pi} n\,dt \end{cases} \tag{31}$$

$$= O(1)\begin{cases} e^{-\delta n} \\ e^{-\delta n}|x-y|^{\alpha}. \end{cases} \tag{32}$$

Using (31) and (32)

$$J = J^{1-\beta/\alpha}J^{\beta/\alpha} = O(1)e^{-\delta n(1-\beta/\alpha)}(|x-y|^{\alpha})^{\beta/\alpha}$$
$$= O(1)(e^{-\delta n})^{1-\beta/\alpha}(|x-y|^{\alpha})^{\beta/\alpha} = O(1)e^{-\delta n}|x-y|^{\beta}. \tag{33}$$

As

$$f(x)-f(y) = O(1) \quad \text{and} \quad f(x)-f(y) = O(|x-y|^{\alpha}),$$

using (26), we get

$$L = O(1)\begin{cases} \dfrac{1}{n^{1/2}} \\ \dfrac{|x-y|^{\alpha}}{n^{1/2}}. \end{cases}$$

Similarly (argue as in J)

$$L = O(1)\frac{|x-y|^{\beta}}{n^{1/2}}. \tag{34}$$

We write

$$I = \frac{2}{\pi}\left[\int_0^{\eta} + \int_{\eta}^{N} + \int_N^{\pi}\right] \equiv I_1 + I_2 + I_3, \text{ say.} \tag{35}$$

Using (20), we get

$$I_1 = \int_0^{\eta} \frac{F(t)}{2\sin\frac{1}{2}t} e^{-n\lambda t^2}\sin\left(n+\frac{1}{2}\right)t\,dt$$
$$= O(1)\int_0^{\eta} t^{\alpha-1}e^{-n\lambda t^2}dt$$
$$= O(1)\int_0^{\eta} t^{\alpha-1}dt = 0(n^{-\alpha}). \tag{36}$$

Using (21), we get

$$I_1 = O(1)\int_0^{\eta} |x-y|^{\alpha} n e^{-n\lambda t^2} dt$$

$$= O(1)|x-y|^{\alpha} n \int_0^{\eta} dt = O(|x-y|^{\alpha}). \tag{37}$$

Using (20), we get

$$I_3 = \frac{2}{\pi}\int_N^{\pi} \frac{F(t)}{2\sin\frac{1}{2}t} e^{-n\lambda t^2} \sin\left(n+\frac{1}{2}\right)t\, dt$$

$$= O(1)\int_N^{\pi} t^{\alpha-1} e^{-n\lambda t^2} dt$$

$$= O(1)e^{-n\lambda N^2}\int_N^{\pi} t^{\alpha-1} dt \quad (\text{as } e^{-n\lambda t^2} \text{ is decreasing})$$

$$= O(e^{-n\lambda N^2})$$

$$= O(1)(e^{-\lambda\pi^2(\log n)^2})$$

$$= O(1)(e^{-\Delta \log n}) \quad (\Delta > 0 \text{ however large})$$

$$= O(1)\left(\frac{1}{n^{\Delta}}\right). \tag{38}$$

Using (21), we get

$$I_3 = O(|x-y|^{\alpha})\int_N^{\pi} \frac{e^{-\lambda n t^2}}{t} dt$$

$$= O(|x-y|^{\alpha})e^{-n\lambda N^2}\int_N^{\pi} \frac{dt}{t}$$

$$= O(|x-y|^{\alpha}e^{-\lambda\pi^2(\log n)^2}\log N$$

$$= O(|x-y|^{\alpha})\left(\frac{1}{n^{\Delta}}\right) \quad (\Delta > 0 \text{ however large}) \tag{39}$$

as in (38).

Now

$$I_2 = \frac{2}{\pi}\int_{\eta}^{N} \frac{F(t)}{2\sin\frac{1}{2}t} e^{-\lambda n t^2} \sin\left(n+\frac{1}{2}\right)t\, dt$$

$$= \frac{2}{\pi}\int_{\eta}^{N} \frac{F(t)}{t} e^{-\lambda n t^2} \sin\left(n+\frac{1}{2}\right)t\, dt$$

$$+ \frac{2}{\pi}\int_{\eta}^{N} F(t)\left[\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right] e^{-\lambda n t^2} \sin\left(n+\frac{1}{2}\right)t\, dt$$

$$= I_{2,1} + I_{2,2}, \text{ say.} \tag{40}$$

Using (20) and the fact that

$$\left[\frac{1}{2\sin\frac{1}{2}t}-\frac{1}{t}\right]=O(t),$$

we get

$$\begin{aligned}
I_{2,2}&=\frac{2}{\pi}\int_{\eta}^{N}F(t)\left[\frac{1}{2\sin\frac{1}{2}t}-\frac{1}{t}\right]e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\
&=O(1)\int_{\eta}^{N}t^{\alpha+1}e^{-\lambda nt^2}dt\\
&=O(n^{-1})\int_{\eta}^{N}t^{\alpha}(nte^{-\lambda nt^2})dt\\
&=O(n^{-1})\int_{\eta}^{N}t^{\alpha}\frac{d}{dt}(e^{-\lambda nt^2})dt\\
&=O(n^{-1-\alpha})\quad\text{(integrating by parts).}
\end{aligned}\tag{41}$$

Next, we write

$$\begin{aligned}
2I_{2,1}&=2\int_{\eta}^{N}\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\
&=\left(\int_{\eta}^{N}+\int_{2\eta}^{N+\eta}+\int_{\eta}^{2\eta}-\int_{N}^{N+\eta}\right)\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\
&=\int_{\eta}^{N}\left(\frac{F(t)}{t}e^{-\lambda nt^2}-\frac{F(t+\eta)}{t+\eta}e^{-\lambda n(t+\eta)^2}\right)\sin\left(n+\frac{1}{2}\right)t\,dt\\
&\quad+\int_{\eta}^{2N}\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\
&\quad-\int_{N}^{N+\eta}\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\
&\qquad\qquad\text{(since }\sin(n+\tfrac{1}{2})(t+\eta)=-\sin(n+\tfrac{1}{2})t\text{)}\\
&=-\int_{\eta}^{N}\frac{F(t+\eta)-F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\
&\quad+\int_{\eta}^{N}F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\
&\quad+\int_{\eta}^{N}\frac{F(t+\eta)}{t+\eta}[e^{-\lambda nt^2}-e^{-\lambda n(t+\eta)^2}]\sin\left(n+\frac{1}{2}\right)t\,dt\\
&\quad-\int_{\eta}^{2\eta}\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\
&\quad-\int_{N}^{N+\eta}\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\
&=M_1+M_2+M_3+M_4+M_5,\text{ say.}
\end{aligned}\tag{42}$$

Using (22), we have

$$M_1 = O(\eta^\alpha)\int_\eta^N \frac{e^{-\lambda n t^2}}{t}dt = O(\eta^\alpha)\int_\eta^N \frac{dt}{t} = O\left(\frac{\log n}{n^\alpha}\right). \tag{43}$$

Using (20) and (23)

$$\begin{aligned} M_3 &= O(1)\int_\eta^N (t+\eta)^{\alpha-1}e^{-\lambda n t^2}(t+\eta)dt \\ &= O(1)\int_\eta^N t^\alpha e^{-n\lambda t^2}dt \\ &= O(n^{-1})\int_\eta^N t^{\alpha-1}\frac{d}{dt}(e^{-\lambda n t^2})dt \\ &= O(n^{-\alpha}) \quad \text{(integrating by parts).} \end{aligned} \tag{44}$$

Using (20), we get

$$M_4 = O(1)\int_\eta^{2\eta} t^{\alpha-1}e^{-\lambda n t^2}dt = O(n^{-\alpha}) \tag{45}$$

and

$$\begin{aligned} M_5 &= O(1)\int_N^{N+\eta} t^{\alpha-1}e^{-\lambda n t^2}dt \\ &= O(1)e^{-\lambda n N^2}\int_N^{N+\eta} t^{\alpha-1}dt \\ &= O(1)e^{-\lambda\pi^2(\log n)^2}N^\alpha \\ &= O(1)n^{-\Delta}\left(\frac{\log n}{n^{1/2}}\right)^\alpha \quad (\Delta \text{ however large}) \\ &= O(1)(n^{-\Delta}). \end{aligned} \tag{46}$$

Now, we write

$$\begin{aligned} 2M_2 &= \int_\eta^N F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-\lambda n t^2}\sin\left(n+\frac{1}{2}\right)t\,dt \\ &= \left(\int_\eta^N + \int_{2\eta}^{N+\eta} + \int_\eta^{2\eta} - \int_N^{N+\eta}\right)F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-\lambda n t^2} \\ &\quad \times \sin\left(n+\frac{1}{2}\right)t\,dt \\ &= \int_\eta^N \left\{F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-\lambda n t^2} - F(t+2\eta)\left(\frac{1}{t+\eta}-\frac{1}{t+2\eta}\right)\right. \\ &\quad \left.\times e^{-n\lambda(t+\eta)^2}\right\}\sin\left(n+\frac{1}{2}\right)t\,dt \\ &\quad + \int_\eta^{2\eta} F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-n\lambda t^2}\sin\left(n+\frac{1}{2}\right)t\,dt \end{aligned}$$

$$- \int_N^{N+\eta} F(t+\eta)\left(\frac{1}{t} - \frac{1}{t+\eta}\right) e^{-n\lambda t^2} \sin\left(n+\frac{1}{2}\right) t \, dt$$

$$= P + Q - R, \text{ say.} \tag{47}$$

Using (20) and the fact that

$$\frac{1}{t} - \frac{1}{t+\eta} = O(\eta/t^2)$$

it can be proved employing the argument used in proving (45) and (46) that

$$Q = O(n^{-\alpha}) \tag{48}$$

$$R = O\left(\frac{1}{n^{\Delta}}\right). \tag{49}$$

By formal computation, we get

$$P = \int_\eta^N (F(t+\eta) - F(t+2\eta))\left(\frac{1}{t} - \frac{1}{t+\eta}\right) e^{-\lambda n t^2} \sin\left(n+\frac{1}{2}\right) t \, dt$$

$$+ \int_\eta^N F(t+2\eta)\left[\frac{1}{t} - \frac{2}{t+\eta} + \frac{1}{t+2\eta}\right] e^{-\lambda n t^2} \sin\left(n+\frac{1}{2}\right) t \, dt$$

$$+ \int_\eta^N F(t+2\eta)\left(\frac{1}{t+\eta} - \frac{1}{t+2\eta}\right)(e^{-\lambda n t^2} - e^{-\lambda n (t+\eta)^2}) \sin\left(n+\frac{1}{2}\right) t \, dt$$

$$= P_1 + P_2 + P_3, \text{ say.} \tag{50}$$

Using (22), we get

$$P_1 = O(\eta^\alpha) \int_\eta^N \frac{\eta}{t(t+\eta)} e^{-\lambda n t^2} \, dt$$

$$= O(\eta^{1+\alpha}) \int_\eta^N \frac{dt}{t^2} = O(n^{-\alpha}). \tag{51}$$

As

$$\left[\frac{1}{t} - \frac{2}{t+\eta} + \frac{1}{t+2\eta}\right] = \frac{2\eta^2}{t(t+\eta)(t+2\eta)}.$$

we obtain using (20)

$$P_2 = O(1) \int_\eta^N \frac{\eta^2}{t^3} (t+2\eta)^\alpha e^{-\lambda n t^2} \, dt$$

$$= O(\eta^2) \int_\eta^N \frac{dt}{t^{3-\alpha}} = O(n^{-\alpha}). \tag{52}$$

Lastly using (20) and (23), we get

$$P_3 = O(\eta) \int_\eta^N t^{\alpha-1} \, dt = O\left(\frac{(\log n)^\alpha}{n^{1+\alpha/2}}\right). \tag{53}$$

Collecting the results of (42)–(53), we get

$$I_{2,1} = O\left(\frac{\log n}{n^{\alpha}}\right). \tag{54}$$

From (40), (41) and (54), we have

$$I_2 = O\left(\frac{\log n}{n^{\alpha}}\right). \tag{55}$$

Using (21), we also get

$$\begin{aligned} I_2 &= O(|x-y|^{\alpha}) \int_{\eta}^{N} \frac{e^{-\lambda n t^2}}{2\sin\frac{1}{2}t} dt \\ &= O(|x-y|^{\alpha}) \int_{\eta}^{N} \frac{dt}{t} = O(|x-y|^{\alpha} \log n). \end{aligned} \tag{56}$$

Writing

$$I_k = I_k^{1-\beta/\alpha} I_k^{\beta/\alpha} \quad (k = 1, 2, 3)$$

and using the estimates (36), (37) for I_1, (55), (56) for I_2 and (38), (39) for I_3 we get

$$I_1 = O(|x-y|^{\beta} n^{\beta-\alpha}) \tag{57}$$

$$I_2 = O(|x-y|^{\beta} n^{\beta-\alpha} \log n) \tag{58}$$

$$I_3 = O\left(|x-y|^{\beta} \frac{1}{n^{\Delta}}\right), \quad \Delta > 0, \quad \text{however large.} \tag{59}$$

From (35), (57), (58) and (59), we get

$$I = O(1)|x-y|^{\beta} n^{\beta-\alpha} \log n. \tag{60}$$

Using (25), we get

$$\begin{aligned} K &= \frac{2}{\pi} \int_0^{\pi} \frac{F(t)}{2\sin\frac{1}{2}t} L_n(t) dt \\ &= O(e^{-\delta n}) \int_0^{\pi} |F(t)| dt = \begin{cases} O(e^{-\delta n}) \\ O(|x-y|^{\alpha} e^{-\delta n}) \end{cases}. \end{aligned} \tag{61}$$

From (61), we get (writing $K = K^{1-\beta/\alpha} K^{\beta/\alpha}$)

$$K = O(e^{-\delta n}|x-y|^{\beta}). \tag{62}$$

Collecting the results of (30), (33), (34), (52) and (62) we get

$$l_n(x) - l_n(y) = O(1) \begin{cases} |x-y|^{\beta} n^{\beta-\alpha} \log n, & 0 < \alpha - \beta \leqslant \frac{1}{2} \\ |x-y|^{\beta} \frac{!}{n}, & \frac{1}{2} < \alpha - \beta \leqslant 1 \end{cases}.$$

Hence

$$\sup_{\substack{x,y \\ x \neq y}} \Delta^{\beta} l_n(x,y) = O(1) \begin{cases} n^{\beta-\alpha} \log n, & 0 < \alpha - \beta \leqslant \frac{1}{2} \\ \frac{1}{\sqrt{n}}, & \frac{1}{2} < \alpha - \beta \leqslant 1 \end{cases}. \tag{63}$$

Again $f \in H_x \Rightarrow \Phi_x(t) = O(|t|^\alpha)$ and so proceeding as above, we get

$$\|l_n\|_c| = \sup_{-\pi \leqslant x \leqslant \pi} |l_n(x)| = \begin{cases} \dfrac{\log n}{n^\alpha}, & 0 < \alpha \leqslant \frac{1}{2} \\ \dfrac{1}{\sqrt{n}}, & \frac{1}{2} < \alpha \leqslant 1 \end{cases} . \tag{64}$$

Theorem 1 is completely proved by combining (63) and (64).

Proof of Theorem 2. We proceed as in the proof of Theorem 1 and retain all the estimates of J, K and L. As regards I, we retain all the estimates of the components of I except the one given in (43) for M_1 which contributes the estimation $O(\log n/n^\alpha)$. By (11) of the hypothesis of Theorem 2

$$\begin{aligned} M_1 &= \int_\eta^N \frac{F(t+\eta) - F(t)}{t} e^{-\lambda n t^2} \sin\left(n + \frac{1}{2}\right) t \, dt \\ &= O(1) \int_\eta^N \frac{|F(t+\eta) - F(t)|}{t} e^{-\lambda n t^2} \left| \sin\left(n + \frac{1}{2}\right) t \right| dt \\ &= O(1) \int_\eta^N \frac{|F(t+\eta) - F(t)|}{t} e^{-\lambda n t^2} dt = O(n^{-\alpha}). \end{aligned} \tag{65}$$

Using (65) instead of (43), it can be proved that

$$I_2 = O(n^{-\alpha}). \tag{66}$$

Now using (56) and (66)

$$\begin{aligned} I_2 &= I_2^{\beta/\alpha} I_2^{1-\beta/\alpha} \\ &= O(1)(|x-y|^\alpha \log n)^{\beta/\alpha} (n^{-\alpha})^{1-\beta/\alpha} \\ &= O(1)|x-y|^\beta (\log n)^{\beta/\alpha} n^{\beta-\alpha}. \end{aligned} \tag{67}$$

Proceeding as in Theorem 1 and using (67) and the estimates of I_1 and I_3 from (57), we obtain

$$\sup_{\substack{x,y \\ x \neq y}} \Delta^\beta l_n(x,y) = O(1) \begin{cases} n^{\beta-\alpha} (\log n)^{\beta-\alpha}, & 0 < \alpha - \beta \leqslant \frac{1}{2} \\ \dfrac{1}{\sqrt{n}}, & \frac{1}{2} < \alpha - \beta \leqslant 1 \end{cases} . \tag{68}$$

Arguing as in Theorem 1 and using (11) as employed above in the estimation of I_2, it can be shown that

$$\|l_n\|_c = \begin{cases} O(n^{-\alpha}), & 0 < \alpha \leqslant \frac{1}{2} \\ O\left(\dfrac{1}{\sqrt{n}}\right), & \frac{1}{2} < \alpha \leqslant 1 \end{cases} . \tag{69}$$

Now Theorem 2 follows at once from (68) and (69).

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The algebra $A_p((0, \infty))$ and its multipliers

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Abstract. Let $I = \{x \in R: 0 \leqslant x < \infty\}$ be the locally compact semigroup with addition as binary operation and the usual interval topology. The purpose of this note is to study the algebra $A_p(I)$ of elements in $L_1(I)$ whose Gelfand transforms belong to $L_p(\hat{I})$, where $\hat{I}$ denotes maximal ideal space of $L_1(I)$. The multipliers of $A_p(I)$ have also been identified.

Keywords. Binary operation; interval topology; Gelfand transforms, maximal ideal space.

1. Introduction

Let G be a locally compact Hausdorff Abelian group and $\hat{G}$ denote the dual group of G. The algebra $A_p(G)$, $1 \leqslant p \leqslant \infty$, of elements in $L_1(G)$ whose Fourier transforms belong to $L_p(\hat{G})$, and the multipliers for these algebras have been studied by various authors including Larsen, Liu and Wong [8], Reiter [10], Figa-Talamanca and Gaudry [3], and Martin and Yap [9]. The algebra $A_p((0, \infty))$ with order convolution, in short, $A_p(I)$ of elements in $L_1(I)$ whose Gelfand transforms belong to $L_p(\hat{I})$ and the multipliers for these algebras, where I is the locally compact idempotent commutative topological semigroup consisting of the open interval $(0, \infty)$ of real numbers from 0 to ∞ equipped with the usual topology and max. multiplication and $\hat{I}$ is the maximal ideal space of $L_1(I)$, have been studied by Kalra, Singh and Vasudeva [6]. The purpose of this note is to study the algebras $A_p(I)$ of elements in $L_1(I)$ whose Gelfand transforms belong to $L_p(\hat{I})$ and the multipliers for these algebras, where $I = \{x \in R: 0 \leqslant x < \infty\}$ is the locally compact semigroup with addition as binary operation and the usual interval topology and $\hat{I}$ is the maximal ideal space of $L_1(I)$. Whereas the algebras $A_p((0, \infty))$ with order convolution studied in [6] are dissimilar to the order convolution algebra $L_1((0, \infty))$, the algebra $A_p(I)$ proposed to be studied in this note show similarities to the algebra $L_1(I)$. In particular we shall see that the maximal ideal space $\Delta(A_p(I))$ of $A_p(I)$ is the same as that of $L_1(I)$. The situation is thus akin to the group algebras $L_1(G)$ and its subalgebras $A_p(G)$ studied by Larsen, Liu and Wong [8]. It turns out that the algebras $A_p(I)$ are not regular, whereas the algebras $A_p(G)$ [8] and $A_p((0, \infty))$ with order convolution [6] are regular. Moreover, the algebras of multipliers of $A_p(I)$ contain the algebras of multipliers of $L_1(I)$. We establish below our notations and then proceed to describe the results.

Let $I = \{x \in R: 0 \leqslant x < \infty\}$ be the locally compact semigroup with addition as binary operation and the usual interval topology. Let $\Sigma = \{z \in C: \mathrm{Re}\, z > 0\}$ and $\bar{\Sigma}$ denote the closure of Σ. The measure associated with Σ or $\bar{\Sigma}$ shall be the usual planar measure. The Fourier transform of a measureable function f, whenever it is meaningful, shall be denoted by $\tilde{f}$. $C_c(I)$ (resp. $C_c^\infty(I)$) shall denote the space of continuous complex-valued functions (resp. infinitely differentiable functions) with compact support in I. The

index conjugate to p, $1 \leqslant p \leqslant \infty$, shall be denoted by p', i.e., p and p' are positive numbers greater than or equal to 1 such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let $M(I)$ denote the Banach algebra of all finite regular Borel measures on I under convolution product $*$ and total variation norm. Then the Banach space $L_1(I)$ of all continuous measures in $M(I)$ which are absolutely continuous with respect to Lebesgue measure on I becomes a commutative semisimple Banach algebra in the inherited product $*$. More specifically, for $f \in L_1(I)$, $g \in L_1(I)$ and $x \in I$,

$$f*g(x) = \int_0^x f(x-y)g(y)\mathrm{d}y, \quad \|f\|_1 = \int_0^\infty |f(x)|\mathrm{d}x$$

satisfy $\|f*g\| \leqslant \|f\|_1 \|g\|_1$. The maximal ideal space $\hat{I}$ of $L_1(I)$ can be identified [4] with $\bar{\Sigma}$ and the Gelfand transform of an $f \in L_1(I)$ is its Laplace transform, i.e.,

$$\hat{f}(z) = \int_0^\infty f(t)\mathrm{e}^{-tz}\mathrm{d}t.$$

The function $\hat{f}$ is analytic in Σ. It, therefore, follows that $L_1(I)$ is not regular, *a fortiori*, no subalgebra of $L_1(I)$ under any norm with the same maximal ideal space can be regular. Clearly, for $x \geqslant 0$, the function $y \to \hat{f}(x+iy)$ is $\sqrt{2\pi}\tilde{f}_x$, where $f_x \in L_1(R)$ is given by $f_x(t) = f(t)\mathrm{e}^{-tx}$ for $t \in I$ and 0 in $R - I$. For these and other results that may be used in the sequel, the reader is referred to [4], [12].

Let $1 \leqslant p \leqslant \infty$. The algebras $A_p(I)$ consist of all those $f \in L_1(I)$ whose Gelfand transforms $\hat{f}$ belong to $L_p(\hat{I})$. $A_p(I)$ form an ascending chain of ideals in $L_1(I)$. $A_p(I)$ equipped with suitable norms become Banach algebras. These algebras do not have bounded approximate identity nor are these algebras regular. However, these algebras are semisimple. The maximal ideal space $\Delta(A_p(I))$ can be identified with $\bar{\Sigma}$. The above and other related results are contained in § 2.

A mapping T on a commutative Banach algebra A to itself is called a multiplier if $T(xy) = (Tx)y$ for $x, y \in A$. For results on multipliers, we refer to Larsen [7] rather than original sources. As $A_p(I)$ is semisimple, every multiplier of $A_p(I)$ is bounded and we may define a multiplier of $A_p(I)$ to be a bounced continuous function ϕ on $\bar{\Sigma}$ such that $\phi\hat{f} \in \hat{A}_p(I)$, whenever $\hat{f} \in \hat{A}_p(I)$, where $\hat{A}_p(I) = \{\hat{f} : f \in A_p(I)\}$ is the Banach algebra under pointwise operations and norm $|||\hat{f}||| = \|f\|_1 + \|\hat{f}\|_p$. In § 3, we prove an analogue of Paley–Wiener theorem. This, in turn, helps us provide a set of sufficient conditions and a set of necessary conditions on ϕ such that $\phi\hat{f} \in \hat{A}_p(I)$ whenever $\hat{f} \in \hat{A}_p(I)$.

2. The Banach algebras $A_p(I)$

As the Gelfand transform of a function in $L_1(I)$ belongs to $C_0(\hat{I})$, it is evident that $A_\infty(I) = L_1(I)$ and each $A_p(I)$ is an ideal in $L_1(I)$. Moreover, $A_p(I) \subseteq A_r(I)$ if $p < r$. Indeed, if $f \in A_p(I)$ and $p < r < \infty$, then

$$\|\hat{f}\|_r^r = \int_{\bar{\Sigma}} |\hat{f}(z)|^r \mathrm{d}z \leqslant \|\hat{f}\|_\infty^{r-p} \int_{\bar{\Sigma}} |\hat{f}(z)|^p \mathrm{d}z = \|\hat{f}\|_\infty^{r-p} \|\hat{f}\|_p^p.$$

The case $r = \infty$ is trivially true. For each p, $1 \leqslant p \leqslant \infty$, we define

$$|||f|||_p = \|f\|_1 + \|\hat{f}\|_p, \quad f \in A_p(I).$$

It can be verified as in Larsen, Liu and Wong [8] that $\|\ \|_p$ defines a norm on $A_p(I)$ and that $A_p(I)$ is a commutative Banach algebra under convolution. As observed earlier the algebras $L_1(I)$ and $A_\infty(I)$ are identical. Since

$$\|f\|_1 \leqslant \|f\|_1 + \|\hat{f}\|_\infty \equiv |||f|||_\infty \leqslant 2\|f\|_1,$$

for $f \in A_\infty(I)$, it follows that $\|\ \|_1$ and $|||\ |||_\infty$ are equivalent norms on $A_\infty(I)$.

The mapping $\Phi: A_p(I) \to L_1(I) \times L_p(\hat{I})$ defined by $\Phi(f) = (f, \hat{f})$, $f \in A_p(I)$ is clearly an isometry of $A_p(I)$ into the Banach space $L_1(I) \times L_p(\hat{I})$ with the norm $\|(f,g)\| = \|f\|_1 + \|g\|_p$. Thus $A_p(I)$ may be regarded as a closed subspace of $L_1(I) \times L_p(\hat{I})$. For each p, $1 \leqslant p < \infty$, the dual $A_p^*(I)$ of $A_p(I)$ is isometrically isomorphic to $L_\infty(I) \times L_{p'}(\hat{I})/\mathrm{Ker}\,\Phi^*$, where Φ^* is the adjoint of the map Φ and p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$ [see Theorem 2 [8]].

PROPOSITION 1

$$A_p(I) \subsetneq A_r(I) \quad \textit{for } 1 \leqslant p < r \leqslant \infty.$$

Proof. Let, for $n \in N$, u_n be the function $n\chi_{[0,1/n)}$. Then $\|u_n\| = 1$. Also for $z \in \bar{\Sigma}$ and $n \in N$, $\hat{u}_n(z) = (e^{-z/n} - 1)/(-z/n) = \hat{u}_1(z/n)$ and therefore. $|\hat{u}_n(z)| \leqslant \min\left\{\frac{2n}{|z|}, 1\right\}$. Further for $\mathrm{Re}\, z \geqslant n$, $|\hat{u}_n(z)| \geqslant n\left(1 - \frac{1}{e}\right)\Big/|z|$. Observe that $u_n \in A_p(I)$ for $p > 2$ and $u_n \notin A_2(I)$. Indeed for $p > 2$,

$$\begin{aligned}
\|\hat{u}_n(z)\|_p^p &= \int_{x \geqslant 1} \int_{y=-\infty}^{\infty} |\hat{u}_n(x+iy)|^p \mathrm{d}x\mathrm{d}y \\
&\quad + \int_{0 \leqslant x \leqslant 1} \int_{|y| \geqslant 1} |\hat{u}_n(x+iy)|^p \mathrm{d}x\mathrm{d}y \\
&\quad + \int_{0 \leqslant x \leqslant 1} \int_{|y| \leqslant 1} |\hat{u}_n(x+iy)|^p \mathrm{d}x\mathrm{d}y \\
&= T_1 + T_2 + T_3,
\end{aligned}$$

where

$$\begin{aligned}
T_1 &\leqslant \int_{x \geqslant 1} \int_{y=-\infty}^{\infty} \frac{(2n)^p}{(x^2+y^2)^{p/2}} \mathrm{d}x\mathrm{d}y \\
&\leqslant (2n)^p \int_1 \frac{\mathrm{d}x}{x^p} \int_{y=-\infty}^{\infty} \frac{\mathrm{d}y}{(1 + y^2/x^2)} = (2n)^p \pi/(p-1),
\end{aligned}$$

$$T_2 \leqslant \int_{0 \leqslant x \leqslant 1} \int_{|y| \geqslant 1} \frac{(2n)^p}{(x^2+y^2)^{p/2}} \mathrm{d}x\mathrm{d}y \leqslant 2 \cdot (2n)^p,$$

and

$$T_3 \leqslant \int_{0 \leqslant x \leqslant 1} \int_{|y| \leqslant 1} |\hat{u}_n(z)|^p \mathrm{d}x\mathrm{d}y \leqslant 2\|\hat{u}_n\|_\infty \leqslant 2\|u_n\|_1 = 2.$$

Thus $\hat{u}_n \in L_p(\hat{I})$, $p > 2$, and consequently $u_n \in A_p(I)$.

We next show that $u_n \notin A_2(I)$. Indeed,

$$\|\hat{u}_n\|_2^2 = \int_{\mathrm{Re}z \geqslant 0} |\hat{u}_n(z)|^2 \mathrm{d}z \geqslant \int_{\mathrm{Re}z \geqslant n} n^2 \left(1 - \frac{1}{e}\right)^2 \Big/ |z|^2 \mathrm{d}z$$

$$\geqslant n^2 \left(1 - \frac{1}{e}\right)^2 \int_{x \geqslant n} \int_{y=-\infty}^{\infty} \frac{\mathrm{d}x\mathrm{d}y}{(x^2+y^2)}$$

$$= n^2 \left(1 - \frac{1}{e}\right)^2 \int_{x \geqslant n} \frac{1}{x} |\tan^{-1}(y/x)|_{-\infty}^{\infty} \mathrm{d}x$$

$$= \pi n^2 \left(1 - \frac{1}{e}\right)^2 \int_{x \geqslant n} \frac{1}{x} \mathrm{d}x = \infty.$$

Again $u_n * u_n \notin A_1(I)$. Indeed, $\widehat{u_n * u_n}(z) = (\hat{u}_n(z))^2$ and $\int_{\mathrm{Re}z \geqslant 0} |(\hat{u}_n(z))^2| \mathrm{d}z = \int_{\mathrm{Re}z \geqslant 0} |(\hat{u}_n(z))|^2 \mathrm{d}z \geqslant \infty$, as shown above.

Also $u_n * u_n \in A_p(I)$ for $p > 1$. Indeed, $\widehat{u_n * u_n}(z) = (\hat{u}_n(z))^2$ and $\int_{\mathrm{Re}z \geqslant 0} |(\hat{u}_n(z))^2|^p \mathrm{d}z = \int_{\mathrm{Re}z \geqslant 0} |(\hat{u}_n(z))|^{2p} \mathrm{d}z = \int_{\mathrm{Re}z \geqslant 0} |(\hat{u}_n(z))|^q \mathrm{d}z$, where $q = 2p > 2$ and the right hand side is finite as shown above. It is also a consequence of above that $u_n * u_n * u_n \in A_1(I)$. So for $1 < p \leqslant 2 < r$, $A_1(I) \subset A_p(I) \subset A_r(I)$.

Now, let $1 < p < r < \infty$. Let, if possible, $A_p(I) = A_r(I)$. Then there exists $K > 0$ such that $|||f|||_p \leqslant K |||f|||_r$ for $f \in A_p(I)$. For $n \in N$ and $1 < s < \infty$, $|||u_n * u_n|||_s = \|u_n * u_n\|_1 + \|\hat{u}_n^2\|_s = \|u_n * u_n\|_1 + n^{2/s} \|\hat{u}_1^2\|_s = 1 + n^{2/s} \|\hat{u}_1^2\|_s$. Consequently, $n^{2/p - 2/r} \|\hat{u}_1^2\|_p \leqslant K(n^{-2/r} + \|\hat{u}_1^2\|_r)$. On letting $n \to \infty$, the left hand side of the preceeding inequality tends to infinity whereas the right hand side tends to a finite limit. This contradiction completes the proof in this case.

Now, let $1 \leqslant p < \infty$. Since $A_p(I) \subset A_{2p}(I) \subset A_\infty(I)$, we cannot have $A_p(I) = A_\infty(I)$.

PROPOSITION 2

Let f be a function defined on $[0, \infty)$ such that f', f'' exists and satisfies $f(0) = f'(0) = f''(0)$ and $0 = \lim_{x \to \infty} f'(x) = \lim_{x \to \infty} f''(x)$. Further suppose that $f, f', f'' \in L_1(I)$. Then

(a) *$f \in A_p(I)$ for $p > 1$,*
(b) *$f \in A_1(I)$ if $f'' = 0$ on $(0, c)$ for some $c > 0$,*
(c) *$f \in A_1(I)$ if f''' exists and is in $L_1(I)$, $f'''(0) = 0$ and $f'''(x) \to 0$ as $x \to \infty$.*

Proof. Observe that for $z \in \bar{\Sigma}$, $z \neq 0$,

$$\hat{f}(z) = \int_0^\infty f(t) \mathrm{e}^{-tz} \mathrm{d}t = \frac{1}{z} \int_0^\infty f'(t) \mathrm{e}^{-tz} \mathrm{d}t = \frac{1}{z} \hat{f}'(z),$$

using $f(t) \to 0$ as $t \to \infty$ and $|\mathrm{e}^{-tz}| = \mathrm{e}^{-tx} \to 0$ as $t \to \infty$. On applying the above argument to f' and in case (c) also to f'', one obtains respectively, $\hat{f}(z) = \frac{1}{z^2} \hat{f}''(z)$ and in case (c) $\hat{f}(z) = \frac{1}{z^3} \hat{f}'''(z)$.

(a) $|\hat{f}(x+iy)| = \frac{1}{|z|^2} |\hat{f}''(z)| \leqslant \frac{1}{|z|^2} \|\hat{f}''\|_1$.

Now,

$$\|\hat{f}\|_p^p = \int_{x\geqslant 1}\int_{y=-\infty}^{\infty} |\hat{f}(x+iy)|^p \mathrm{d}x\mathrm{d}y + \int_{0\leqslant x\leqslant 1}\int_{|y|\geqslant 1} |\hat{f}(x+iy)|^p \mathrm{d}x\mathrm{d}y$$
$$+ \int_{0\leqslant x\leqslant 1}\int_{|y|\leqslant 1} |\hat{f}(x+iy)|^p \mathrm{d}x\mathrm{d}y = T_1 + T_2 + T_3,$$

where

$$T_1 \leqslant \|f''\|_1^p \int_{x\geqslant 1}\int_{y=-\infty}^{\infty} \frac{\mathrm{d}x\mathrm{d}y}{(x^2+y^2)^p} \leqslant \|f''\|_1^p \int_{x\geqslant 1} \frac{1}{x^{2p-1}} |\tan^{-1}(y/x)|_{-\infty}^{\infty} \mathrm{d}x$$
$$= \pi \|f''\|_1^p/2(p-1)$$

$$T_2 \leqslant \int_{0\leqslant x\leqslant 1}\int_{|y|\geqslant 1} \frac{\|f''\|_1^p}{|z|^{2p}} \mathrm{d}x\mathrm{d}y \leqslant 2\int_{0\leqslant x\leqslant 1} \|f''\|_1^p \mathrm{d}x = 2\|f''\|_1^p$$

and

$$T_3 \leqslant \int_{0\leqslant x\leqslant 1}\int_{|y|\leqslant 1} |\hat{f}(x+iy)|^p \mathrm{d}x\mathrm{d}y \leqslant 2\|\hat{f}\|_\infty^p \leqslant 2\|f\|_1^p.$$

Thus $f\in A_p(I)$ for $p>1$.

(b) $$|\hat{f}(x+iy)| = \frac{1}{|z|^2}|\hat{f}''(z)| = \frac{1}{|z|^2}\left|\int_0^\infty f''(t)\mathrm{e}^{-tz}\mathrm{d}t\right|$$
$$\leqslant \frac{1}{|z|^2}\int_0^\infty |f''(t)|\mathrm{e}^{-tx}\mathrm{d}t \leqslant \mathrm{e}^{-cx}\frac{1}{|z|^2}\int_0^\infty |f''(t)|\mathrm{d}t$$
$$= \frac{\mathrm{e}^{-cx}}{|z|^2}\|f''\|_1.$$

The proof from now onwards is the same as in case (a).

(c) $$|\hat{f}(x+iy)| = \frac{1}{|z|^3}|\hat{f}'''(z)| = \frac{1}{|z|^3}\|f'''\|_1.$$

If we write $\|\hat{f}\|_1 = T_1 + T_2 + T_3$ as above, then

$$T_1 \leqslant \int_{x\geqslant 1}\int_{y=-\infty}^{\infty} \frac{\|f'''\|_1}{x^3(1+(y/x)^2)}\mathrm{d}x\mathrm{d}y = \pi\|f''\|_1,$$

$$T_2 \leqslant 2\|f'''\|_1 \int_{x=0}^1 \int_{y=1}^\infty \frac{\mathrm{d}x\mathrm{d}y}{(x^2+y^2)^{3/2}} \leqslant 2\|f'''\|_1 \int_{x=0}^1\int_{y=1}^\infty \frac{\mathrm{d}x\mathrm{d}y}{y^3} = \|f'''\|_1,$$

and

$T_3 \leqslant 2\|f\|_1$, as above. This completes the proof.

Theorem 3. *$A_p(I)$ is $\|\ \|_1$-dense ideal in $L_1(I)$.*

Proof. $A_p(I)$ is an ideal in $L_1(I)$ was observed in the beginning paragraph of § 2. That it is dense in $L_1(I)$ follows from Proposition 1 on noting that $C_c^\infty(I)$ is dense in $L_1(I)$.

DEFINITION 4 [BURNHAM]

Let $(A, \|\ \|_A)$ be a Banach algebra. The subalgebra B of A is an A-Segal algebra in case

(i) B is a dense left ideal of A,
(ii) B is a Banach space with respect to norm $\| \ \|_B$,
(iii) There exists $C>0$ such that $\|f\|_A \leqslant C\|f\|_B$ for all $f\in B$, and
(iv) There exists $K>0$ such that $\|fg\|_B \leqslant K\|f\|_A\|g\|_B$ for all $f,g\in B$.

Remark 5. (i) It is clear from the foregoing that $A_p(I)$ is an $L_1(I)$-Segal algebra. The above proofs have been included in view of their intrinsic value even though Burnham ([2], ex. 19) cites an example of a Segal algebra which includes the one studied in this note.
(ii) In view of the fact that for $1 \leqslant p < \infty$, $A_p(I)$ is an $L_1(I)$-Segal algebra, the following results follow from the general theory of A-Segal algebras [2]. Let $1 \leqslant p < \infty$. (a) The maximal ideal space $\Delta(A_p(I))$ of $A_p(I)$ is homeomorphic to $\bar{\Sigma}$, (b) $A_p(I)$ has no bounded approximate identity.
(iii) It follows from (ii) (a) that for each p, $1 \leqslant p < \infty$, $A_p(I)$ is a semisimple commutative Banach algebra.

Our next result provides a characterization of $A_2(I)$.

Theorem 6. *Let $f\in L_1(I)$. Then $f\in A_2(I)$ iff $t\to f(t)/\sqrt{t}$ is in $L_2(I)$.*

Proof.

$$\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty |\hat{f}(x+iy)|^2\,dx\,dy = \int_0^\infty\int_{-\infty}^\infty |\tilde{f}_x(y)|^2\,dx\,dy$$

$$= \int_0^\infty\int_{-\infty}^\infty |\tilde{f}_x(t)|^2\,dt\,dx = \frac{1}{2}\int_0^\infty \left|\frac{f(t)}{\sqrt{t}}\right|^2 dt,$$

using Plancherel Theorem.

Our next result shows that $C_c(I)$ is contained in $A_p(I), p>2$.

PROPOSITION 7

Suppose $2<p<\infty$ and p' is such that $\frac{1}{p}+\frac{1}{p'}=1$. Choose u and u' such that $1<u'<p/p'$ and $\frac{1}{u}+\frac{1}{u'}=1$. Then $L_1(I)\cap L_p(I)\cap L_{up'}(I)\subseteq A_p(I)$. So $L_1(I)\cap L_\infty(I)$ is contained in $A_p(I)$ for $p>2$. In particular, $C_c(I)$ is contained in $A_p(I), p>2$.

Proof. Suppose $f\in L_1(I)\cap L_{p'}(I)\cap L_{up'}(I)$. Let $x\in I$. Since $f\in L_{p'}(I)$, $f_x\in L_{p'}(R)$. So $\|\tilde{f}_x\|_p \leqslant \|f_x\|_{p'}$, using Hausdorff–Young inequality. Moreover, $\|f_x\|_{p'} \leqslant \|f\|_{p'}$. Also

$$\|f_x\|_{p'}^{p'} = \int_0^\infty |f(t)|^{p'} e^{-txp'}\,dt$$

$$\leqslant \left(\int_0^\infty |f(t)|^{up'}\,dt\right)^{1/u}\left(\int_0^\infty e^{-txu'p'}\,dt\right)^{1/u'}.$$

Consequently,

$$\|\tilde{f}_x\|_p \leqslant \|f\|_{up'}\left(\frac{1}{xu'p'}\right)^{1/u'p'}.$$

Hence

$$(2\pi)^{-p/2}\|\hat{f}\|_p^p = \int_{x=0}^{1}\|\tilde{f}_x\|_p^p dx + \int_{x=1}^{\infty}\|\tilde{f}_x\|_p^p dx$$

$$\leqslant \|f\|_{p'}^p + \frac{\|f\|_{up'}^p}{\left(\frac{p}{p'u'}-1\right)(p'u')^{p/p'u'}}.$$

This completes the proof.

Remark 8. Though $A_p(I)$ does not possess a bounded approximate identity, yet it does always have an approximate identity as the following theorem shows.

Theorem 9. *The sequences $\{u_n\}, \{u_n * u_n\}$ and $\{u_n * u_n * u_n\}$, where u_n denotes the function $n\chi_{[0,1/n)}$, act as approximate identities for $A_p(I)$ with $2 < p \leqslant \infty$, $1 < p \leqslant \infty$ and $1 \leqslant p \leqslant \infty$, respectively. In particular each $A_p(I)$ possesses an approximate identity present in $A_1(I)$.*

Proof. It is well-known that $\{u_n\}$ is an approximate identity for $L_1(I)$. We shall show that $v_n = u_n * u_n * u_n, n = 1,2,\ldots$ is an approximate identity for $A_1(I)$. It has been observed that $\{v_n\}$ is contained in $A_1(I)$ [Proposition 1]. Suppose $f \in A_1(I)$. Then

$$\|f * u_n * u_n * u_n - f\|_1 \leqslant (\|u_n * u_n\|_1 + \|u_n\|_1 + 1)\|f * u_n - f\|_1$$

$$\leqslant 3\|f * u_n - f\|_1.$$

So $\|f * v_n - f\|_1 \to 0$ as $n \to \infty$.

There exists a compact set $K \subset \bar{\Sigma}$ such that $\|\hat{f}|_K c\|_1 < \varepsilon/4$.

Now,

$$\|(\hat{f}\hat{v}_n - \hat{f})\chi_K\|_1 \leqslant (1 + \|\hat{u}_n\|_\infty + \|\hat{u}_n\|_\infty^2)\|(\hat{f}\hat{u}_n - \hat{f})\chi_K\|_1$$

$$\leqslant 3\|\hat{f}\hat{u}_n - \hat{f}\|_\infty \mu(K)$$

$$\leqslant 3\|f * u_n - f\|_1 \mu(K),$$

where $\mu(K)$ denotes the planar measure of K. Also,

$$\|(\hat{f}\hat{v}_n - \hat{f})\chi_{\bar{\Sigma}-K}\|_1 \leqslant (1 + \|\hat{u}_n\|_\infty^3)\|(\hat{f}\chi_{\bar{\Sigma}-K}\|_1$$

$$\leqslant 2\|\hat{f}\chi_{\bar{\Sigma}-K}\|_1 < \varepsilon/2.$$

Choose n so large that $\|f * v_n - f\|_1 < \varepsilon/6\mu(K)$. Consequently, for this n, $\|\hat{f} * v_n - \hat{f}\|_1 < \varepsilon$. This completes the proof in the case $p = 1$. The proof for the case $p > 1$ is similar and is, therefore, not included.

Remark 10. (a) It follows from Theorem 9 above that for $1 \leqslant p \leqslant \infty$, $A_p(I) * A_1(I)$ is dense in $A_p(I)$. Since $A_1(I)$ is an ideal in $L_1(I)$ this gives that $A_1(I)$ is dense in $A_p(I)$ which, in turn, gives that for $1 \leqslant r < p \leqslant \infty$, $A_r(I)$ is dense in $A_p(I)$.

(b) It follows from Theorem 9 that $L_1(I) * A_p(I)$ is dense in $A_p(I)$. This observation together with ([5], 32.22) implies that $L_1(I) * A_p(I) = A_p(I)$. Now ([5], 32.33(a)) implies that $\{e_\alpha\}$ is an approximate identity in $L_1(I)$ which is present in $A_p(I)$, then it is an approximate identity for $A_p(I)$ as well.

Finally, we state without proof the following result regarding ideals in $A_p(I)$. The proof follows from the fact that $A_p(I)$ is an $L_1(I)$-Segal algebra and Burnham ([12], Th. 13) result on ideal theory of Segal algebras.

Theorem 11. *For each p, $1 \leqslant p < \infty$, the following statements hold:*

(i) *If J_1 is a closed ideal in $L_1(I)$, then $J = J_1 \cap A_p(I)$ is a closed ideal in $A_p(I)$.*

(ii) *If J is a closed ideal in $A_p(I)$ and J_1 is the closure of J in $L_1(I)$, then J_1 is a closed ideal in $L_1(I)$ and $J = J_1 \cap A_p(I)$.*

3. Multipliers of $A_p(I)$

In this section we attempt to identify the multipliers of $A_p(I)$. In view of ([7], 1.2.2) and results proved in § 2 above, every multiplier T of $A_p(I)$ is bounded and corresponds to a bounded continuous function ϕ on $\bar{\Sigma}$. It follows from the analyticity of $\hat{f}, f \in A_p(I)$, that ϕ is analytic on Σ. Since for $f \in A_p(I)$ and ϕ bounded and continuous on $\bar{\Sigma}$, $\psi = \phi\hat{f}$ is in $L_p(\Sigma)$ with $\|\psi\|_p \leqslant \|\phi\|_\infty \|\hat{f}\|_p$, the problem reduces to that of identifying those ϕ's which are bounded continuous on $\bar{\Sigma}$ and analytic on Σ such that for each $f \in A_p(I)$, $\phi\hat{f}$ is $\hat{h}$ for some $h \in L_1(I)$. Remark 10(b) further reduces it to requiring $\phi\hat{f}$ to be $\hat{\mu}$ for some $\mu \in M([0, \infty))$. Thus a multiplier on $A_p(I)$ into $M([0, \infty))$ is in fact a multiplier of $A_p(I)$ and keeping in view 10(a), we have that for $1 \leqslant p < r \leqslant \infty$, a non-zero multiplier of $A_r(I)$ induces a non-zero multiplier of $A_p(I)$ via restriction. The following analogue of Paley–Wiener Theorem ([11], Th. 19.2) helps us in expressing, for $f \in A_p(I)$ and certain ϕ's, $\phi\hat{f}$ and $\hat{h}$ for some $h \in L_1(I)$.

Theorem 12 (Paley–Wiener). *Let $1 \leqslant r \leqslant 2$ and r' be the number given by $\frac{1}{r} + \frac{1}{r'} = 1$. Let $\psi \in L_r(\Sigma)$ be analytic on Σ. Then there exists a $g \in L^1_{\mathrm{loc}}(I)$ such that $\psi = \hat{g}$ on Σ. This $g \in L_1(I)$ iff $a = \operatorname{ess}\lim_{x \to 0+} \inf \|{}_x\tilde{\psi}\|_1 < \infty$ and then $\sqrt{2\pi}\|g\|_1 = a$ and if $b = \operatorname{ess}\lim_{x \to \infty +} \inf \|{}_x\psi\|_r < \infty$ then $g \in L_{r'}(I)$ with $\|g\|_{r'} \leqslant \frac{1}{\sqrt{2\pi}} b$, where, for $x \in I$, ${}_x\psi(y) = \psi(x + iy)$.*

Proof. We shall modify the detailed proof given in ([11], Th. 19.2). We fix x, x' in I with $x < x'$ and write $J = [x, x']$. For $\alpha \in I$, let Γ_α be the rectangular path with vertices at $x + i\alpha$, $x - i\alpha$, $x' - i\alpha$ and $x' + i\alpha$. It follows from Cauchy theorem that $\int_{\Gamma_\alpha} \psi(z)e^{tz}\,dz = 0$ for $t \in R$. For $\alpha, \beta \in R$, let $\Psi(t, \beta) = \int_J \psi(u + i\beta)e^{t(u+i\beta)}\,du$, $\Lambda(\beta) = \int_J |\psi(u + i\beta)|^r\,du$ and $M(t) = |x' - x|^{1/r'} \max\{e^{tx}, e^{tx'}\}$ with the convention that $r^0 = 1$ for $r > 0$. Then $|\Psi(t, \beta)| \leqslant \Lambda(\beta)^{1/r} M(t)$. Now, by Tonelli's theorem

$$\int_R \Lambda(\beta)\,d\beta = \int_J \left(\int_R |\psi(u + i\beta)|^r\,d\beta \right) du = \|\psi\|_r^r < \infty.$$

So $\Lambda(\beta) + \Lambda(-\beta)$ is not bounded below away from zero as $\beta \to \infty$ and, therefore, there is a sequence $\{\alpha_j\}$ in I such that $\alpha_j \to \infty$ and $\Lambda(\alpha_j) + \Lambda(-\alpha_j) \to 0$ and $j \to \infty$. We note that $\{\alpha_j\}$ is independent of t and for $t \in R$, $\Psi(t, \alpha_j) \to 0$ and $\Psi(t, -\alpha_j) \to 0$ as $j \to \infty$. Now since $\psi \in L_r(\Sigma)$, ${}_u\psi \in L_r(R)$ for almost all $u \in I$, say, in a set S with $m(I - S) = 0$. Then for $u \in S$,

${}_u\psi\chi_{[-\alpha_j,\alpha_j]} \to {}_u\psi$ in $L_r(R)$ and therefore, by Hausdorff–Young inequality, $({}_u\psi\chi_{[-\alpha_j,\alpha_j]})^\sim \to ({}_u\psi)^\sim$ in $L_{r'}(R)$. So for $x, x' \in S$, there exists a strictly increasing sequence $\{n_j\}$ in N with $({}_x\psi\chi_{[-\alpha_{n_j},\alpha_{n_j}]})^\sim \to ({}_x\psi)^\sim$ and $({}_{x'}\psi\chi_{[-\alpha_{n_j},\alpha_{n_j}]})^\sim \to ({}_{x'}\psi)^\sim$ almost everywhere in R. But for each $n_j \in N$ and $t \in R$, $\sqrt{2\pi}({}_x\psi\chi_{[-\alpha_{n_j},\alpha_{n_j}]})^\sim(-t)e^{tx}$ and $\sqrt{2\pi}({}_{x'}\psi\chi_{[-\alpha_{n_j},\alpha_{n_j}]})^\sim(-t)e^{tx'}$ are the integrals of $\psi(z)e^{tz}$ along the vertical lines of $\Gamma_{\alpha_{n_j}}$ and therefore for each $t \in R$,

$$\lim_{j\to\infty} \sqrt{2\pi}|({}_x\psi\chi_{[-\alpha_{n_j},\alpha_{n_j}]})^\sim(-t)e^{tx} - ({}_{x'}\psi\chi_{[-\alpha_{n_j},\alpha_{n_j}]})^\sim(-t)e^{tx'}|$$

$$= \lim_{j\to\infty} |\Psi(t,\alpha_{n_j}) - \Psi(t,-\alpha_{n_j})| = 0.$$

So, $({}_x\psi)^\sim(-t)e^{tx} = ({}_{x'}\psi)^\sim(-t)e^{tx'}$ for almost all $t \in R$. We take g as the function $t \to \frac{1}{\sqrt{2\pi}}({}_x\psi)^\sim(-t)e^{tx}$ which is independent of $x \in S$.

If $r = 1$, then for each $x \in S$, $({}_x\psi)^\sim \in C_0(R)$ and $\sqrt{2\pi}\|({}_x\psi)^\sim\|_\infty \leqslant \|{}_x\psi\|_1$. Since for each $t \in R$, $e^{-tx} \to 1$ as $x \to 0+$ in S, we have $2\pi\|g\|_\infty \leqslant b = \operatorname{ess}\lim_{x\to 0+} \inf \|{}_x\psi\|_1$ and $\sqrt{2\pi}\|g\|_1 = \operatorname{ess}\lim_{x\to 0+} \inf \sqrt{2\pi}\int e^{-tx}|g(t)|\,dt = \operatorname{ess}\lim_{x\to 0+} \inf \|({}_x\psi)^\sim\|_1$.

Now, for $\alpha \in I$, $t < 0$

$$\frac{|g(t)|}{t}(1 - e^{-\alpha t}) = \int_0^\alpha e^{-ut}|g(t)|\,du = \frac{1}{\sqrt{2\pi}}\int_0^\alpha |({}_u\psi)^\sim(-t)|\,du$$

$$\leqslant \int_0^\alpha \|{}_u\psi\|_1\,du \leqslant \|\psi\|_1 < \infty.$$

So letting $\alpha \to \infty$, we obtain that $g(t) = 0$ for $t < 0$. By continuity of g, $g(0) = 0$. Thus $g \in L^1_{\mathrm{loc}}(I)$. Further $|g(t)| \leqslant t\|\psi\|_1$ for $t \in I$. Thus for $x \in S$, $({}_x\psi)^\sim \in L_1(R)$ and as ${}_x\psi$ is also in $L_1(R)$, $\psi(x+iy) = {}_x\psi(y) = ({}_x\psi)^\sim(-y) = \int_0^\infty g(t)e^{-xt}e^{-iyt}dt$. Since ψ as well as the function $z = x + iy \to \int_0^\infty g(t)e^{-tz}dt = \hat{g}(z)$, ([12], Th. 6.3) are both continuous on Σ, we conclude that $\psi = \hat{g}$ on Σ.

If $r > 1$, then, on using Hausdorff–Young inequality, we get

$$(2\pi)^{r/2}\int_0^\infty \left(\int_R e^{-r'xt}|g(t)|^{r'}\,dt\right)^{r/r'} dx = \int_0^\infty \left(\int_R |({}_x\psi)^\sim(-t)|^{r'}\,dt\right)^{r/r'} dx$$

$$\leqslant \int_0^\infty \|{}_x\psi\|_r^r\,dx = \|\psi\|_r^r < \infty.$$

So the function $x \to \int_R e^{-r'xt}|g(t)|^{r'}\,dt$ is not bounded below away from zero as $x \to \infty$. So there is a sequence $\{x_j\}$ in I with $x_j \to \infty$ and $\int_R e^{-r'tx_j}|g(t)|^{r'}\,dt \to 0$ as $j \to \infty$. Since $\int_{-\infty}^0 |g(t)|^{r'}\,dt \leqslant \int_{-\infty}^0 e^{-r'tx_j}|g(t)|^{r'}\,dt$ for each j, we conclude that $g = 0$ a.e. on $(-\infty, 0)$. Now, for $\alpha \in I$, $x \in S$,

$$\sqrt{2\pi}\int_0^\alpha |g(t)|\,dt = \int_0^\alpha \left(\int_R |({}_x\psi)^\sim(-t)|e^{tx}\,dt \leqslant \|({}_x\psi)^\sim\|_{r'}\left(\int_0^\alpha e^{rxt}dt\right)^{1/r}\right.$$

$$\leqslant \|{}_x\psi\|_{r'}\alpha^{1/r}e^{\alpha x} < \infty.$$

So $g \in L^1_{\mathrm{loc}}(I)$. Since $I - 2S = 2(I - S)$ has measure zero, we have that $m(I - S \cap 2S) = 0$

as well. For $x \in S \cap 2S$,

$$\frac{1}{\sqrt{2\pi}}({}_x\psi)^{\sim}(-t) = g_x(t) = g_{x/2}(t)e^{-(x/2)t} = \frac{1}{\sqrt{2\pi}}({}_{x/2}\psi)^{\sim}(-t)e^{-tx/2}.$$

Since $x/2 \in S$, $({}_{x/2}\psi)^{\sim} = L_r(R)$ and therefore $t \to g_x(t) = \frac{1}{\sqrt{2\pi}}({}_x\psi)^{\sim}(-t)$ is in $L_1(I)$. Since $m(I - S \cap 2S) = 0$, $S \cap 2S$ is dense in I. So $g_x \in L_1(I)$ for all $x \in I$. Consequently $\hat{g}$ is defined on Σ and is given by

$$\hat{g}(x+iy) = \int_0^\infty g(t)e^{-xt}e^{-iyt}dt.$$

which for $x \in S$ is

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty ({}_x\psi)^{\sim}(-t)e^{-iyt}dt = ({}_x\psi)^{\tilde{}}(-y) = {}_x\psi(y)$$

$$= \psi(x+iy),$$

since ${}_x\psi \in L_r(R)$ and $({}_x\psi)^{\sim} \in L_1(R)$ ([11], Th. 9.11). Because ψ and $\hat{g}$ are both continuous, we conclude that $\psi = \hat{g}$ on Σ.

Further $g \in L_1(R)$ iff $a = \operatorname{ess}\lim_{x \to 0+} \inf \|({}_x\psi)^{\sim}\|_1 < \infty$ and then $\sqrt{2\pi}\|g\|_1 = a$ simply because for any sequence $\{x_j\}$ in S with $x_j \to 0$,

$$\sqrt{2\pi}\int_0^\infty |g(t)|dt = \sqrt{2\pi}\lim_{j\to\infty}\int_0^\infty e^{-x_j t}|g(t)|dt = \lim_{j\to\infty}\|({}_{x_j}\psi)^{\sim}\|_1.$$

We note that in this case $\hat{g}$ is defined on $\bar{\Sigma}$ and extends ψ to a bounded continuous function on $\bar{\Sigma}$. Finally, for any sequence $\{x_j\}$ in S, $x_j \to 0$, we have, by Monotone Convergence Theorem,

$$\sqrt{2\pi}\|g\|_{r'} = \sqrt{2\pi}\lim_{j\to\infty}\|g_{x_j}\|_{r'} = \lim_{j\to\infty}\inf\|({}_{x_j}\psi)^{\sim}\|_{r'} \leqslant \lim_{j\to\infty}\inf\|{}_{x_j}\psi\|_{r'},$$

and, therefore, $\sqrt{2\pi}\|g\|_{r'} \leqslant b = \operatorname{ess}\lim_{j\to\infty}\inf\|{}_x\psi\|_r$.

COROLLARY 13

Let $1 \leqslant p \leqslant \infty$, ϕ a bounded continuous function on $\bar{\Sigma}$, which is analytic on Σ and $f \in A_p(I)$. Then $\phi\hat{f} = \hat{h}$ for some $h \in L^1_{\text{loc}}(I)$ if $1 \leqslant p \leqslant 2$ or $p > 2$ and $\phi \in L_{2p/p-2}(\Sigma)$. This $h \in L_1(I)$ iff

$$\operatorname{ess}\lim_{x\to 0+}\inf\int_0^\infty \left|\int_R \phi(x+iy)\hat{f}(x+iy)e^{iyt}dy\right|dt < \infty.$$

Proof. We take $r = p$ if $1 \leqslant p \leqslant 2$ and $r = 2$ if $p > 2$ so that $\phi\hat{f} \in L_r(\Sigma)$ under the stated conditions and then apply Theorem 12 above.

Theorem 14. *Let $1 \leqslant p < \infty$ and ϕ be a bounded continuous function on $\bar{\Sigma}$ which is analytic on Σ.*

(i) *ϕ induces a multiplier of $A_p(I)$ if (a) $1 \leqslant p \leqslant 2$ or $p > 2$ together with $\phi \in L_{2p/p-2}(\Sigma)$ and (b)* $\operatorname{ess} \lim_{x \to 0+} \inf \int_0^\infty |\int_R \phi(x+iy)\hat{f}(x+iy)e^{iyt}dy|dt < \infty$ *for all $f \in A_p(I)$.*

(ii) *ϕ induces a multiplier of $A_p(I)$ only if*

$$\operatorname{ess} \lim_{x \to 0+} \inf \int_0^\infty \left| \int_R \phi(x+iy) \left(\frac{n(1-e^{-(x+iy)/n})}{x+iy} \right)^s e^{iyt} dy \right| dt < \infty$$

for each $n \in N$, where $s = 2$ if $p = 2$, 1 if $p > 2$ and 3 if $p < 2$.

Proof. We apply the above corollary and use the fact that $u_n^{*s} \in A_p(I)$ for each n, where $u_n = n\chi_{(0,1/n]}$ and $*s$ denote the sth convolution power.

Remark 15. Even though $A_p(I)$ is not regular, it contains functions f with $\hat{f}$ vanishing nowhere on Σ, for instance, u_n^{*s} in the proof above. Such $\hat{f}$ are bounded below away from zero on compact subsets of Σ and thus the strict topology τ on $M(A_p(I))$ (i.e., strong topology on a subalgebra of the algebra $B(A_p(I))$ of bounded linear operators on $A_p(I)$ to itself) is stronger than the topology of uniform convergence on compact subsets of Σ. By ([7], 1.1.6) and Theorem 9 above $A_p(I)$ is dense in $(M(A_p(I), \tau_s))$, where $g \in A_p(I)$ is identical with the multiplication operator M_g given by $M_g(f) = g * f$ and, *a fortiori*, $(\widehat{A_p(I)})$ is dense in $(M(\widehat{A_p(I)}))$, topology of compact convergence on Σ).

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Reflection of P-waves in a prestressed dissipative layered crust

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Abstract. The paper deals with overall reflection and transmission response of seismic P-waves in a multilayered medium where the whole medium is assumed to be dissipative and under uniform compressive initial stress. The layers are assumed to be homogeneous, each having different material properties. Using Biot's theory of incremental deformation, analytical solutions are obtained by matrix method. Numerical results for a stack of four layers – modelling earth's upper layers, show a decreasing trend in both the Reflection Coefficients R_D^{PP} and R_D^{PS} of the reflected P and S-waves.

Keywords. Reflection; P-wave; S-wave; dissipative; homogeneous layers; Biot's theory; matrix method; reflection coefficients.

1. Introduction

The study of reflection and transmission of seismic body waves through multilayered media is an important part of seismic sounding techniques. It is recognized that these studies provide a very convenient method of investigating the earth's interior. Although other approximations are possible, the simplest representation of the system of rocks beneath the earth's surface might be supposed to consist of a series of plane, parallel layers, each having its own characteristic – but constant within the layer – parameters of velocity and density [12]. Observation of propagation of stress waves in solids (or fluids) show that dissipation of strain energy occurs even when the waves have small amplitude. This dissipation results from imperfection in elasticity, loss by radiation, by geometrical spreading and scattering [5, 7–11, 13, 14]. A convenient measure of attenuation in waves is the dimensionless loss factor (or specific dissipation constant) Q^{-1}. It is related to the rate at which the mechanical energy of vibration is converted irreversibly into heat energy and does not depend on the detailed mechanism by which energy is dissipated. For P-waves Q_α^{-1} is given by [12].

$$Q_\alpha^{-1} = \frac{2v}{V}$$

where V and v are the real and imaginary parts of the complex P-wave velocity. It is also known that, surprisingly, Q^{-1} is independent of frequency, pressure and temperature [5].

In the focal region, prior to an earthquake, considerable tectonic thrust builds up as a uniaxial stress system. It is of some interest to investigate reflection characteristics, through a theoretical model of a stack of layers under uniaxial compressive prestress. Biot [2] has provided a detailed theory of incremental deformation of a medium in a state of prestress brought about by even arbitrary finite deformation. Later, Dahlen [3], in a limited context of *initial elastic deformation* arrives at identical set of equations, excepting the constitutive equations for the incremental stresses. If restricted to

two-dimensions, Dahlen's equations fail to reduce to the equations for incompressible medium derived elaborately by Biot. Secondly, the elastic moduli in the transverse direction also change due to the uniaxial prestress. Consequently, we adhere completely to Biot's theory.

For treatment of the equations for a stack of layers, we adopt a simple matrix method based on Kennett [6]. In this paper we restrict to two-dimensional propagation.

2. Formulation of the problem

Consider an initially stressed, dissipative medium consisting of 'n' parallel homogeneous layers overlying a half-space. The interfaces are ordered as $Z_1, Z_2, \ldots, Z_n$ where the origin $Z = 0 < Z_1$ is on a hypothetical free surface from which P-wave originate and travel downwards, ultimately as plane waves. The reflected waves are received at the same surface. To keep the analysis simple in the first instance, as is often done, we disregard stress-free condition on $Z = 0$, that is to say, regard the top layer $Z < Z_1$ as semi-infinite. The topmost layer is layer number 1 and the bottom layer $n+1$ and thicknesses of the intermediate layers are designated as $H_2, H_3, \ldots, H_n$ (figure 1). The physical quantities associated with layer number 'm' will be denoted by symbols with suffix m.

In general, if we have an isotropic elastic solid under uniform initial horizontal compression $-S_{11}$ (tensile $S_{11} < 0$) parallel to x-axis, which undergoes additional infinitesimal deformation, then according to Biot [2], the incremental stresses consist of two parts: one part due to additional deformation and the other due to infinitesimal rotation ω_2 acting to rotate the initial stress system:

$$\bar{\sigma}_{11} = S_{11} + s_{11}, \quad \bar{\sigma}_{33} = s_{33}, \quad \bar{\sigma}_{13} = s_{13} + S_{11}\omega_2 \tag{1}$$

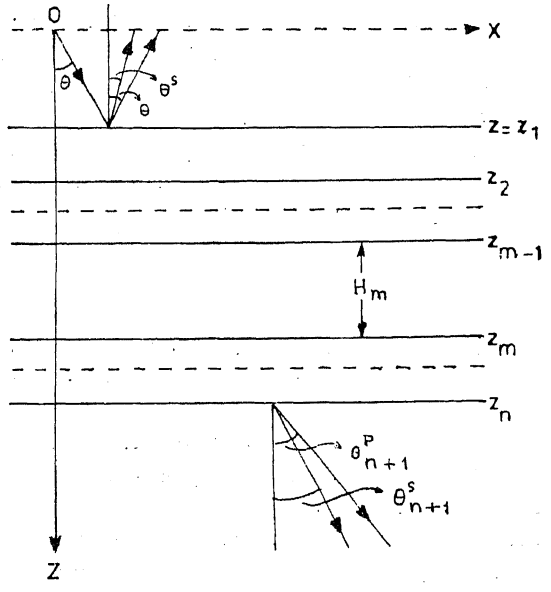


Figure 1. Geometry and schematic of the problem.

where s_{ij} are incremental stresses referred to axes which rotate with the medium (Biot [2], eq. (4.13)) and

$$\omega_2 = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right). \tag{2}$$

For infinitesimal incremental strain e_{ij}, the incremental stress s_{ij} will be linear functions of e_{ij}. Assuming these to be orthotropic in nature we can write

$$\begin{aligned} s_{11} &= B_{11}e_{xx} + B_{13}e_{zz}, \quad e_{xx} = \frac{\partial u}{\partial x} \\ s_{33} &= B_{31}e_{xx} + B_{33}e_{zz}, \quad e_{zz} = \frac{\partial w}{\partial z} \\ s_{31} &= 2Qe_{zx}, \quad e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right). \end{aligned} \tag{3}$$

Also, after careful consideration of existence of strain-energy,

$$B_{31} - B_{13} = S_{11} \tag{4}$$

(Biot [2], eq. (6.2)). The elastic constants $B_{11}, \ldots, Q$ in general may depend on the initial stress S_{11}. Biot ([2], eq. (8.31e)) after analysis of an incompressible medium, selects for an original isotropic compressible medium (Lamé constants λ, μ), relations equivalent to

$$\begin{aligned} &B_{11} = \lambda + 2\mu - S_{11}, \quad B_{13} = \lambda - S_{11} \\ &B_{31} = \lambda, \quad B_{33} = \lambda + 2\mu, \quad Q = \mu. \end{aligned} \tag{5}$$

A salient feature of these relations is that the moduli in the x-direction (the direction of initial stress) increases due to the initial compressive stress while those in the transverse z-direction remain unchanged. To account for dissipation in the medium λ and μ are to be regarded complex: $\lambda = \lambda_r + i\lambda_i$, $\mu = \mu_r + i\mu_i$.

The two-dimensional dynamical equations of motion as obtained by Biot [2] are

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{13}}{\partial z} - S_{11}\frac{\partial \omega_2}{\partial z} &= \rho\frac{\partial^2 u}{\partial t^2} \\ \frac{\partial s_{31}}{\partial x} + \frac{\partial s_{33}}{\partial z} - S_{11}\frac{\partial \omega_2}{\partial x} &= \rho\frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{6}$$

For time-harmonic plane wave propagation of frequency $f = \omega/2\pi$, we may assume a factor $\exp[i(\omega t - kx)]$. Insertion of (3) with (5) in (6) results in two O.D.E's for u and w, the displacement components. However, for developing a matrix method we introduce stresses

$$\begin{aligned} &\tau_{11} = s_{11}, \quad \tau_{33} = s_{33} \\ &\tau_{13} = s_{13} - S_{11}\omega_2 \end{aligned} \tag{7}$$

and the quantities [6]

$$W = iw, \quad U = u, \quad T = i\tau_{33}, \quad S = \tau_{13}. \tag{8}$$

Constructing the stress-displacement vector

$$\mathbf{b} = [W, U, T, S]^T \tag{9}$$

eq. (6), with the aid of (7) and (3) can be written as a first order system. For subseq computational purpose we nondimensionalize all quantities: the displacements b (thickness traversed by the waves in the top layer) and stresses by μ_{1r}, the real pa shear modulus of the top layer. Denoting the respective nondimensional quantitie superscript *, the first order system can be written as

$$\frac{d\mathbf{b}^*}{dz^*} = A^*\mathbf{b}^*$$

where

$$A^* = \begin{bmatrix} 0 & -\dfrac{B^*_{31}}{B^*_{33}}kz_1 & \dfrac{1}{B^*_{33}} & 0 \\ \dfrac{Q^* + 0{\cdot}5S^*_{11}}{Q^* - 0{\cdot}5S^*_{11}}kz_1 & 0 & 0 & \dfrac{1}{Q^* - 0{\cdot}5S^*_{11}} \\ -\dfrac{(kz_1)^2}{(p\beta_1)^2}\dfrac{\rho}{\rho_1} & 0 & 0 & -kz_1 \\ 0 & \left[B^*_{11} - \dfrac{B^*_{13}B^*_{31}}{B^*_{33}} - \dfrac{\rho/\rho_1}{(p\beta_1)^2}\right]k^2z_1^2 & \dfrac{B^*_{13}}{B^*_{33}}kz_1 & 0 \end{bmatrix}$$

is the coefficient matrix. $p = k/\omega$ is the wave slowness (reciprocal of phase velocit propagation in the x-direction) and $\beta_1 = (\mu_{1r}/\rho_1)^{1/2}$ is the shear wave velocity in topmost layer. For reflection and transmission of body waves, p remains constant i the layers. Finally, $S^*_{11} = S_{11}/\mu_{1r}$.

The incremental boundary forces have also been carefully examined by Biot ([2], 17.56)). In our case, where the boundaries are $z = \text{const.}$, the components turn out t τ_{13} and τ_{33}, so that at an interface $z = z_m$ $\mathbf{b}^*$ is a continuous vector when per bonding is assumed.

3. Propagation in the stack

In an *intermediate* mth layer, the solution of (10) is

$$\mathbf{b}^* = e^{A^*_m(z^* - z^*_{m-1})}\mathbf{b}^*_{m-1}$$

where $\mathbf{b}^*_{m-1}$ is the stress-displacement vector at the interface $z^* = z^*_{m-1}$. Hence at $z^* =$

$$\mathbf{b}^*_m = e^{A^*_m H^*_m}\mathbf{b}^*_{m-1}.$$

where $H^*_m = z^*_m - z^*_{m-1}$ is the nondimensional thickness of the mth layer. Her recursively

$$\mathbf{b}^*_n = e^{A^*_n H^*_n}e^{A^*_{n-1}H^*_{n-1}}, \ldots, e^{A^*_2 H^*_2}\mathbf{b}^*_1 \equiv E\mathbf{b}^*_1.$$

All the exponentials involved above are 4×4 matrix exponentials.

For $\mathbf{b}^*_1$ we note that it consists of the down going incident P type wave and reflec up going P and S type waves (figure 1). We construct the contributions from eac these separately and superpose. Suppressing the time harmonic term, we can write the down going incident wave

$$u^*_1 = A_1 e^{-iqz}e^{-ikx}, \quad w^*_1 = e^{-iqz}e^{-ikx}$$

where the predominant z-component of the amplitude has been taken to be unity. Inserting in the equations of motion (6) with (3) and (5) and assuming

$$k = K\sin\theta, \quad q = K\cos\theta \tag{16}$$

so that θ is the angle of incidence, we get

$$A_1 = -\frac{(\lambda_1^* + 2\mu_1^*)\cos^2\theta + \{\mu_1^* + 0{\cdot}5S_{11}^* - 1/(p\beta_1)^2\}\sin^2\theta}{(\lambda_1^* + \mu_1^* - 0{\cdot}5S_{11}^*)\sin\theta\cos\theta} \tag{17}$$

and the velocity of propagation ω/k is given by a quadratic equation whose roots are

$$\rho_1\frac{\omega^2}{K^2} = \frac{1}{2}(R_1 \pm \sqrt{S_1}) \tag{18}$$

where

$$R_1 = \lambda_1 + 3\mu_1 - 0{\cdot}5S_{11} + \frac{S_{11}}{\rho}p^2(\lambda_1 + \mu_1 - 0{\cdot}5S_{11})$$

$$S_1 = R_1^2 - 4(\lambda_1 + 2\mu_1)(\mu_1 - 0{\cdot}5S_{11}). \tag{19}$$

If S_{11} is neglected, the positive sign in (18) yields P-waves and the negative sign, S-waves. In the presence of S_{11}, the velocities are p, that is, direction dependent and the waves are not pure, in the sense that P-waves are accompanied by some transverse component and S-waves by some longitudinal component [4]. Stresses corresponding to (15) can be readily calculated from (3) and (5). We thus obtain

$$\mathbf{b}_{1D}^{*P} = e^{-iKz_1\cos\theta}\begin{bmatrix} i \\ A_1 \\ Kz_1\{A_1\lambda_1^*\sin\theta + (\lambda_1^* + 2\mu_1^*)\cos\theta\} \\ -iKz_1\{A_1(\mu_1^* - 0{\cdot}5S_{11}^*)\cos\theta + (\mu_1^* + 0{\cdot}5S_{11}^*)\sin\theta\} \end{bmatrix} \tag{20}$$

p – the constant for all the layers – in (17), can be computed from the equation

$$\sin\theta = (p\beta_1)\left(\frac{R_1^* + \sqrt{S_1^*}}{2}\right)^{1/2} \tag{21}$$

which is arrived at from (16) and (18). Here θ is given so $(p\beta_1)$ is to be obtained by solving the above nonlinear equation.

For up going reflected P type wave, we have to use the representations

$$u_1^* = A_2 e^{iq(z-z_1)}e^{-ikx}, \quad w_1^* = B_2 e^{iq(z-z_1)}e^{-ikx}. \tag{22}$$

Analysis similar to the above leads to

$$\mathbf{b}_{1U}^{*P} = B_2\begin{bmatrix} i \\ -A_2/B_2 \\ -Kz_1\left\{\frac{A_2}{B_2}\lambda_1^*\sin\theta + (\lambda_1^* + 2\mu_1^*)\cos\theta\right\} \\ -iKz_1\left\{\frac{A_2}{B_2}(\mu_1^* - 0{\cdot}5S_{11}^*)\cos\theta + (\mu_1^* + 0{\cdot}5S_{11}^*)\sin\theta\right\} \end{bmatrix} \tag{23}$$

where $A_2/B_2 = -A_1$ is obtained from (17). For up going reflected S type wave we again use representation of the type (22) with amplitudes A_3, B_3 instead of A_2, B_2. We thus obtain $\mathbf{b}_{1U}^{*S}$ similar to (23) with A_2/B_2 replaced by $A_3/B_3 = -A_1$ and θ replaced by θ^s given by

$$\sin\theta^s = \mathrm{Re}\left\{(p\beta_1)\left(\frac{R_1^* - \sqrt{S_1^*}}{2}\right)^{1/2}\right\} \tag{24}$$

appropriate for S type waves. Here Re means real part of. The total stress-displacement vector in the top layer is thus

$$\mathbf{b}_1^* = \mathbf{b}_{1D}^{*P} + \mathbf{b}_{1U}^{*P} + \mathbf{b}_{1U}^{*S}. \tag{25}$$

Finally, for the bottom most $(n+1)$th layer, only down going P and S type waves are sustained. For the former we take

$$u_{n+1}^* = A_4 e^{-iq(z-z_n)} e^{-ikx}, \quad w_{n+1}^* = B_4 e^{-iq(z-z_n)} e^{-ikx}. \tag{26}$$

As in the case of $\mathbf{b}_{1D}^{*P}$ we obtain

$$\mathbf{b}_{n+1,D}^{*P} = B_4 \begin{bmatrix} i \\ A_4/B_4 \\ Kz_1\left\{\dfrac{A_4}{B_4}\lambda_{n+1}^* \sin\theta_{n+1}^P + (\lambda_{n+1}^* + 2\mu_{n+1}^*)\cos\theta_{n+1}^P\right\} \\ -iKz_1\left\{\dfrac{A_4}{B_4}(\mu_{n+1}^* - 0{\cdot}5S_{11}^*)\cos\theta_{n+1}^P + (\mu_{n+1}^* + 0{\cdot}5S_{11}^*)\sin\theta_{n+1}^P\right\} \end{bmatrix} \tag{27}$$

where

$$\sin\theta_{n+1}^P = \mathrm{Re}\left[(p\beta_1)\left(\frac{R_{n+1}^* + \sqrt{S_{n+1}^*}}{2}\right)\right] \tag{28}$$

R_{n+1}^* and S_{n+1}^* are quantities identical to R_1^* and S_1^* (cf. eq. (19)), save that λ_1 and μ_1 are to be replaced λ_{n+1} and μ_{n+1}. Similarly A_4/B_4 is given by an expression like that of A_1 (eq. 17)) save for λ_1, μ_1, θ we have to write $\lambda_{n+1}, \mu_{n+1}, \theta_{n+1}^P$. For the down going S type waves we get in a similar manner $\mathbf{b}_{n+1,D}^{*S}$ with a form similar to (27) except that A_4, B_4 are to be replaced by similar amplitudes A_5, B_5 and θ_{n+1}^P replaced by θ_{n+1}^S given by

$$\sin\theta_{n+1}^S = \mathrm{Re}\left[(p\beta_1)\left(\frac{R_{n+1}^* - \sqrt{S_{n+1}^*}}{2}\right)\right] \tag{29}$$

and A_5/B_5 given by right hand side of (17) with λ_1, μ_1, θ replaced by $\lambda_{n+1}, \mu_{n+1}, \theta_{n+1}^S$. Thus,

$$\mathbf{b}_{n+1}^* = \mathbf{b}_{n+1,D}^{*P} + \mathbf{b}_{n+1,D}^{*S}. \tag{30}$$

The expressions for $\mathbf{b}_1^*$ and $\mathbf{b}_{n+1}^*$ from (25) and (30) can now be inserted in (14). If we denote the successive vectors [ ] in the expressions for $\mathbf{b}_{1D}^{*P}, \mathbf{b}_{1U}^{*P}, \mathbf{b}_{1U}^{*S}, \mathbf{b}_{n+1,D}^{*P}, \mathbf{b}_{n+1,D}^{*S}$ by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and $\mathbf{v}_5$, we get the system of equations

$$[-\mathbf{v}_2, -\mathbf{v}_3, E^{-1}\mathbf{v}_4, E^{-1}\mathbf{v}_5][B_2, B_3, B_4, B_5]^T = e^{-iKz_1\cos\theta}\mathbf{v}_1. \tag{31}$$

Solving these equations we get "reflection coefficients", $R_D^{PP} = B_2$, $R_D^{PS} = B_3$ and "transmission coefficients", $T_D^{PP} = B_4$, $T_D^{PS} = B_5$.

4. Numerical calculations for model crust

In general the earth's continental crust consists of three layers: granitic, basaltic and a thin sedimentary layer at the top. For computations of reflection (and transmission) coefficients we consider the earth's crust beneath the Indo-Gangetic plain, which lies between the Himalayas and the Peninsula. Surface wave dispersion across this region has been investigated by several investigators [1]. Inversion of these data gives the crustal and upper mantle structure of the region. Such a model of crust is given by Bhattacharya [1] and is given below:

Region	Thickness of layer (km)	P-wave velocity (km/sec)	S-wave velocity (km/sec)	Density (gm/cm^3)
1. Sedimentary	3·5	3·40	2·00	2·00
2. Granitic	16·5	6·15	3·55	2·60
3. Basaltic	23·0	6·58	3·80	3·00
4. Upper Mantle	∞	8·19	4·603	3·30

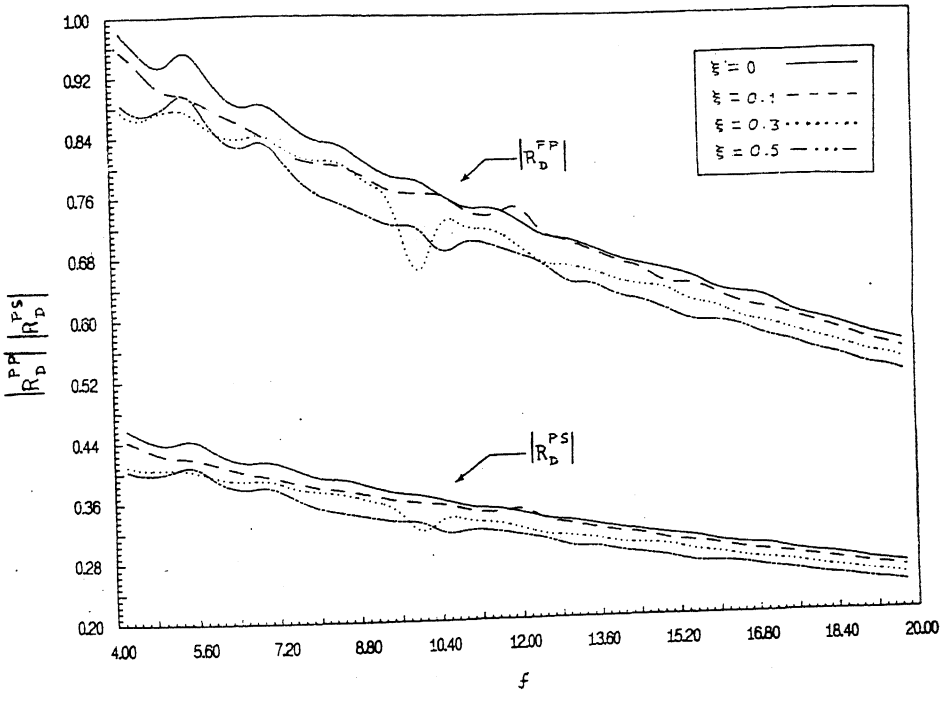
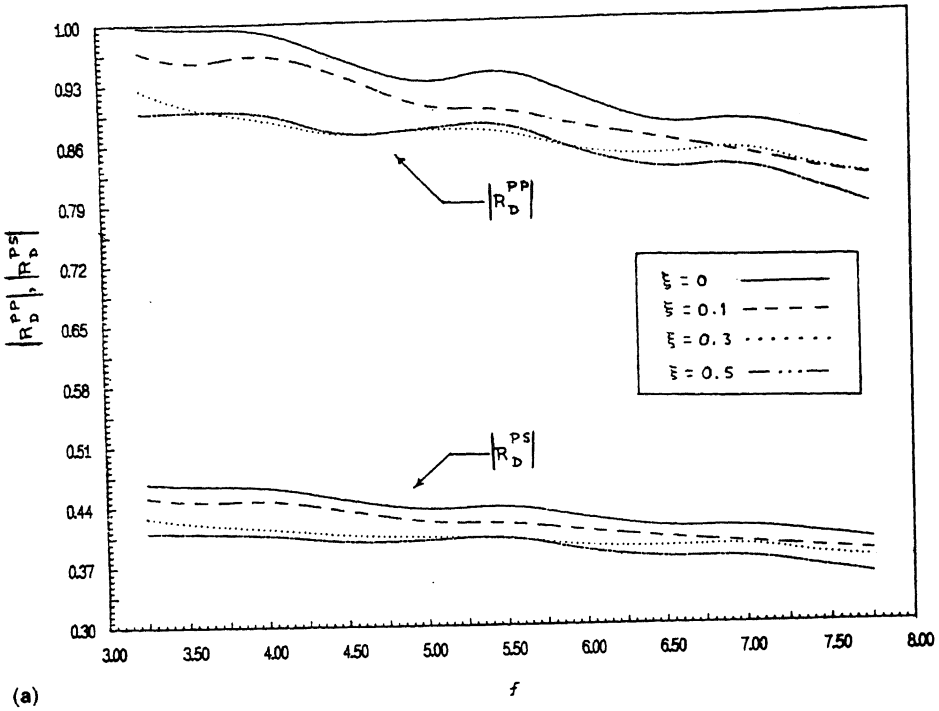
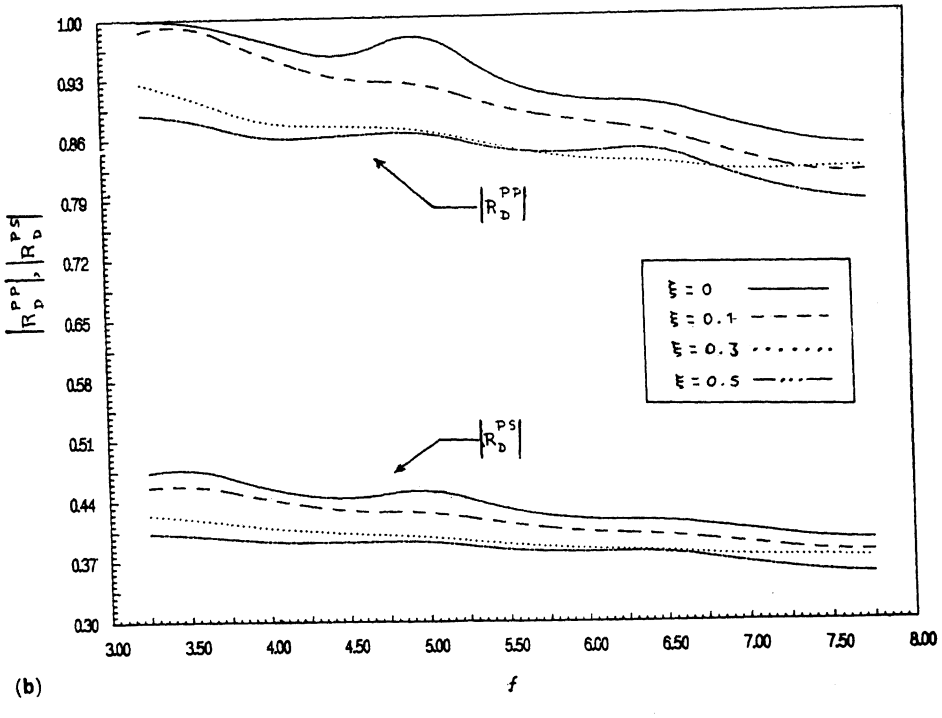


Figure 2. Amplitudes of reflection coefficients $|R_D^{PP}|$ and $|R_D^{PS}|$ for near vertical propagation: $\theta = 1°$.


1.00
0.93
0.86
0.79
0.72
0.65
0.58
0.51
0.44
0.37
0.30
$|R_D^{PP}|, |R_D^{PS}|$
$|R_D^{PP}|$
$|R_D^{PS}|$
ξ = 0
ξ = 0.1
ξ = 0.3
ξ = 0.5
3.00
3.50
4.00
4.50
5.00
5.50
6.00
6.50
7.00
7.50
8.00
f

(a)


1.00
0.93
0.86
0.79
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0.65
0.58
0.51
0.44
0.37
0.30
$|R_D^{PP}|, |R_D^{PS}|$
$|R_D^{PP}|$
$|R_D^{PS}|$
ξ = 0
ξ = 0.1
ξ = 0.3
ξ = 0.5
3.00
3.50
4.00
4.50
5.00
5.50
6.00
6.50
7.00
7.50
8.00
f

(b)

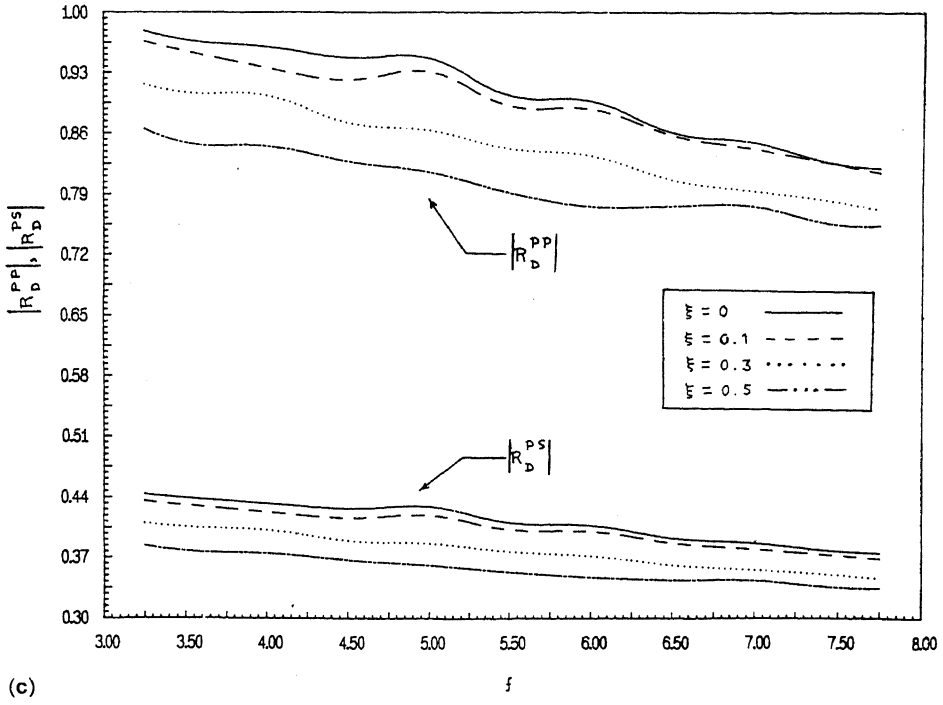

Figure 3 (*Continued*). Amplitudes of reflection coefficients $|R_D^{PP}|$ and $|R_D^{PS}|$ for wide angle propagation: (a) $\theta = 2°$ (b) $\theta = 5°$ (c) $\theta = 10°$.

The above yield the real part of Lamé constants of each layer. For the imaginary parts, the loss factors Q_α^{-1} of P-waves as given in Waters [12]

$$Q_\alpha(\text{granite}) = 311, \quad Q_\alpha(\text{basalt}) = 561$$

Q_α for sedimentary rocks is highly disperse, so, as an example we take old red sandstone for which $Q_\alpha = 93$ – a figure nearing the mean of dispersal of the values. Since the role of dissipation is small, the computed values are not expected to change very much on account of actual deviation. For the upper mantle we take $Q_\alpha = 849$ from data discussed in Ewing *et al* ([5], p. 278). Further data on imaginary part of shear modulus are provided by loss factor Q_β^{-1} of S-waves:

$$Q_\beta = \frac{4}{3}\left(\frac{\beta^2}{\alpha^2}\right) Q_\alpha$$

which is obtained from the often used assumption of zero dilatational viscosity [12, 5].

For initial stress-free basalt rock, strength $\leqslant 11{,}000$ atmospheres and if we consider hydrostatic pressure at a depth of 40 km to be present, the approximate range of the compressive initial stress $\xi = -S_{11}^*$ could be (0, 0·3). We therefore consider the parametric values $\xi = 0, 0{\cdot}1, 0{\cdot}3$ and $0{\cdot}5$, over a slightly enhanced range.

For selecting suitable frequency range, we consider the cases of seismic prospecting method of weight-dropping devices in which near vertical propagation takes place and

explosion seismology technique where it is wide angle propagation. In the former case, f is taken within the range of 4–20 Hz [12] with $\theta = 1°$. In the second case the range chosen is 3–8 Hz ([5], p. 202) with θ ranging from 2° to 10°.

In the numerical treatment of (31) we use Gauss's method for matrix inversion. The computation of the matrix exponentials in E (eq. (14)) is performed using the Cayley–Hamilton theorem. The latter requires the eigenvalues of matrices like A^* (eq. (11)), which is a simple task, because of the fact that the characteristic equation for the eigenvalues Λ of A^* reduces to a quadratic in Λ^2. The solution of (21) is performed by Muller's method.

We restrict presentation of the results to R_D^{PP} and R_D^{PS} only. In figures 2 and 3, we present the variation of the amplitudes of these quantities with frequency f, for different values of initial stress parameter ξ. In figure 2, the results for near vertical propagation are presented. There is a general trend of *diminution* in the reflection coefficients for increasing ξ, which becomes significant towards the higher frequencies in the band. The results for wide angle propagation for $\theta = 2°, 5°$ and 10° are presented in figures 3(a), (b) and (c) respectively. Here too, is a general trend of diminution in the reflection coefficients for increasing ξ. The trend of diminution increases with increasing θ.

It may be mentioned here that when P-waves propagate *vertically* in an unbounded initially stressed *homogeneous* medium, there is no effect of initial stress on the velocity of propagation [4]. This fact can be verified from (18), (19), with $k = 0 = p = \theta$ for the case. For reflections from the stack, there are no up going S-waves, $A_1 = 0$ (verifiable by the limit $\theta \to 0$ in (17)), $A_2 = A_4 = 0$ and the reflection and transmission coefficients B_2, B_4 are given by a pair of equations similar to (31).

5. Conclusion

The focal regions at plate boundaries of the earth prior to earthquakes are at considerable thrust due to tectonic movement. For understanding the reflection and transmission characteristics of body waves in such regions appropriate mathematical model studies are required. Herein, is considered, a stack of dissipative layers under uniaxial thrust to which the theory of incremental deformation given by Biot [2] is applicable. The governing equations can be compactly treated by matrix method, as in the case of initial stress free case, for the reflection and transmission of body waves. A numerical model study of a stack of four layers – sedimentary, granitic, basaltic and upper mantle, for near vertical as well as wide angle reflections, shows significant diminution in the magnitudes of both P and S waves.

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Computer extended series solution to viscous flow between rotating discs

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Abstract. The problem of injection (suction) of a viscous incompressible fluid through a rotating porous disc onto a rotating co-axial disc is studied using computer extended series. The universal coefficients in the low Reynolds number perturbation expansion are generated by delegating the routine complex algebra to computer. Various cases leading to specific types of flows are studied. Analytic continuation of the series solution yields results which agree favourably with pure numerical findings up to moderately large Reynolds number. The precise variation of lift as a function of R is established in each case.

Keywords. Series solution; Pade' approximants; reversion of series; Euler transformation; analytic continuation; Brown's method.

1. Introduction

Flows driven by rotating discs have constituted a major field of study in fluid mechanics for the later part of this century. These flows have technical applications in many areas, such as rotating machinery, lubrication, viscometry, computer storage devices and crystal growth processes. However, they are of special theoretical interest, because they represent one of the few examples for which there is an exact solution to the Navier–Stokes equations. This problem was first discussed by Batchelor [1] who generalized the solution of von Karman [2] and Bodewadt [3] for the flow over a single infinite rotating disc. Further this problem was discussed by Stewartson [4] who obtained approximate perturbation solution for the small Reynolds number. Later, Hoffman [5] has studied this problem using computer extended series. The numerical solutions for this problem have been obtained by Lance and Rogers [6], Mellor *et al* [7] and Brady and Durlofsky [8]. Flow between rotating and a stationary disk has been studied by Phan-Thein and Bush [9]. The problem of injection of a viscous incompressible fluid through a rotating porous disc onto a rotating co-axial disc was studied by Wang and Watson [10]. Through this span of a period of half a century, since the Batchelor–Stewartson contributions, the interaction between physically based conjectures, numerical calculations, formal asymptotic expansions and rigorous mathematical treatment has been quite intensive. In the present paper we have used semi-analytical numerical technique to understand the effect of both injection and suction separately. For simple geometries the semi-analytical numerical method proposed here provides accurate results and have advantages over pure numerical methods like finite differences, finite elements, etc. In numerical methods a separate scheme is to be developed for calculating derived quantities. If the computation of derivatives are required the numerical scheme to be used will be very sensitive to the grid/step size. This itself will be an elaborate numerical scheme. However, this difficulty is not there in the case of series solution method. A single computer run yields the solution for a large range of the expansion quantity rather than a solution for a single

value. In addition the method reveals an analytical structure of the solution which is absent in numerical solution. Van Dyke [11] and his associates have successfully used these series methods in unveiling important features of various types of fluid flows. Recently, in the analysis of thrust bearings, Bujurke and Naduvinamani [12] have used series analysis satisfactorily.

The physical problem considered in this paper is of great importance in lubrication theory. So calculation of lift is of interest in all these cases. The present analysis is primarily concerned with possible extension of Wang's [10] low Reynolds number perturbation series by computer and its analysis. The forms of the few manually calculated functions in low Reynolds number perturbation solution of two point boundary value problem allows to propose the generation of universal functions in compact form which are solutions of infinite sequence of linear problems. Using these universal coefficient functions we obtain series solution and calculate various physical parameters of interest. The present series, which is expected to be limited in convergence by the presence of a singularity, may be extended to moderately high Reynolds number by analytic continuation.

The aims of the present work are two folds. First, to calculate enough terms of the low-Reynolds number perturbation series by computer so that the nature and location of the nearest singularity (which limits the convergence) can be determined accurately, second, to show that the analytic continuation can be used effectively to extend the validity of perturbation series to moderately high Reynolds number.

2. Formulation

As shown in figure 1 we denote the spacing between the discs by 'd', the angular velocity of bottom disc by Ω_1, and that of the upper disc by Ω_2. Let the injection (or suction) at the lower disc be W ($-W$ for suction) and let u, v, w be the velocity components in the direction r, θ, z respectively (figure 1). The governing equations of the problem are

$$uu_r + wu_z - \frac{v^2}{r} = -\frac{p_r}{\rho} + \nu\left(\nabla^2 u - \frac{u}{r^2}\right) \tag{1}$$

$$uv_r + wv_z + \frac{uv}{r} = \nu\left(\nabla^2 v - \frac{v}{r^2}\right) \tag{2}$$

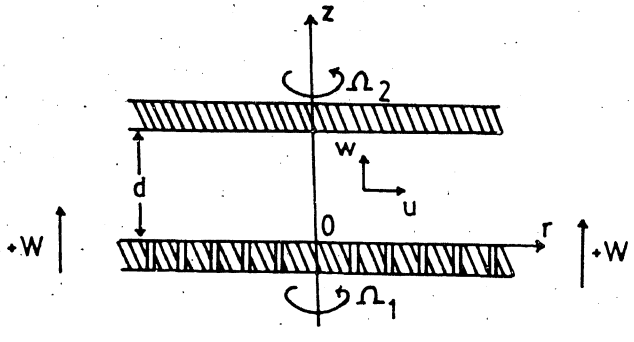

Figure 1. Schematic diagram of the problem.

$$uw_r + ww_z = -\frac{p_z}{\rho} + \nu \nabla^2 w \tag{3}$$

$$(ru)_r + rw_z = 0 \tag{4}$$

where

$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$, subscripts denote p.d.e. w.r. to the variable, p is the pressure, ρ the density and ν is the kinematic viscosity.

The boundary conditions are

$$u = 0, \quad v = r\Omega_1, \quad w = +W \quad \text{at } z = 0, \tag{5}$$

$$u = 0, \quad v = r\Omega_2, \quad w = 0. \qquad \text{at } z = d. \tag{6}$$

For similarity solution, the boundary conditions and the continuity equation suggest the transformations [10]

$$u = rf'(\eta)\frac{W}{d}, \quad v = rg(\eta)\frac{W}{d}, \quad W = -2f(\eta)\, W \tag{7}$$

$$p = \rho r^2 A \frac{W^2}{(2d^2)} + P(\eta) \tag{8}$$

where $\eta = z/d$ and A is a constant to be determined. With these transformations the equations of motion reduce to

$$(f')^2 - 2ff'' - g^2 = A + \frac{1}{R}f''' \tag{9}$$

or after differentiation, we have

$$-2R(ff''' + gg') = f'''' \tag{10}$$

$$2R(f'g - fg') = g'' \tag{11}$$

$$P(\eta) = -\rho\left(-2f^2 W^2 - 2\nu W\frac{f'}{d}\right) + P_0 \tag{12}$$

Here $R = (Wd/\nu)$ is the cross flow Reynolds number. The constant P_0 is determined by the pressure at the edge of the discs. The boundary conditions take the forms

$$f'(0) = 0, \quad g(0) = \frac{\Omega_1 d}{W} = \alpha, \quad f(0) = -\frac{1}{2} \tag{13}$$

$$f'(1) = 0, \quad g(1) = \frac{\Omega_2 d}{W} = \beta, \quad f(1) = 0 \tag{14}$$

In order to investigate the mutual interaction of rotation and injection (suction), we shall assume α and β to be of order of unity. This includes many interesting cases where both rotation and injection (suction) are not minor perturbations. Differential eqs (10) and (11) are solved usually by direct integration which frequently involves more than one integration process because of the two point nature of the boundary conditions. The use of series solution provides an attractive alternative approach. Not only the difficulties associated with two point boundary value problems are relieved, but also

the terms of series method are capable of providing results to any desired degree of accuracy with minimum time and less storage requirement of computer.

3. Method of solution

We seek the solution of (10) and (11) in power series of R in the forms

$$f(\eta) = f_0(\eta) + \sum_{n=1}^{\infty} R^n f_n(\eta) \tag{15}$$

$$g(\eta) = g_0(\eta) + \sum_{n=1}^{\infty} R^n g_n(\eta). \tag{16}$$

Substituting (15), (16) into (10), (11) and comparing like powers of R on both sides, we get

$$f_n'''' = -2 \sum_{n=1}^{\infty} (f_r f_{n-1-r}''' + g_r g_{n-1-r}') \tag{17}$$

$$g_n'' = 2 \sum_{n=1}^{\infty} (f_r' g_{n-1-r} - f_r g_{n-1-r}') \tag{18}$$

$$n = 0, 1, 2, \ldots$$

The relevant boundary conditions are

$$f_0'(0) = 0, \quad f_0(1) = 0, \quad f_0(0) = -\frac{1}{2}, \quad f_0'(1) = 0$$

$$f_n'(0) = 0, \quad f_n'(1) = 0, \quad f_n(0) = 0, \quad f_n(1) = 0 \tag{19}$$

$$g_0(0) = \alpha, \quad g_0(1) = \beta$$

$$g_n(0) = 0, \quad g_n(1) = 0 \tag{20}$$

$$n = 1, 2, 3, \ldots$$

The solutions of above equations are

$$f_0 = -\left(\frac{1}{2} - \frac{3}{2}\eta^2 + \eta^3\right)$$

$$f_1 = -\frac{11}{70}\eta^2 + \frac{13}{35}\eta^3 - \frac{1}{4}\eta^4 + \frac{1}{20}\eta^6 - \frac{1}{70}\eta^7$$

$$-\frac{\alpha}{12}(\beta - \alpha)(\eta^2 - 2\eta^3 + \eta^4) - \frac{(\beta - \alpha)^2}{60}(2\eta^2 - 3\eta^3 + \eta^5) \tag{21}$$

$$g_0 = (\beta - \alpha)\eta + \alpha$$

$$g_1 = -\left[\frac{(\beta - \alpha)}{20}(4\eta^5 - 5\eta^4 - 10\eta^2 + 11\eta) + \frac{\alpha}{2}(\eta^4 - 2\eta^3 + \eta)\right].$$

The slow convergence of the series ((15), (16)) requires large number of terms for obtaining the approximate sum. As we proceed for higher approximations, the algebra becomes cumbersome and it is difficult to calculate the terms manually. We propose

a systematic series expansion scheme with polynomial coefficients so that whole process can be made automatic using computer. For this purpose, we consider f_n and g_n to be of the forms

$$f_n(\eta) = (1-\eta)^2 \sum_{k=2}^{4n+1} A_{n(k)} \eta^k \tag{22}$$

$$g_n(\eta) = (1-\eta) \sum_{n=1}^{4n} B_{n(k)} \eta^k \tag{23}$$

in (15) and (16) respectively. This expression yields exactly the above calculated terms f_1 and g_1 besides this it enables us to find f_i and g_i for $i \geqslant 2$ using computer. We substitute (22), (23) into (17), (18) and equate various powers of η on both sides and obtain two recurrence relations for unknowns $A_{n(k)}$ and $B_{n(k)}$ in the forms

$$\begin{aligned} A_{n(k)} = {} & 2A_{n(k+1)} - A_{n(k+2)} + \frac{1}{(k+2)(k+1)k(k-1)} \\ & \times \left\{ \sum_{i=1}^{6} A_{(n-1)(k+2-i)} P_i(k+2-i) + \sum_{i=1}^{3} B_{(n-1)(k-i)} Q_i(k-i) \right. \\ & + \sum_{r=1}^{n-2} \left[\sum_{i=0}^{4} \sum_{j=1}^{nk+i} A_{r(l+1-nk-i)} A_{(m-1)(t-2)} P_{7+i}(t-2) \right. \\ & \left. \left. + \sum_{i=0}^{2} \sum_{j=1}^{nk+i} B_{r(l-nk-i)} B_{(m-1)(t-3)} Q_{4+i}(t-3) \right] \right\} \\ & K = 2, 3, \ldots, (4n+1) \end{aligned} \tag{24}$$

$$\begin{aligned} B_{n(k)} = {} & B_{n(k+1)} - \frac{1}{k(k+1)} \left[\sum_{i=1}^{5} B_{(n-1)(k+1-i)} T_i(k+1-i) \right. \\ & + \sum_{i=1}^{4} A_{(n-1)(k+1-i)} T_i'(k+1-i) \\ & \left. + \sum_{r=1}^{n-2} \sum_{i=0}^{3} \sum_{j=1}^{nk+i} A_{r(l+1-nk-i)} B_{(m-1)(t-3)} T_{6+i}(l+1-nk-i, t-3) \right] \\ & K = 1, 2, \ldots, 4n \end{aligned} \tag{25}$$

where

$$m = n - r, \quad l = 4r + j, \quad t = 4m - j, \quad nk = 4n - 2 - k,$$

$$P_1(k) = k(k-1)(k-2),$$

$$P_2(k) = -2(k+1)k(k-1),$$

$$P_3(k) = -(3k(k-1)(k-2) - (k+2)(k+1)k),$$

$$P_4(k) = (2k(k-1)(k-2) + 6(k+1)k(k-1) + 12),$$

$$P_5(k) = -(4(k+1)k(k-1) + 3(k+2)(k+1)(k-1) + 24),$$

$$P_6(k) = (2(k+2)(k+1)k + 12),$$

$$P_7(k_1) = -2k_1(k_1-1)(k_1-2),$$

$$P_8(k_1) = 4(k_1+1)k_1(k_1-1) + 4k_1(k_1-1)(k_1-2),$$

$$P_9(k_1) = -2(k_1+2)(k_1+1)k_1 - 8(k_1+1)k_1(k_1-1) - 2k_1(k_1-1)(k_1-2),$$
$$P_{10}(k_1) = 4(k_1+2)(k_1+1)k_1 + 4(k_1+1)k_1(k_1-1),$$
$$P_{11}(k_1) = -2(k_1+2)(k_1+1)k_1,$$
$$Q_1(k) = -2k\alpha,$$
$$Q_2(k) = -2k(\beta-\alpha) + 2(k+1)\alpha - 2(\beta-\alpha),$$
$$Q_3(k) = 2(k+1)(\beta-\alpha) + 2(\beta-\alpha),$$
$$Q_4(k_1) = -2k_1,$$
$$Q_5(k_1) = 2(2k_1+1),$$
$$Q_6(k_1) = -2(k_1+1),$$
$$T_1(k) = k, \quad T_2(k) = -(k+1), \quad T_3(k) = -(3k-6), \quad T_4(k) = (5k-9),$$
$$T_5(k) = -(2k-4), \quad T_6(k,k_1) = 2k - 2k_1, \quad T_7(k,k_1) = 2(3k_1 - 3k - 1),$$
$$T_8(k,k_1) = 2(3k - 3k_1 + 2), \quad T_9(k,k_1) = 2(-k + k_1 - 1),$$
$$T'_1(k) = 2k\alpha, \quad T'_2(k) = 2(\beta-\alpha)k - 4(k+1)\alpha - 2(\beta-\alpha),$$
$$T'_3(k) = -4(\beta-\alpha)(k+1) + 2(k+2)\alpha + 4(\beta-\alpha),$$
$$T'_4(k) = 2(k+2)(\beta-\alpha) - 2(\beta-\alpha),$$
$$A_{12} = -\frac{11}{70} - \frac{\alpha}{12}(\beta-\alpha) - \frac{(\beta-\alpha)^2}{30},$$
$$A_{13} = \frac{4}{70} - \frac{(\beta-\alpha)^2}{60}, \quad A_{14} = \frac{3}{140}, \quad A_{15} = -\frac{1}{70},$$
$$B_{11} = -\left(\frac{\alpha}{2} + \frac{11(\beta-\alpha)}{20}\right), \quad B_{12} = -\left(\frac{\beta+9\alpha}{20}\right),$$
$$B_{13} = \left(\frac{\alpha}{2} - \frac{(\beta-\alpha)}{20}\right), \quad B_{14} = \frac{(\beta-\alpha)}{5}.$$

or the radial velocity profile $f'(\eta)$, we have

$$f'(\eta) = -(3\eta^2 - 3\eta) + \sum_{n=1}^{\infty} R^n \sum_{k=2}^{4n+1} A_{n(k)}(k\eta^{k-1} - 2(k+1)\eta^k + (k+2)\eta^{k+1}) \tag{26}$$

he constant A in (9) which is proportional to the lift is given by

$$A = -\frac{1}{R} f'''(1) - \beta^2$$
$$= -\frac{1}{R}\left[-6 + \sum_{n=1}^{\infty} R^n \sum_{k=2}^{4n+1} 6k A_{n(k)}\right] - \beta^2$$
$$= -\frac{1}{R}\left[-6 + \sum_{n=1}^{\infty} a_n R^n\right] - \beta^2 \tag{27}$$

Case (1): $\alpha = 0, \beta = 0$ which corresponds to the case when both discs are stationary and the flow is due to injection only. In this case the coefficients of the series for $f'''(1)$, which is used to calculate A, has terms which are all positive after third term (table 1). Using the computed coefficients we draw Domb–Sykes plot (figure 2) for $f'''(1)$ (series (27)) to find the nature and location of the nearest singularity which restricts the convergence of the series. In this case singularity is found to be a square root singularity at $R = 17{\cdot}9826$. This singularity on the positive real axis is not a real singularity, but an indication of double valuedness of the function. This artificial restriction on convergence can be eliminated by reverting the series. This type of reversion was successfully employed earlier by Richardson [13] and Schwartz [14]. Towards this goal the reversion of the series (27) for $f'''(1)$ is performed as follows. Consider

$$f'''(1) = -6 + \sum_{n=1}^{\infty} a_n R^n. \tag{28}$$

Let

$$Y = f'''(1) + 6 = \sum_{n=1}^{\infty} a_n R^n$$

Reverting the above series, we have

$$R(Y) = \sum_{n=1}^{\infty} B_n Y^n \tag{29}$$

where

$$B_1 = \frac{1}{e(1,1)}$$

$$B_m = -\frac{1}{e(1,m)} \sum_{i=0}^{m-2} B_{(i+1)} e(m-i, i+1) \quad m = 2, 3, \ldots, n$$

$$e(1, \chi) = (a_1)^{\chi}$$

$$e(k+1, \chi) = \frac{1}{K a_1} \sum_{i=0}^{k-1} \left\{ (k-i)\chi - i \right\} e(i+1, \chi) a_{k-i+1}$$

$$K = 1, 2, \ldots, n; \quad \chi = 1, 2, \ldots, n.$$

Table 1. The coefficients a_n of the series (27) for $f'''(1)$ in the case of $\alpha = 0, \beta = 0$.

No	a_n	No	a_n
1	−6·00000000000000E−00	14	3·6058013122139E−011
2	−7·7142857142857E−001	15	2·8303130882864E−012
3	−3·2003710575139E−002	16	2·1564675357185E−013
4	5·6407538040317E−004	17	1·6028392195738E−014
5	9·4914049735636E−003	18	1·1664258107783E−015
6	2·2591450404529E−003	19	8·3319098951704E−017
7	3·7542463046543E−004	20	5·8565624104072E−018
8	5·0483529518823E−005	21	4·0599933881679E−019
9	5·8577314766192E−006	22	2·7794211813578E−020
10	6·0972460009191E−007	23	1·8811986978930E−021
11	5·8354641682513E−008	24	1·2608050796818E−022
12	5·2237099823177E−009	25	8·3754097096218E−024
13	4·4337512718581E−010	26	5·5163592628791E−025

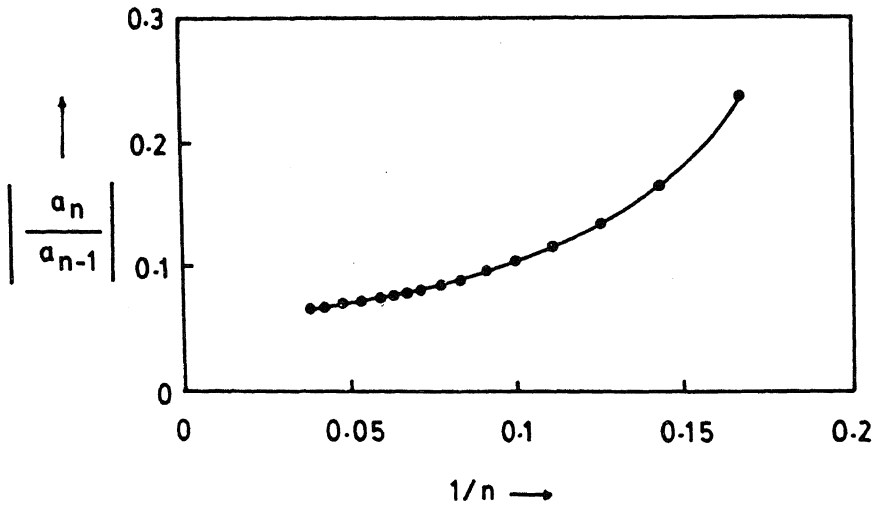

Figure 2. Domb–Sykes plot for series (27) in the case of $\alpha = 0$, $\beta = 0$.

Besides reversion we use Pade' approximants for summing the reverted series (29) which yields analytic continuation. The details about Pade' approximants are given in Appendix. These results are shown in figure 3.

Case (2): $\alpha = 4$, $\beta = 0$, lower disc is rotating and the upper disc is stationary. The coefficients (a_n) of the series (27) for $f'''(1)$ are listed in table 2. They are decreasing in magnitude and have no regular pattern of sign. We invoke Pade' approximants to achieve analytic continuation of the series (27) [11] and the corresponding results are shown in figure 4.

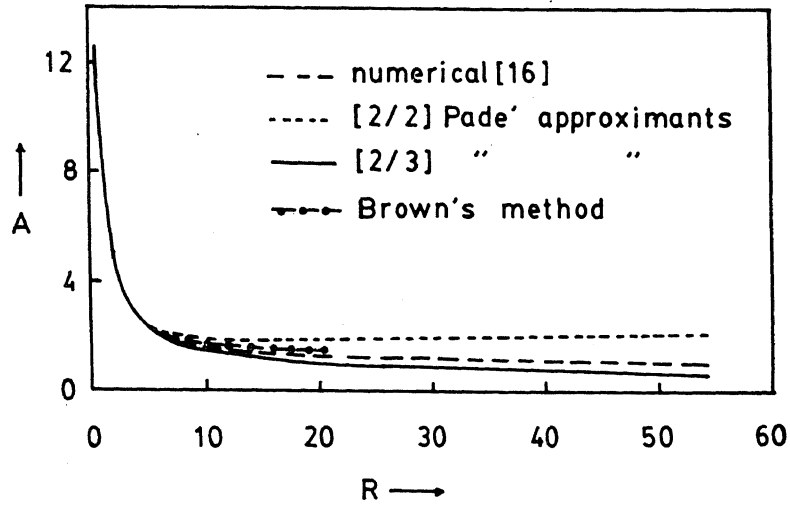

Figure 3. Values of A as a function of $R(\alpha = 0, \beta = 0)$.

Table 2. The coefficients a_n of the series (27) for $f'''(1)$ in the case of $\alpha = 4, \beta = 0$.

No	a_n	No	a_n
1	−6·00000000000000E−00	14	4·3651028454742E−007
2	4·02857142857140E−00	15	−3·8077592733234E−007
3	−2·6924345495784E−002	16	1·1081633809731E−008
4	−1·2306126753066E−001	17	2·7544302221770E−008
5	1·3750340358019E−001	18	−5·0110664501736E−009
6	9·5410260700099E−002	19	−1·1312815967417E−009
7	8·2631707941841E−003	20	4·6747879187209E−010
8	−1·7355299293887E−002	21	2·6531328428935E−011
9	2·9717102085349E−005	22	−3·8735009780784E−011
10	2·9683069164996E−004	23	3·8598232577803E−012
11	−3·0384182421661E−005	24	2·0690368287353E−012
12	−1·8230118955959E−005	25	−5·1032148957020E−013
13	5·0851049208551E−006	26	−8·7096012910697E−014

Case (3): $\alpha = 0$, $\beta = 0{\cdot}5$ in this case the upper disc is rotating and the lower one is stationary. The coefficients (a_n) of the series (27) for $f'''(1)$ are listed in table 3. They are decreasing in magnitude but have no regular pattern of sign. So, as in the previous case we use Pade' approximants to sum the series. The results obtained are shown in figure 5.

Case (4): $\alpha = 1, \beta = 1$ in this case discs are corotating (with same speed). The coefficients (a_n) of the series (27) for $f'''(1)$ are listed in table 4. They are decreasing in magnitude and alternate in sign after 11th term. Using the computed coefficients we draw Domb–Sykes plot (figure 6) for $f'''(1)$ (series (27)) to find the nature and location of the nearest singularity which restricts the convergence of the series. In this case singularity is found to be at $R = 2{\cdot}579849$ on the negative real axis. The bilinear Euler transformation will help in recasting the series into new series whose region of validity is increased

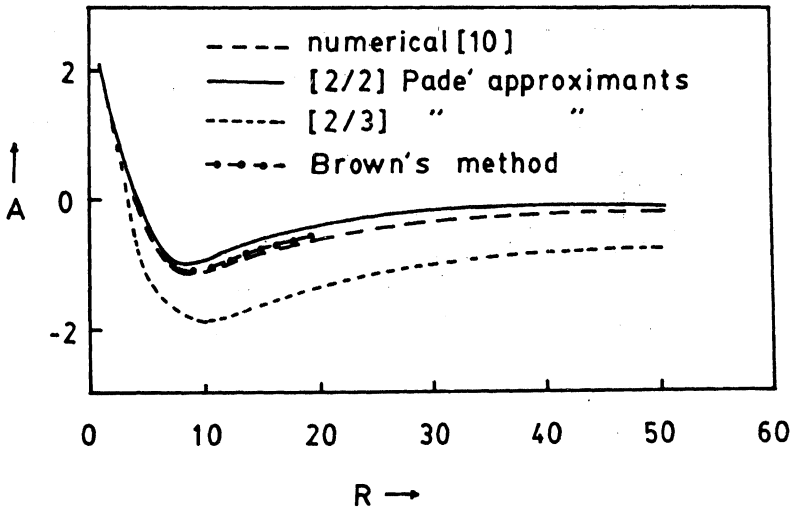

Figure 4. Values of A as a function of $R(\alpha = 4, \beta = 0)$.

Table 3. The coefficients a_n of the series (27) for $f'''(1)$ in the case of $\alpha = 0$, $\beta = 0.5$.

No	a_n	No	a_n
1	−6·00000000000000E−000	14	3·1356929508470E−009
2	−9·4642857142857E−001	15	3·3582404952799E−011
3	−6·1091012162441E−002	16	−1·3299119511885E−010
4	3·7858935530174E−002	17	1·8233492599838E−011
5	6·0657643681146E−003	18	2·0308395587595E−012
6	6·4380238514181E−004	19	−6·6509147430890E−013
7	5·9793189211569E−004	20	−6·1403723963494E−014
8	7·6918357469503E−005	21	4·7938734279652E−014
9	−5·9467326297197E−006	22	−7·0845088542235E−015
10	8·8622406112891E−007	23	6·2683407300982E−018
11	5·0127735327728E−007	24	1·3799804166256E−017
12	−5·5929925021510E−008	25	6·2326271557991E−017
13	−7·7492242493884E−009	26	−2·3804925454199E−017

compared to the original series (27). Consider the Euler transformation

$$\omega = R/(R + R_0)$$

then $R = \omega R_0/(1-\omega)$ and

$$f'''(1) = -6 + \sum_{n=1}^{\infty} a_{n+1}(\omega R_0/(1-\omega))^n = \sum_{n=1}^{\infty} D_n \omega^{n-1} \tag{30}$$

where

$$D_1 = -6,$$

$$D_2 = a_2 R_0,$$

$$D_3 = a_2 R_0 + a_3 R_0^2,$$

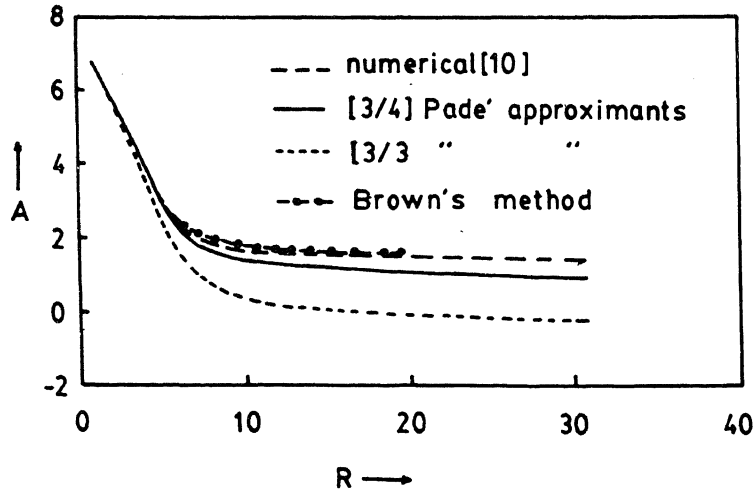

Figure 5. Values of A as a function of $R(\alpha = 0, \beta = 0{\cdot}5)$.

Table 4. The coefficients a_n of the series (27) for $f'''(1)$ in the case of $\alpha = 1, \beta = 1$.

No	a_n	No	a_n
1	−6·000000000000000000	14	1·3686820085834E−006
2	−7·7142857142857E−001	15	−1·9036823867936E−007
3	−2·7486085343228E−001	16	8·9756722699421E−008
4	1·4814962475167E−001	17	−6·0920523719445E−008
5	−1·1322403383611E−002	18	2·2416431500647E−008
6	−4·6167826401268E−003	19	−5·3363130655523E−009
7	−2·9912751714986E−005	20	1·4329467638064E−009
8	1·4864726973501E−003	21	−6·7360950377740E−010
9	−4·7373348018575E−004	22	2·9287892083304E−010
10	2·8876377167786E−005	23	−9·3478798663218E−011
11	4·8008012782765E−007	24	2·6176956621819E−011
12	1·0028862586723E−005	25	−9·1174262387237E−012
13	−5·9189631736138E−006	26	3·7433180023877E−012

$$D_4 = a_2 R_0 + 2a_3 R_0^2 + a_4 R_0^3,$$
$$D_5 = a_2 R_0 + 3a_3 R_0^2 + 3a_4 R_0^3 + a_5 R_0^4,$$
$$D_n = (-1)^{n-1} \Delta^{n-2} e_2 \qquad (31)$$

with

$$\Delta e_j = e_{j+1} - e_j$$
$$e_j = (-R_0)^{j-1} a_j.$$

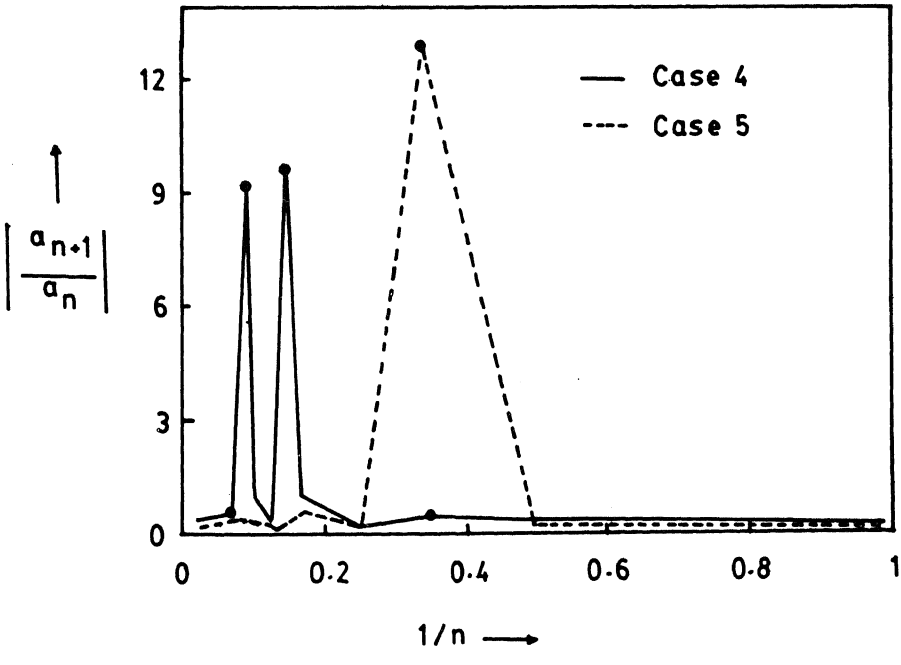

Figure 6. Domb–Sykes plot for series (27) in the case of $\alpha = 1, \beta = 1$ and $\alpha = 1, \beta = -1$.

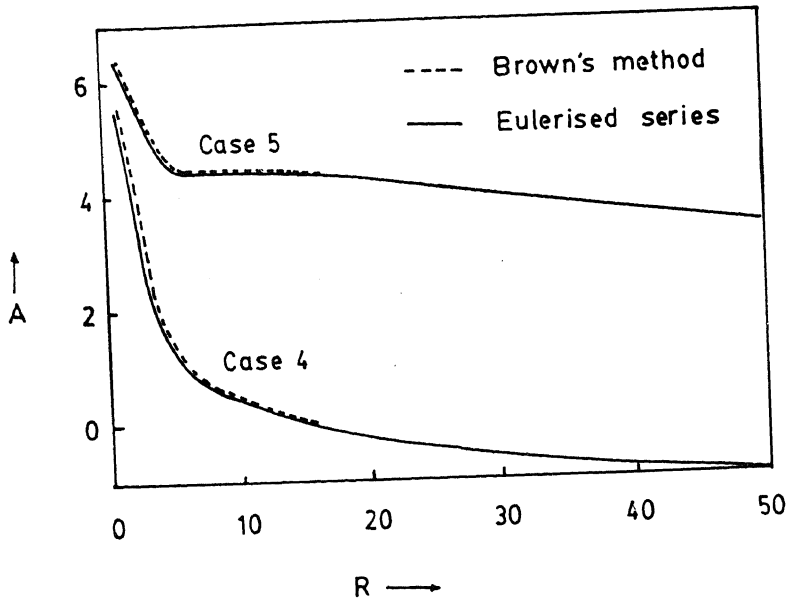

Figure 7. Values of A as a function of $R(\alpha = 1, \beta = 1)$ and $(\alpha = 1, \beta = -1)$.

This transformation maps the dominant singularities to infinity while the origin remains fixed. Points close to the dominant singularities are mapped far from the origin and hence outside the unit circle in the transformed plane [11]. The results obtained are shown in figure 7.

Case (5): $\alpha = 1$, $\beta = -1$ in this case discs are counterotating (with same speed). The coefficients (a_n) of the series (27) for $f'''(1)$ are listed in table 5. They are decreasing in magnitude and alternate in sign after 10th term. Using the computed coefficients we draw Domb–Sykes plot (figure 6) for $f'''(1)$ (series (27)) to find the nature and location of the nearest singularity which restricts the convergence of the series. In this case

Table 5. The coefficients a_n of the series (27) for $f'''(1)$ in the case of $\alpha = 1$, $\beta = -1$.

No	a_n	No	a_n
1	−6·000000000000000000	14	−9·2298357740712E−008
2	−1·571428571428600000	15	3·3221879279623E−008
3	−2·1210059781490E−002	16	−1·8249751078357E−008
4	−2·7858353362435E−001	17	5·3594587988598E−009
5	−1·4495845921663E−002	18	−1·1691594163933E−009
6	−3·2775029096463E−003	19	2·6218335178891E−010
7	2·0976832622669E−003	20	−9·8211386423030E−011
8	−4·9288165348855E−004	21	3·5209099021094E−011
9	−7·5443633168820E−006	22	−1·0026682644541E−011
10	−8·6986972673174E−006	23	2·4337119787355E−012
11	7·6675014841131E−006	24	−6·7971322153644E−013
12	−3·5515893049259E−006	25	2·2411087467669E−013
13	5·6651432024791E−007	26	−7·1355223824464E−014

singularity is found to be at $R = 2{\cdot}623517$ on the negative real axis. As in the previous case we have used Euler Transformation to increase the region of validity. So

$$f'''(1) = -6 + \sum_{n=1}^{\infty} a_{n+1}(\omega R_0/(1-\omega))^n = \sum_{n=1}^{\infty} D'_n \omega^{n-1} \tag{32}$$

The variation of A with R is shown in figure 7.

Equations (10)–(14) are also solved by power series method.

4. Power series method

We assume power series solution to (10)–(14) in the forms

$$f = \sum_{n=1}^{\infty} d_n(1-\eta)^{n+1} \tag{33}$$

$$g = \beta + \sum_{n=1}^{\infty} b_n(1-\eta)^n \tag{34}$$

where

$$\sum_{n=1}^{\infty} d_n + \frac{1}{2} = 0$$

$$\sum_{n=1}^{\infty} (n+1)d_n = 0$$

$$\sum_{n=1}^{\infty} b_n + \beta = \alpha \tag{35}$$

and

$$b_2 = 0$$

$$b_3 = -2R\beta d_1/3$$

$$d_3 = R\beta b_1/12$$

$$b_{n+3} = \frac{-2R\beta d_{n+1}}{(n+3)} - \frac{2R}{(n+3)(n+2)} \sum_{m=1}^{n} (2m-n)d_m b_{n-m+1} \tag{36}$$

$$d_{n+3} = \frac{2R\beta b_{n+1}}{(n+4)(n+3)(n+2)} + \frac{2R}{(n+1)(n+2)(n+3)(n+4)} \times \left\{ \sum_{m=1}^{n} m b_m b_{n-m+1} + m(m^2-1)d_m d_{n-m+1} \right\} \tag{37}$$

Expression (35) comes from the boundary conditions at $\eta = 0$ and (36) and (37) are obtained from (10) and (11) respectively. If b_1, d_1 and d_2 are known then rest of $\{b_n\}$ and $\{d_n\}$ can be found from the recursive relations (36) and (37).

Effectively we have transformed a two point boundary value problem into solving a system of nonlinear equations. We wish to find b_1, d_1 and d_2 such that conditions (35) are satisfied. To solve this system of nonlinear equations Brown's method is useful. The details of this procedure are given in Byrne [18]. It is found that the series (33), (34) converge much faster and also more accurate solution with very little computer time can be obtained. It is implemented in analysing all the five cases considered. The first two coefficients of the series and lift at different Reynolds numbers are calculated. All

these values are accurate to six significant figures. The number of significant figures for accuracy was determined by increasing the number of terms in the series from 30 to 350. The time taken by the computer is also comparatively less whereas other methods [9, 10, 16] require more computer time and large storage.

5. Discussion of results

Here the problem of injection (suction) of a viscous incompressible fluid through a rotating porous disc onto a rotating co-axial disc is studied using computer extended series analysis. The motion of the fluid is governed by a pair of coupled nonlinear ordinary differential (10) and (11) together with the boundary conditions (13) and (14). The series expansion scheme with polynomial coefficients ((22), (23)) proposed enables in obtaining recurrence relations (24) and (25). Using these interactive relations we generate large number ($n = 25$) of universal coefficients (($A_{n(k)}$, $k = 2, 3, \ldots, 4n + 1$), $n = 1, 2, \ldots, 25$) and (($B_{n(k)}$, $k = 1, 2, \ldots, 4n$), $n = 1, 2, \ldots, 25$). To this order there are 1300 coefficients $A_{n(k)}$ and 1300 coefficients $B_{n(k)}$. A careful FORTRAN program consisting of number of DO loops makes it possible in performing complex algebra involved. Using the universal coefficients of the series ((22), (23)) we obtain series expansion for A which is directly proportional to the lift. The coefficients a_n of the series (27) for A in the case of $\alpha = 0$, $\beta = 0$ are listed in table 1. They decrease in magnitude and have same sign after third term. Figure 2 shows the Domb–Sykes plot for series (27) in the case of $\alpha = 0$, $\beta = 0$. The slope of the curve indicates square root singularity corresponding to double valuedness of the solution (by using rational extrapolation exact position of the singularity is found to be at $R = 17{\cdot}9826$ with an error of order 10^{-5}). So the region of validity of the series (27) for A in the case of $\alpha = 0$, $\beta = 0$ will be increased by reverting the series (by changing the role of dependent and independent variables). We use Pade' approximants for summing the reverted series (29) which accelerates the convergence and yields its analytic continuation. The results agree most favourably with results of Wang [16] (numerical), Bujurke and Naduvinamani [12] (semi-numerical) and Phan-Thien ad Bush [9] (power series). It is of interest to note that [2/2] and [2/3] Pade' approximants bracket [The Pade' approximants $P_2^2(1)$ and $P_3^2(1)$ form upper and lower bounds for the numerical value of lift force [15]] the Numerical results of Wang [16] (figure 3). Double precision arithmetic used guarantees the accuracy of Pade' approximants. Also, the round off errors will be of negligible order as the Pade' approximants bracketing the numerical results are of the form where denominators are polynomials of degree $\leqslant 4$ [17]. Table 2 contains the list of coefficients a_n of the series (27) for the case of $\alpha = 4$, $\beta = 0$. These coefficients decrease in magnitude but have no regular sign pattern. We invoke Pade' approximants to achieve analytic continuation of the series (27). The results agree favourably with earlier numerical findings [10]. Also, we observe that [2/2] and [2/3] Pade' approximants bracket the numerical results which are given in figure 4. The coefficients a_n for the case of $\alpha = 0$, $\beta = 0{\cdot}5$ are listed in table 3. In this case also coefficients are decreasing in magnitude and have no regular sign pattern. As in the previous case analytic continuation of the series (27) is achieved by using Pade' approximants. The [3/4] Pade' approximant is found to be very near to the numerical results [10] which are shown in figure 5. The coefficients a_n of the series (27) for A in the case of $\alpha = 1$, $\beta = 1$ are listed in table 4. They decrease in magnitude and have alternate sign after 11th term. Figure 6, the Domb–Sykes plot for series (27) in the

Table 6. Comparison of Brown's method with optimization method.

R	Lift (Brown's Method)	Terms (N) (required for the convergence of Brown's Method)	Lift (Optimization)	Terms (N) (required for the convergence of the optimization method)
1	6·80278	30	6·80278	50
5	2·10850	50	2·10850	100
10	1·58847	75	1·58847	200
15	1·42121	150	1·42846	500
18	1·38127	200	1·39342	750
22	1·35841	350	—	—

case of $\alpha = 1, \beta = 1$ shows the singularity on the negative real axis after extrapolation at $R = 2{\cdot}579849$ with an error of 10^{-4}. The region of validity of the series is increased by Euler Transformation. The results obtained are shown in figure 7. In case 5 the analytic continuation is achieved exactly in the way like case 4. The results obtained are shown in figure 7. This problem is also solved by power series in conjunction with Brown's method for different cases [$\alpha = 0\,\beta = 0$, $\alpha = 4\,\beta = 0$, $\alpha = 0\,\beta = 0{\cdot}5$, $\alpha = 1\,\beta = 1$, $\alpha = 1\,\beta = -1$] and the results obtained are shown in figures (3–5 and 7). Details of case 1 ($\alpha = 0$, $\beta = 0$) (table 6) corresponding to stationary disks with injection shows the efficiency of Brown's method. The series (26) representing radial velocity profiles in various cases ($\alpha = 0$, $\beta = 0$; $\alpha = 4$, $\beta = 0$; $\alpha = 0$, $\beta = 0{\cdot}5$) are analysed using Pade' approximants and these results are shown in figures 8 and 9). It is observed that velocity attains peak values for $\alpha = 0$, $\beta = 0{\cdot}5$ and it is much higher than first two cases.

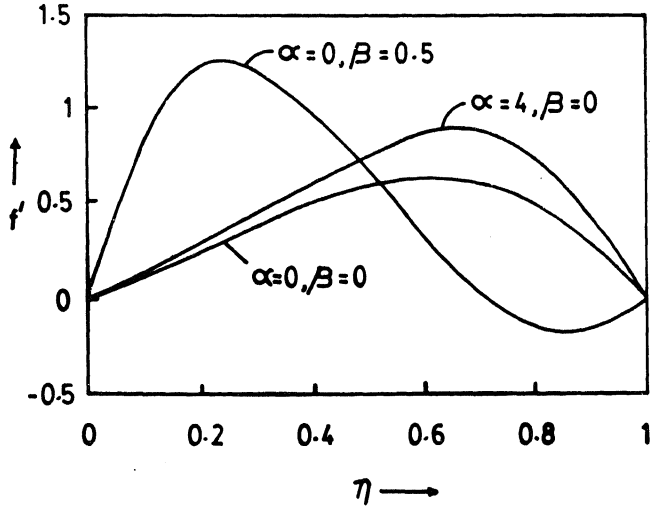

Figure 8. Radial velocity distribution $f'(\eta)$ at $R = 16$.

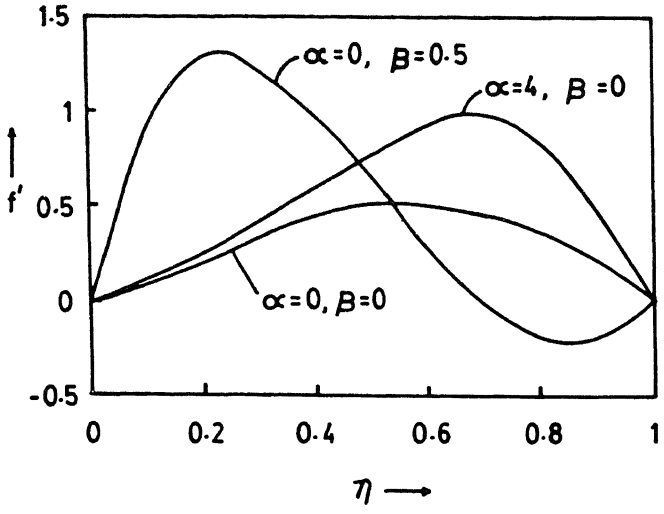

Figure 9. Radial velocity distribution $f'(\eta)$ at $R = 20$.

The method proposed here is quite flexible and efficient in implementing on computer compared with the pure numerical methods. Once the universal coefficients are generated rest of the analysis can be done at a stretch requiring hardly any computer time and storage. Whereas other methods [9, 10, 16] require more computer time and large storage.

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Appendix

Pade' Approximants

The basic idea of Pade' summation is to replace a power series

$$\sum C_n R^n$$

by a sequence of rational functions of the form

$$P_M^N(R) = \frac{\sum_0^N A_n R^n}{\sum_0^M B_n R^n}$$

where we choose $B_0 = 1$ without loss of generality. We determine the remaining $(M + N + 1)$ coefficients $A_0, A_1, A_2, \ldots A_N; B_1, B_2, \ldots B_M$ so that the first $(M + N + 1)$ terms in the Taylors series expansion of $P_M^N(R)$ match with first $(M + N + 1)$ terms of the power $\Sigma C_n R^n$. The resulting rational function $P_M^N(R)$ is called a Pade' approximant. If $\Sigma C_n R^n$ is a power series representation of the function $f(R)$ than in favourable cases

$P_M^N(R) \to f(R)$, pointwise as $N, M \to \infty$. There are many methods for the construction of Pade' approximants. One of the efficient methods for constructing Pade' approximants is recasting of the series into continued fraction form. A continued fraction is an infinite sequence of fractions whose $(N+1)$th member has the form

$$F_N(R) = \cfrac{D_0}{1 + \cfrac{D_1 R}{1 + \cfrac{D_2 R}{\ddots \cfrac{}{1 + D_N R}}}} \qquad \text{(E)}$$

The coefficients D_n are determined by expanding the terminated continued fraction $F_N(R)$ in a Taylor series and comparing with those of the power series to be summed. An efficient procedure for calculating the coefficients D_n's of the continued fraction (E) may be derived from the algebraic identities (8.4.2a)–(8.4.2c) [15]. Contrary to representations by power series, continued fraction representations may converge in regions that contain isolated singularities of the function to be represented, and in many cases convergence is accelerated. Based on these D_n's we get terminated continued fractions of various order from other algorithms ((8.4.7), (8.4.8a) and (8.4.8b) [15]).

Pade' approximants perform an analytic continuation of the series outside its radius of convergence. It is clear that it can approximate a pole by zeros of the denominator. With branch points it extracts a single-valued function by inserting branch cuts, which it simulates by lines of alternating poles and zeros [19].

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The Hodge conjecture for certain moduli varieties

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Abstract. For smooth projective varieties X over $\mathbb{C}$, the Hodge Conjecture states that every rational Cohomology class of type (p, p) comes from an algebraic cycle. In this paper, we prove the Hodge conjecture for some moduli spaces of vector bundles on compact Riemann surfaces of genus 2 and 3.

Keywords. Chow groups; Abel-Jacobi maps; moduli spaces; normal functions; Hecke correspondences.

Introduction

For smooth projective varieties X over $\mathbb{C}$, the field of complex numbers, the Hodge conjecture states that every rational cohomology class of type (p, p) comes from an algebraic cycle. More precisely, consider the Hodge decomposition

$$H^i(X, \mathbb{C}) = \sum_{p+q=i} H^{p,q}(X).$$

Let $C^p(X)$ denote the Chow group of algebraic cycles of codimension p on X, modulo rational equivalence. Then one has the 'class map'

$$\lambda_X^p : C^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X).$$

Then the Hodge (p, p) conjecture states that λ_X^p is surjective.

Let C be an irreducible smooth projective curve if genus $g \geqslant 2$, and let $M(n, \xi)$ be the moduli space of stable vector bundles V on C, of rank n, $\det V \simeq \xi$, ξ a line bundle of degree d such that $(n, d) = 1$. The aim of this paper is to prove the Hodge (p, p) conjecture in the case when $g = 2$, $n = 3$ ($\dim M(3, \xi) = 8$). In the case when $n = 2$, $g = 2, 3, 4$, the Hodge conjecture can be proved by elementary means which we indicate at the end of the paper.

The case we consider is of interest, as it gives a non-trivial family of examples where the general method of normal functions is used to prove the conjecture. Geometric descriptions given in [T] in the rank 2 case lead to elementary proofs of the Hodge conjecture. In the rank 3 case, any such description does not give elementary proofs of the Hodge conjecture. (cf. Remark 4.3, 4.4)

The Poincaré–Lefschetz theory of normal functions was generalized and developed by Griffiths and Zucker and had the proof of the Hodge (p, p) conjecture as a primary goal. In this paper we give a natural construction of a smooth projective variety and a proper generically finite morphism onto the moduli of rank n, degree $(ng - n)$ bundles which plays the role of the Lefschetz pencil in the context of normal functions. From the remarks of Zucker (cf. [Z-2], pp. 266) all the known examples where normal functions have been used to prove the Hodge conjecture, more elementary methods have been

successful (cf. [M], [Z-2], and [Sh] for a full survey of the Hodge conjecture); however, in the present case this seems unlikely.

In § 1, we recall some general facts. Section 2, contains a theorem giving a criterion for a variational Hodge (p, p) conjecture to hold under some stringent conditions. In § 3, we give a pencil type construction in the context of moduli. Section 4, gives the proof of the conjecture for $M(3, \xi)$.

Some notations. Let X be a smooth projective variety defined over $\mathbb{C}$ the field of complex numbers. We state at the outset that our base field is $\mathbb{C}$. Let $C^p(X)$ denote the Chow group of cycles of codimension p modulo rational equivalence and $A^p(X) \subset C^p(X)$ the subgroup of cycle classes algebraically equivalent to zero.

1. Preliminaries

Lemma 1.1. (*cf.* [Z-1] *A*.2) *Let X and Y be smooth projective varieties, $f: X \to Y$ be a proper generically finite surjection. If the Hodge (p, p) conjecture is true for X, then it is true for Y.*

Proof. We note that, $f_* f^* =$ multiplication by d, both on cycles and cohomology, where $d = [k(X):k(Y)]$. Therefore, if $\gamma \in H^{p,p}(Y, \mathbb{Q})$, $f^*\gamma \in H^{p,p}(X, \mathbb{Q})$; so if $f^*\gamma$ is a rational cycle Z, then

$$d\gamma = (f_* f^* \gamma) = f_* Z$$

implying γ is a rational cycle $1/d(f_* z)$ on Y.

Lemma 1.2. *Let E be a vector bundle of rank $r = e + 1$, and let $P = P(E)$. Let $f: P \to X$ be the associated projective bundle. Then the Hodge (p, p) conjecture is true for X if and only if it is true for P.*

Proof. Let h be the relative ample class $\mathcal{O}_P(1)$, and $\hat{h} = c_1(\mathcal{O}_P(1))$. Then we have the well-known decompositions of the Chow groups and cohomology groups of P, and we have the diagram:

$$\begin{array}{ccccccc} C^p(P) & = & f^*C^p(X) & \oplus & hf^*C^{p-1}(X) & \oplus \cdots \oplus & h^e f^* C^{p-e}(X) \\ \downarrow \lambda_P^p & & \downarrow \lambda_X^p & & \downarrow \lambda_X^{p-1} & & \downarrow \lambda_X^{p-e} \\ H^{2p}(P) & = & f^*H^{2p}(X) & \oplus & \hat{h} f^* H^{2p-2}(X) & \oplus \cdots \oplus & \hat{h}^e f^* H^{2p-2e}(X) \end{array}$$

From this diagram, the proof follows easily, noting the fact that f^* is an injection both on cycles and cohomology.

Lemma 1.3. *Let X be a smooth projective variety, $Y \hookrightarrow X$ a smooth closed subvariety of codimension r; let $U \hookrightarrow X$ be $X - Y$, i (resp. j) the inclusion of Y (resp. U) in X. Then we have the following commutative diagram:*

$$\begin{array}{ccccccc} C^{q-r}(Y) & \xrightarrow{i_*} & C^q(X) & \xrightarrow{j_*} & C^q(U) & \to 0 \\ \downarrow \lambda_Y & & \downarrow \lambda_X & & \downarrow & \\ H^{2q-2r}(Y) & \xrightarrow{\text{Gysin}} & H^{2q}(X) & \longrightarrow & H^{2q}(U) & \end{array}$$

Proof. This follows from the existence of the Gysin map i_* which is functorial with respect to the class map λ. (cf. J Milne, Etale Cohomology, Proposition 9.3, Ch. VI).

DEFINITION 1.4

Let $J^p(X)$ be the pth Griffiths-intermediate Jacobian of X based on $H^{2p-1}(X)$ ([G], [Z-2]) and let

$$\theta^p: A^p(X) \to J^p(X)$$

be the Abel–Jacobi map on codimension p-cycles algebraically equivalent to 0.

We say, X has the Abel–Jacobi property for p, if θ^p is surjective.

Lemma 1.5. *Let X be a smooth projective variety and E a vector bundle of rank $r = e+1$ on X. Let $P = \mathbb{P}(E)$ be the associated projective bundle and $f: P \to X$ the projection. Then $A^i(X)$ has the Abel–Jacobi property for all i if and only if $A^p(P)$ has it for all p.*

Proof. Let $\theta^p: A^p(X) \to J^p(X)$ be the Abel–Jacobi map. Then by assumption, θ^p is surjective for all p. By the standard decomposition theorems for Chow groups, and cohomology of a projective bundle we have

$$H^{2p-1}(P) \simeq f^*H^{2p-1}(X) \oplus \sum_{i=1}^{e} f^*H^{2p-2i-1}(X)\hat{h}^i \qquad (*)$$

$$\hat{h} = c_1(\mathcal{O}_P(1)).$$

We note that this decomposition is true for cohomology with integer coefficients, further, since $\hat{h}$ is of type $(1,1)$, the isomorphism $(*)$ preserves the Hodge decomposition. Hence, the complex structure on the Griffiths Jacobian on P is canonically isomorphic to the one induced by $(*)$. Therefore, one has

$$J^p(P) \simeq J^p(X) \oplus J^{p-1}(X) \oplus \cdots \oplus J^{p-e}(X).$$

Further, one has a similar decomposition for the Chow groups

$$A^p(P) \simeq A^p(X) \oplus A^{p-1}(X) \oplus \cdots \oplus A^{p-e}(X).$$

Combining this with the functoriality of the Abel–Jacobi maps, we get

$$\theta^p: A^p(P) \to J^p(P)$$

is surjective, since it is so in all the terms in the decomposition. The proof of the converse is similar.

2. Normal functions

Let $f: X \to S$ be a proper smooth morphism, with X, a smooth projective variety, and S a non-singular complex curve. In this section, we prove a theorem which under some very strong assumptions on the fibres of f give the Hodge (p,p) conjecture for X. The basic ideas in this theorem come from the work of Griffiths and Zucker ([Z-1], [Z-2], [Z-3], [Z-4]).

Theorem 2.1. *Let $f: X \to S$ be as above. Let $X_s = f^{-1}(s) \forall s \in S$. Suppose that the following conditions hold:*

(a) *Hodge (p,p) and Hodge $(p-1, p-1)$ are true for $X_s \forall s \in S$.*

(b) *X_s has the Abel–Jacobi property in codimension p, i.e. the map*

$$\theta^p: A^p(X_s) \to J^p(X_s)$$

is surjective $\forall s \in S$.

Then Hodge (p,p) holds for X.

Proof. Consider the Leray filtration $\{L^p\}$ on $H^*(X)$ associated to the morphism f. Since the spectral sequence degenerates (cf. [G]), we have:

$$L^0 \supset L^1 \supset L^2.$$

We need the following description of the Leray filtration from ([Z-3], pp. 194):

$$L^1 = \ker\{H^{2p}(X) \to H^{2p}(X_s)\}$$

$$L^2 = \ker\{H^{2p}(X) \to H^{2p}(X - X_s)\}$$

$$= \mathrm{Im}\{H^{2p-2}(X_s) \xrightarrow{\text{Gysin}} H^{2p}(X)\} \quad \text{(cf. Lemma 1.3)}$$

for any $s \in S$, and

$$L^0/L^1 \simeq H^0(S, R^{2p}f_*\mathbb{Q}).$$

We need to handle the (p,p) classes in the rational cohomology of X, which come from the various parts of the Leray filtration.

The *primitive class* i.e. the (p,p) classes lying in L^1 can be dealt with as follows:

(i) Observe firstly that $L^1/L^2 \simeq H^1(S, R^{2p-1}f_*\mathbb{Q})$. Integral (p,p) classes in L^1/L^2, thus arise as cohomology classes of normal functions i.e. holomorphic sections of the intermediate Jacobian bundle, $J^p(X_s) \to S$. This is a consequence of Theorem 2.13 of [Z-4]. Our assumption (b) then ensures by [Z-1], that this normal function comes from a relative algebraic cycle on X.

(ii) (p,p) classes which lie in L^2: Note that

$$L^2 = \mathrm{Im}\{H^{2p-2}(X_s) \xrightarrow{\text{Gysin}} H^{2p}(X)\}$$

and by assumption (a) and Lemma 1.3 of §1, since Hodge $(p-1, p-1)$ holds for X_s, (p,p) classes in L^2 come from algebraic cycles.

Now for the remaining classes, in L^0/L^1, let γ be a (p,p) class in $H^{2p}(X)$, which restricts to non-zero classes γ_s on X_s for all $s \in S$. Let $\sum^d_{X/S}$ denote the Chow variety (or reduced Hilbert scheme) of relative codimension p cycles of degree d on X. By the theory of Hilbert schemes, for some $d \gg 0$, the natural morphism

$$\phi_d : \sum_{X/S}^{d} \to S$$

is a surjection. Hence for all $\lambda \geqslant 1$, $\sum^{\lambda d}_{X/S} \to S$ is surjective.

Let $V^{\lambda d}$ be the non-empty open subset of S for all $\lambda \geqslant 1$, such that

$$\phi_{\lambda d} : \phi^{-1}_{\lambda d}(V^{\lambda d}) \to V^{\lambda d}$$

is *flat*. (Such a non-empty $V^{\lambda d}$ exists since ϕ_α is a proper surjective morphism.) By a Baire argument, it is easy to see that $\bigcap_{\lambda \geqslant 1} V^{\lambda d} \neq \phi$; choose an $s \in \bigcap_{\lambda \geqslant 1} V^{\lambda d}$ and fix this s. Consider $\gamma|_{X_s} = \gamma_s$; then by (a) of Theorem 2.1, since Hodge (p,p) is true for X_s, express $\gamma_s = \alpha_s - \beta_s$, where α_s and β_s are effective codim p-cycles on X_s of degree l and m respectively. Since we are interested only in rational cohomology, we may assume, without loss of generality that l and m are multiples of d.

Therefore, by choice $s \in V^l \cap V^m$, and $\alpha_s \in \phi_l^{-1}(s)$. Since ϕ_l is flat over V^l, all irreducible components of $\phi_l^{-1}(V^l)$ dominate V^l (S being a smooth curve). Choose an irreducible

component of $\phi_l^{-1}(V^l)$ which contains α_s. Then it is easy to see, (by choosing a curve C through α_s and taking its closure in $\sum^l_{X/S}$), that we get a curve S' and a finite morphism $S' \to S$, such that (we could assume S' is also smooth without loss of generality by going to the normalization if need be).

$$\begin{array}{ccc} \Sigma' & \to & \Sigma^l \\ \downarrow & & \downarrow \\ S' & \to & S \end{array}$$

and there is a section for Σ' over S', which passes through α_s. That is if

$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

then, there exists an effective codimension p-cycle α of degree l on X', such that $\alpha'_{X_{s'}} = \alpha_s$; where $s' \to s$. We can similarly get a β of deg m over another finite extension, and we can therefore get T, a smooth curve, with a finite morphism

$$T \to S,$$

such that

$$\begin{array}{ccc} Y & \xrightarrow{\mu} & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \\ t & \longmapsto & s \end{array}$$

and α and β give codimension p-cycles on Y of degree l and m respectively, s.t.

$$\alpha|_{Y_t} = \alpha_s, \quad \beta|_{Y_t} = \beta_s. \tag{*}$$

Thus,

$$\varepsilon = [\mu^*\gamma - (\alpha - \beta)] \in H^{2p}(Y, \mathbb{Q})$$

is a cohomology class which (by (*)) lies in

$$\ker (H^{2p}(Y) \to H^{2p}(Y_t)).$$

Hence ε is a *primitive cohomology* class on Y; observe that fibres of $Y \to T$ are the same as those of $X \to S$, and hence the hypotheses of Theorem 2.1, hold for the fibres of $Y \to T$ as well. So by the first part of our proof, ε comes from a codimension p-algebraic cycle ε' on Y, i.e.

$$\mu^*\gamma - (\alpha - \beta) = \varepsilon' \Rightarrow \mu^*\gamma = \varepsilon' + (\alpha - \beta)$$

is algebraic. Since $\mu: Y \to X$ is a proper finite surjection, by Lemma 1.1, it follows that γ itself is algebraic.

3. A pencil-type construction for moduli

In the discussion that follows, we describe a pencil-type construction in the context of moduli spaces of vector bundles. We remark that, in general, the geometry of a general

hyperplane section in the moduli space is not very transparent and so the usual theory of normal functions and Lefschetz pencil cannot be applied in this setting. We begin by proving a lemma which is essential in the construction.

Lemma 3.1. *Let W be a stable vector bundle of rank 2 and degree 3. Let V be a non-split extension*

$$0 \to \mathcal{O} \to V \xrightarrow{v} W \to 0.$$

Then V is semi-stable.

Proof. This is an elementary consequence of Propositions 4.3, 4.4. and 4.6 of [N-S]. To see this, suppose that V is not semistable, then by Proposition 4.6 there exists an F, stable of rank $\leqslant 2$ such that

$$\mu(F) \geqslant \mu(V) = 1$$

and a non-zero element $f \in \mathrm{Hom}(F, V)$. Thus $\mu(F) \geqslant \mu(W)$. Thus $v \circ f \in \mathrm{Hom}(F, W)$. If $v \circ f$ is zero, f must factor through $\mathcal{O}$ which gives an immediate contradiction. If $v \circ f$ is non-zero, by Proposition 4.4 of [N-S], if W_1 is the subbundle of W generated by $\mathrm{Im}(v \circ f)$ then $\mu(W_1) \geqslant \mu(W)$.

Since W is stable, it implies $W_1 \simeq W$ and $v \circ f$ is an isomorphism, which gives a splitting for v, q.e.d.

Let $M_L = M(3, L)$, be the moduli space of semi-stable bundles of rank 3, deg $3g - 3$, $\wedge^n V \simeq L$, g being the genus of C, i.e. $\deg(L) = 3g - 3$, $g = 2$.

Consider the Θ-divisor in M_L which is defined as follows:

$$\Theta = \{V \in M_L | h^0(V) > 0\}.$$

More generally, we can define for all $\xi \in J^0(C)$, the divisor

$$\Theta_\xi = \{V \in M_L | h^0(V \otimes \xi) > 0\}.$$

Let $\mathcal{W}_\xi$ be the universal family on $C \times M(2, \xi \otimes L)$ and consider the bundle of extensions given by

$$P_\xi = \mathbb{P}(R^1 p_* \mathcal{W}_\xi^*),$$

where $p: C \times M(2, \xi \otimes L) \to M(2, \xi \otimes L)$. Observe that, if $W \in M(2, \xi \otimes L)$, then the points of P_ξ lying above W are given by non-split extensions

$$0 \to \mathcal{O} \to V \to W \to 0. \tag{1}$$

By Lemma 3.1, we see that bundles V obtained above are semistable. Thus we can define a morphism

$$\phi_\xi : P_\xi \to M(3, L)$$
$$V \mapsto V \otimes (\xi^{1/3})^*.$$

Note that since $\det V \simeq \xi \otimes L$, $\det(V \otimes (\xi^{1/3})^*) = L$. Also this map is well-defined since P_ξ parameterizes a universal family and $M(3, L)$ has the coarse moduli property.

It is easy to see that $\mathrm{Im}\, \phi_\xi \subset \Theta_\eta$ (when $\eta = \xi^{1/3}$). Further, by ([S] Theorem IV, 2.1), the component of $\mathrm{Im}\, \phi_\xi$ in θ_η is of codimension at least 2 (in general for rank n it is $n - 1$)

and therefore contains a non-empty open subset of Θ_η, hence by the properness of ϕ_ξ,

$$\operatorname{Im} \phi_\xi = \Theta_\eta$$

(in fact by [S], ϕ_ξ is birational).

The above construction of P_ξ can be globalized as follows:

Let $M(2,3)$ be the moduli space of vector bundles of rank 2 and degree 3. Let $\mathscr{W} \to C \times M(2,3)$ be the universal family. Define $P = \mathbb{P}(R^1 p_* \mathscr{W}^*)$. Then the morphism ϕ_ξ globalizes to give:

$$\phi: P \to M(3, L)$$

(the ambiguity of 'cube roots' can be resolved to by pulling back P by the following diagram:

$$\begin{array}{ccc} & P' & \longrightarrow & P \\ L^{-1} \otimes \det \big\downarrow & & & \big\downarrow \det \otimes L^{-1} \\ & J & \longrightarrow & J \\ & & \eta \mapsto \eta^3 & \end{array}$$

(so in fact, ϕ is well-defined as a morphism $\phi: P' \to M(3, L)$). Define P_C by the following base-change diagram:

$$\begin{array}{ccc} P_C & \to & P' \\ \pi \downarrow & & \downarrow \\ C & \hookrightarrow & J \end{array}$$

where $C \hookrightarrow J$ by mapping a base point x_0 to the fixed degree 3 line bundle L. (Note that C is in fact connected). Then ϕ induces a morphism

$$\phi: P_C \to M(3, L) = M_L.$$

We claim that ϕ is surjective. This is not hard to see since Im ϕ contains the Θ-divisor; further, one can easily get a point in $M_L - \Theta$ in Im ϕ. Now surjectivity follows from the fact that P_C and M_L are irreducible and ϕ is a proper morphism, such that Im ϕ properly contains a divisor.

Since

$$\dim P_C = \dim \Theta + \dim C = \dim M_L,$$

ϕ gives a generically finite proper surjection.

Remark 3.2. We remark that the above construction can be done for all ranks by using the construction of desingularization of the Θ-divisor in [RV]. Our variety P can be related to their $\tilde{\Theta}$ but we would not go into it here.

4. Proof of the Hodge conjecture for $M(3, \eta)$

In this section we complete the proof of the Hodge (p,p) conjecture for $M(3,\eta)$, where $\deg \eta = 1$ or 2, $g = 2$. The strategy is to relate the geometry of $M(3,\eta)$ and $M(3, L)$ by the Hecke correspondence (cf. [B]).

PROPOSITION 4.1

Let $M_L = M(3, L)$, $\deg L = 3g - 3$. *Let* $g = 2$ *and consider the moduli space* P_C *constructed in* §3. *Then Hodge* (p, p) *is true for* P_C *for all* p.

Proof. By Theorem 2.1, it is enough to prove the properties (a) and (b) in its statement for $\pi^{-1}(y)$ for all $y \in C$, where

$$\pi: P_C \to C.$$

By §3, $\pi^{-1}(y)$'s are the moduli spaces P_ξ. Since P_ξ is a projective bundle on $M(2, \xi \otimes L)$ associated to a vector bundle, to prove (a) and (b) of Theorem 2.1 for P_ξ, it is enough to check them for $M(2, \xi \otimes L)$ because of Lemma 1.2 and Lemma 1.5. Since $M(2, \xi \otimes L)$ is a 3-fold, the Hodge conjecture follows from the Lefschetz (1, 1) theorem. That $A^2(M(2, \xi \otimes L))$ has the Abel–Jacobi property follows from ([B-M] pp. 78) since $M(2, \xi \otimes L)$ is a rational 3-fold.

We could also prove the above Proposition for P_C more directly by using the following fact:

By Thaddeus [T], (cf. also [N]), we could, consider the variety obtained by blowing up the curve C embedded in a suitable projective space of extensions. It corresponds to the variety M_1 in [T]. Denote this by $M'(2, \xi \otimes L)$. Then, when $g = 2$, it is easy to see that

$$M'(2, \xi \otimes L) \to M(2, \xi \otimes L)$$

is a birational morphism. Since $M'(2, \xi \otimes L)$ also parameterizes family of vector bundles (in fact a family of pairs!), we have a variety P'_ξ, a projective bundle associated to a vector bundle on $M'(2, \xi \otimes L)$ and a birational morphism

$$P'_\xi \to P_\xi.$$

Properties (a) and (b) of Theorem 2.1 are fairly simple for P'_ξ. Now construct globally the variety P'_C such that

$$\begin{array}{ccc} P'_C & \to & P_C \\ \downarrow & & \downarrow \\ C & \to & C \end{array}$$

Observe that by Theorem 2.1, Hodge (p, p) is true for P'_C. Since $P'_C \to P_C$ is a generically finite surjection, Hodge (p, p) for P_C follows from Hodge (p, p) for P'_C, by Lemma 1.1.

Theorem 4.2. *The Hodge* (p, p) *conjecture is true for* $M(3, \eta)$, *where* $\deg \eta = 1$ *and* 2. $(g = 2)$.

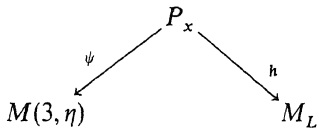
Proof. We prove it for $\deg \eta = d = 1$. Proof for $d = 2$ follows along identical lines.

Let P_x be the moduli space of parabolic stable bundles, (V, Δ), V of rank 3, $\deg 3g - 3 = 3$, $\det V \simeq L$, with parabolic structure Δ at $x \in C$ given by

$$0 \neq F^2 V_x \subset V_x.$$

$F^2 V_x$ a subspace of dim 1, and weights taken sufficiently small (cf. [B], ...). Then, we

have the Hecke correspondence



where η is a line bundle of $\deg \eta = 3g - 5 = 1$. The morphisms ψ and h are given by

$$\psi(V, \Delta) = W,$$

$$h(V, \Delta) = V$$

where W is obtained from the following exact sequence.

$$0 \to W \to V \to T \to 0,$$

T being a torsion sheaf of height 2 given by

$$T = \begin{cases} V_x/F^2V_x & \text{at } x \\ 0 & \text{elsewhere.} \end{cases}$$

Then it is known that ψ is a projective bundle associated to a vector bundle on $M(3, \eta)$ (cf. [B]) and the map h (in (*) above) is generically a projective bundle over the stable points of M_L. Therefore by Lemma 1.2, it is enough to prove the theorem for P_x.

Now P_C by construction parameterizes a universal family $\mathscr{V} \to C \times P_C$. By the definition of ϕ and h, it is easy to see that $\tilde{P}_x \simeq \mathbb{P}(\mathscr{V}_x^*)$, where $\mathscr{V}_x$ is the bundle on P_C obtained by restriction of $\mathscr{V}$ to $x \times P_C$, and $\mathscr{V}_x^*$ its dual. Thus by the coarse moduli property of parabolic bundles for P_x, we have a morphism $\tilde{\phi}\colon \tilde{P}_x \to P_x$ and the following commutative diagram:

$$\begin{array}{ccc} \tilde{P}_x & \xrightarrow{\tilde{\phi}} & P_x \\ {\scriptstyle \tilde{h}}\downarrow & & \downarrow{\scriptstyle h} \\ P_C & \xrightarrow{\phi} & M_L \end{array}$$

By Proposition 4.1, Hodge (p, p) is true for P_C and hence by Lemma 1.2, it is true for $\tilde{P}_x$. Thus by Lemma 1.1, since $\tilde{\phi}$ is a generically finite surjection, Hodge (p, p) is true for P_x for all p, which proves the theorem.

To prove it when $\deg \eta = 2$, we modify the parabolic structure by giving $F^2V_x \subset V_x$, as a subspace of dim 2 and the rest of the argument is similar.

Remark 4.3. (The Hodge (p, p) conjecture for rank 2 moduli when $g = 3, 4$).

In these cases when rank is 2, there is a geometrical picture due to Thaddeus (cf. [T]); in his notation, if $d > 2g - 2$, d being the degree, then the moduli space of stable pairs P_i, $i = (d-1)/2$, dominates $M(2, \xi)$ $d(\xi) = d$. Further, when $d = 2g - 1$, P_i, $i = (d-1)/2$, has the property that

$$\phi\colon P_i \to M(2, \xi)$$

is a birational surjection. Thus, in the case when $g = 3$, (resp. 4) $d = 5$ (resp. 7), the index, $i = 2$ (resp. 3).

Now, the variety P_2 (resp. P_3) is obtained by a sequence of blow-ups and blow-downs where the centres are smooth and Hodge conjecture is easily verified by using the

'formule-clef' which expresses the Chow ring (resp. cohomology) of the blow-up i terms of the Chow ring (resp. cohomology) of the base and the centre of the blow-up Then by Lemma 1.1, using ϕ, Hodge (p,p) follows for $M(2,\xi)$. When $g=5$, the centre blown-up are projective bundles over S^4C, the 4th symmetric power of C and henc Hodge (p,p) would follow, once it is known for S^nC, $n \geqslant 4$.

Remark 4.4. In the rank 3 case, even when $g=2$, the centres of blow-ups in any attemp at such descriptions seem much more complicated, vis-a-vis the Hodge conjecture Also, it is not clear if the centres are smooth in the first place. Our proof, which i inductive, uses the simple nature of the geometry of rank 2 moduli spaces.

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Equivariant cobordism of Grassmann and flag manifolds

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Abstract. We consider certain natural $(\mathbb{Z}_2)^n$ actions on real Grassmann and flag manifolds and S^1 actions on complex Grassmann manifolds with finite stationary point sets and determine completely which of them bound equivariantly.

Keywords. Equivariant cobordism; Grassmann manifold; flag manifold; tangential representation.

1. Introduction

Let G be a compact Lie group. A smooth G-manifold M with a given action of G will be denoted by (M, ϕ), where $\phi: G \times M \to M$ denotes the action map. An element $x \in M$ is called a stationary point if $\phi(g, x) = x$ for all $g \in G$. We shall be concerned with actions of the groups $(\mathbb{Z}_2)^n$ and S^1 with finite stationary point sets. A smooth closed n-dimensional G-manifold (M^n, ϕ) with finite stationary point set, is said to bound equivariantly if and only if there is an action (W^{n+1}, Φ) on a compact $(n+1)$-manifold for which the induced action $(\partial W^{n+1}, \Phi/\partial W^{n+1})$ is equivariantly diffeomorphic to (M^n, ϕ). Two smooth closed G-manifolds (M_1^n, ϕ_1) and (M_2^n, ϕ_2), having finite stationary point sets, are said to be unoriented G-cobordant if and only if the disjoint union $(M_1^n \cup M_2^n, \phi_1 \cup \phi_2)$ bounds equivariantly. This is an equivalence relation and the resulting set of equivalence classes is denoted by $Z_n(G)$. The equivalence class of (M^n, ϕ) is denoted by $[M^n, \phi]_2$. By disjoint union this becomes an abelian group. The cartesian product of G-manifolds with diagonal action makes the direct sum $Z_*(G) = \sum_{n \geq 0} Z_n(G)$ a graded commutative algebra. For a smooth closed oriented n-dimensional G-manifold (M^n, ϕ) (so that for every $g \in G$, $\phi_g: M \to M$, $x \mapsto gx$ is orientation preserving) having finite stationary point set, we say (M^n, ϕ) is an oriented equivariant boundary if and only if there is an action (W^{n+1}, Φ) on a compact oriented $(n+1)$-manifold as a group of orientation preserving diffeomorphism for which the induced action $(\partial W^{n+1}, \Phi/\partial W^{n+1})$ is equivariantly diffeomorphic to (M^n, ϕ) by orientation preserving diffeomorphism. We take $-(M^n, \phi)$ to be $(-M^n, \phi)$, by just reversing the orientation of M^n. Two smooth closed oriented n-dimensional G-manifolds (M_1^n, ϕ_1) and (M_2^n, ϕ_2), having finite stationary point sets, are said to be oriented G-cobordant if and only if the disjoint union $(M_1^n \cup -M_2^n, \phi_1 \cup \phi_2)$ is an oriented equivariant boundary. This is again an equivalence relation and the resulting set of equivalence classes is denoted by $\mathcal{F}_n(G)$, which is an abelian group by disjoint union. The equivalence class of (M^n, ϕ) is denoted by $[M^n, \phi]$. As in the unoriented case we have oriented G-cobordism algebra $\mathcal{F}_*(G)$. Let MO_* and MSO_* denote respectively the unoriented and oriented cobordism algebra, notations are as in [3]. We have forgetful homomorphisms $\varepsilon: Z_*(G) \to MO_*$ and $\tilde{\varepsilon}: \mathcal{F}_*(G) \to MSO_*$ given by $[M, \phi]_2 \mapsto [M]_2$ and $[M, \phi] \mapsto [M]$ respectively. It may be noted here that in [9] (Theorem 4.1), it was proved that any element $\alpha \in MSO_*$ admits

a representative M on which there exists an action of S^1 with finitely many stationary points. Thus in the case $G = S^1$, the map $\tilde{\varepsilon}$ is surjective.

The aim of this paper is to consider certain natural $(\mathbb{Z}_2)^n$ actions on real Grassmann and flag manifolds and S^1 actions on complex Grassmann manifolds with finite stationary point sets and generate elements in the kernel of ε and $\tilde{\varepsilon}$. Group actions with finite stationary point sets are particularly interesting, as in this case, the tangential representations of the group $G = (\mathbb{Z}_2)^n$, at stationary points, completely determine the equivariant cobordism class of manifolds [3]. In case $G = S^1$, although the tangential representations do not determine the equivariant cobordism class of a manifold completely, they carry lot of information about the bordism structure of the manifold. As for example, Atiyah-Singer [1] and Bott [2] have shown that if S^1 acts on an oriented compact manifold M with a finite stationary point set S, then the oriented $\mathbb{R}S^1$-modules $\{T_x M : x \in S\}$ determine the Pontrjagin numbers of M, (also cf. § 2).

For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, Conner and Floyd have described the structure of $Z_*(G)$ completely (cf. [3]). Stong and Kosniowski [4], have also derived this result from a more general consideration. They showed that $Z_*(G)$ is the polynomial algebra over $\mathbb{Z}_2$ generated by the class $[\mathbb{R}P^2, \phi]_2$, where ϕ is given by the generators T_1 and T_2 as follows. $T_1([x, y, z]) = [-x, y, z]$ and $T_2([x, y, z]) = [x, -y, z]$. In particular, the kernel of ε is trivial in this case. No neat description of $Z_*(G)$ for $G = (\mathbb{Z}_2)^n$, $n > 2$, is known. Our results show that in general the kernel of ε is nontrivial. a cobordism class $[M^d, \phi]_2 \in Z_d((\mathbb{Z}_2)^n)$ is equivariantly decomposable if (M^d, ϕ) is equivariantly cobordant to a disjoint union of products of lower dimensional manifolds with $(\mathbb{Z}_2)^n$ action with finite stationary point sets, otherwise it is equivariantly indecomposable. The first step towards understanding the structure of $Z_*(G)$ in general, would be to know the indecomposable elements in $Z_*(G)$, which may be considered as the generators. Unfortunately, there is no indecomposability criterion known in the equivariant case. Clearly if $[M]_2 \in MO_*$ is indecomposable (in the non-equivariant sense) and M admits an action of $(\mathbb{Z}_2)^n$, with finite stationary point set, then $[M, \phi]_2$ is indecomposable. But there exist some elements in the kernel of ε which are indecomposable in $Z_*((\mathbb{Z}_2)^n)$. For example, it is easy to argue that $[\mathbb{R}P^3, \phi]_2$ is indecomposable in $Z_*((\mathbb{Z}_2)^3)$, where ϕ is given by the generators as follows. $T_1([x, y, z, w]) = [-x, y, z, w]$, $T_2([x, y, z, w]) = [x, -y, z, w]$ and $T_3([x, y, z, w]) = [x, y, -z, w]$. By knowing enough elements in the kernel, perhaps it would be possible to get an idea about the indecomposable elements in general. We believe that all the elements in the kernel given by Theorem 1.1 and Theorem 1.2 are indecomposable. This motivates our study of these actions.

To determine which real flag manifolds bound, in [9] the authors gave a partial answer to this question. Real Grassmann and flag manifolds come equipped with certain natural $(\mathbb{Z}_2)^n$ actions having finite stationary point sets, to be made precise later. Although, it seems difficult to determine the unoriented cobordism class of flag manifolds, the determination of $(\mathbb{Z}_2)^n$-cobordism class of flag manifolds is easy. In the present paper, which real flag manifolds and Grassmann manifolds bound equivariantly, is completely determined. More precisely, we prove

Theorem 1.1. **(a)** *$(G_{n,k}, \phi)$ bounds equivariantly if $n = 2k$.*
(b) *$(G_{n,k}, \phi)$ does not bound equivariantly if $n \neq 2k$.*

Theorem 1.2. *$(G(n_1, n_2, \ldots, n_s), \phi)$ bounds equivariantly if and only if $n_i = n_j$ for some i, j, $i \neq j$.*

Precise definitions of the actions ϕ on Grassmann and flag manifolds are given in the subsequent sections. Perhaps, by knowing sufficiently many elements in the kernel of ε it would be possible to determine whether the unoriented $(\mathbb{Z}_2)^n$-cobordism class of flag manifolds lie in the kernel of ε or not, and that might lead to a complete answer to the question, which real flag manifolds bound? We also consider certain natural S^1-actions on complex Grassmann manifolds to produce nontrivial elements in the kernel of $\tilde{\varepsilon}$ (cf. Theorem 3.4). In this case, our result produce an infinitely many nontrivial elements in the kernel of $\tilde{\varepsilon}$. As a consequence, we deduce that for each $d>1$, $\mathscr{F}_{2d}(S^1)$ is not finitely generated as abelian group.

2. Representation and cobordism

In this section we briefly recall [3] the relation between tangential representations at stationary points and cobordism and a result of Stong.

Let G be a finite group. Let $R_n(G)$ denote the vector space over the field $\mathbb{Z}_2$, with basis the set of representation classes of degree n. The elements in $R_n(G)$ are formal sums of n-dimensional representation classes with coefficients in $\mathbb{Z}_2$. If $R_*(G)=\sum R_n(G)$, then $R_*(G)$ admits a graded commutative algebra structure with unit over $\mathbb{Z}_2$. The product is given as follows. Suppose $(V_1, G), (V_2, G)$ are representations. We take $(V_1 \oplus V_2, G)$ to be $g(v_1, v_2)=(gv_1, gv_2)$. Then the product is $(V_1, G)\cdot(V_2, G)=(V_1 \oplus V_2, G)$. The identity element is the representation class of degree 0. In fact, $R_*(G)$ is the graded polynomial ring over $\mathbb{Z}_2$ generated by the set of isomorphism classes of irreducible finite dimensional real representations of G.

Consider now an action (M^n, ϕ) with finite stationary point set S. For each $x\in S$, we have a real linear representation of G on the tangent space to M^n at x. We denote the resulting representation class by $X(x)\in R_n(G)$. Since x is an isolated stationary point, it is clear that $X(x)$ contains no trivial summand. To (M^n, ϕ) we assign the element $\sum_{x\in S} X(x)\in R_n(G)$. This element is zero in $R_n(G)$ if and only if each tangential representation class which occurs is present at an even number of stationary points. The correspondence $(M^n, \phi)\mapsto\sum_{x\in S} X(x)$ induces an algebra homomorphism $\eta: Z_*(G)\to R_*(G)$ with image $S_*(G)$. Stong [11] showed that for $G=(\mathbb{Z}_2)^n$, $Z_*(G)\cong S_*(G)$. In other words, (M_1, ϕ_1) and (M_2, ϕ_2) are G-cobordant if and only if $\sum_{x\in S_1} X(x)=\sum_{y\in S_2} X(y)$, where $\sum_{x\in S_1} X(x)$ and $\sum_{y\in S_2} X(y)$ correspond to (M_1, ϕ_1) and (M_2, ϕ_2) respectively. In particular, if $\sum_{x\in S} X(x)=0$ for (M^d, ϕ), then $[M^d, \phi]_2=0$ in $Z_d((\mathbb{Z}_2)^n)$. Thus the unoriented cobordism class $[M]_2$ of a manifold M on which there exists an action of $(\mathbb{Z}_2)^n$ with finite stationary point set S is determined by the tangential $(\mathbb{Z}_2)^n$-modules $\{T_xM: x\in S\}$.

To deal with the oriented case of S^1 action on complex Grassmann manifolds, we need an 'oriented' version of representation ring, which is briefly introduced.

Let G be a compact connected Lie group. For our purpose G will be the circle group S^1. Let V be a (finite dimensional) oriented real representation space. If $\dim_{\mathbb{R}} V>0$, then denote by $-V$ the same $\mathbb{R}G$-module but with opposite orientation on it. If V and W are oriented $\mathbb{R}G$-modules, then $V\oplus W$ is the oriented $\mathbb{R}G$-module where G acts diagonally and the orientation is the 'direct sum' orientation. We regard the 0-dimensional vector space as having a unique orientation. Then for any two oriented $\mathbb{R}G$-modules V and W, $V\oplus W\cong(-1)^{\dim V\cdot\dim W}(W\oplus V)$ as oriented $\mathbb{R}G$-modules, and if $\dim V$ and $\dim W$ are positive, $(-V)\oplus W\cong V\oplus(-W)\cong -(V\oplus W)$ as oriented $\mathbb{R}G$-modules. Note that if $\dim V$ is odd, then $V\cong -V$ as oriented $\mathbb{R}G$-modules because

$-id: V \to V$ is an orientation reversing isomorphism. It is now easy to check that for any two oriented $\mathbb{R}G$-modules V and W, $V \oplus W \cong W \oplus V$.

We now define the graded ring $\tilde{R}_*(G)$ which is the analogue in the oriented case of $R_*(G)$ defined above. For $n \geq 1$ denote by $\tilde{R}_n(G)$ the free abelian group on the isomorphism classes of oriented $\mathbb{R}G$-modules of (real) dimension n modulo the subgroup generated by elements of the form $[V] + [-V]$; $[V]$ stands for the isomorphism class of the oriented $\mathbb{R}G$-module V. $\tilde{R}_0(G)$ is defined to be the free abelian group on $[0]$, the class of the 0-dimensional $\mathbb{R}G$-module. Let $\tilde{R}_*(G) = \sum_{n \geq 0} \tilde{R}_n(G)$ and define as before $[V] \cdot [W]$ to be $[V \oplus W]$, where $V \oplus W$ is given the direct sum orientation and diagonal G action. It is straightforward to check that this gives rise to a well-defined multiplication which makes $\tilde{R}_*(G)$ a commutative graded ring with unit $[0]$. Note that $2x = 0$ for all $x \in \tilde{R}_n(G)$ if n is odd. Let B be the set of all isomorphism classes of irreducible oriented $\mathbb{R}G$-modules, and let $B_i = \{x \in B : \dim x = i \bmod 2\}$, $i = 0, 1$. Then it can be shown that

$$\tilde{R}_*(G) \cong \mathbb{Z}[B]/\langle \{2b : b \in B_1\} \rangle,$$

the quotient of the polynomial ring over integers $\mathbb{Z}$ in the variable B by the ideal generated by $\{2b : b \in B_1\}$.

Now suppose that (M^n, ϕ), $n \geq 1$ is a smooth closed oriented G-manifold with a finite stationary point set S. Let $x \in S$, then the tangent space $T_x M$ at x to M, which is an oriented vector space, is an $\mathbb{R}G$-module. Since x is an isolated stationary point $T_x M$ does not contain any trivial $\mathbb{R}G$-submodule other than 0. To (M, ϕ) we associate the element $\tilde{\eta}(M, \phi) = \sum_{x \in S} [T_x M] \in \tilde{R}_n(G)$. For a 0-dimensional manifold X, the only G-action is the trivial one. We define $\tilde{\eta}(X, trivial) = |X| \cdot [0] \in \tilde{R}_0(G)$. We now state a result, which may be well-known to the experts but an explicit reference is not known and which says that the function $\tilde{\eta}$ behaves well with respect to G-cobordism relation.

PROPOSITION 2.1

Suppose (M, ϕ) and (M', ϕ') are equivariantly cobordant as oriented G-manifolds with finite stationary points. Then $\tilde{\eta}(M, \phi) = \tilde{\eta}(M', \phi')$ in $\tilde{R}_*(G)$. ∎

The proof of the above result goes along the line of the proof of the corresponding result in unoriented case, (cf. § 32 of [3]). Thus by Proposition 2.1, we obtain a well-defined map $\tilde{\eta}: \mathcal{F}_*(G) \to \tilde{R}_*(G)$. It is straightforward to check that the map $\tilde{\eta}$ is a homomorphism of graded rings. Moreover, it can be shown that kernel of $\tilde{\eta}$ consists of elements having representatives (M, ϕ) where G acts without fixed point on M. In fact, for $G = S^1$, kernel of $\tilde{\eta}$ is precisely the inverse image of Torsion (MSO_*) under the map $\tilde{\varepsilon}$ (cf. [10]).

3. Action on Grassmann manifolds

Let $O(n)$ denote the orthogonal group of $n \times n$ matrices. The subgroup of $O(n)$ consisting of diagonal matrices can be identified with $(\mathbb{Z}_2)^n$. Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$, and T_j be the involution

$$T_j(e_i) = \begin{cases} -e_i & \text{if } i = j \\ e_i & \text{if } i \neq j \end{cases}$$

Then there exists an action of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$ given by the pairwise commuting actions of T_is. This action induces an action of $(\mathbb{Z}_2)^n$ on $G_{n,k}$, the real Grassmann manifold of k-dimensional subspaces in $\mathbb{R}^n$, and this action has finite stationary point set. A k-plane X in $\mathbb{R}^n$ is fixed by this action if and only if $X = \langle e_{i_1}, e_{i_2}, \ldots, e_{i_k} \rangle =: E_\alpha$ where $\alpha =: 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. Thus there are $\binom{n}{k}$ stationary points for this action. A Grassmann manifold $G_{n,k}$ along with this action of $(\mathbb{Z}_2)^n$ will be denoted by $(G_{n,k}, \phi)$.

In [8], [12], it was proved that $G_{n,k}$ bounds if and only if $\nu(n) > \nu(k)$ where for a positive integer n, $\nu(n)$ denotes the integer such that $2^{\nu(n)}$ divides n and $2^{\nu(n)+1}$ does not divide n. In this section, the Grassmann manifolds $(G_{n,k}, \phi)$, which bounds equivariantly, that is, $[G_{n,k}, \phi]_2 = 0$ in $Z_{k(n-k)}((\mathbb{Z}_2)^n)$ is determined completely. We need the following lemma.

Lemma 3.1. *Let G be a compact Lie group, X a closed smooth G-manifold. Let $t: X \to X$ be a smooth fixed point free involution on X such that $gt(x) = t(gx)$ for all $g \in G$. Then X bounds equivariantly. Moreover, if X is a smooth closed oriented G-manifold and $t: X \to X$ is a smooth fixed point free orientation reversing involution on X such that $gt(x) = t(gx)$ for all $g \in G$, then X is an oriented equivariant boundary.*

Proof. Let $W = X \times [-1, 1]/\sim$, where $\sim$ is given by $(x, s) \sim (t(x), -s)$. Then W is a compact manifold. An element of W is an equivalence class $[x, s]$, $x \in X$, $s \in [-1, 1]$. Define an action of G on W as follows. For $g \in G$, $[x, s] \in W$, $g[x, s] = [gx, s]$. Note that $(gt(x), -s) = (t(gx), -s) \sim (gx, s)$. Thus the above definition makes sense. Hence W is a smooth compact G-manifold with boundary and $\partial W = X \times \{-1, 1\}/\sim$ is G-diffeomorphic to X by the map $[x, s] \mapsto x$ when $s = 1$. Moreover, if X is oriented, then $(x, s) \mapsto (t(x), -s)$ is an orientation preserving fixed point free involution on $X \times [-1, 1]$ (as t is orientation reversing), hence W becomes an oriented G-manifold. ∎

Proof of Theorem 1.1. (*a*) Suppose $n = 2k$. Then if X is a k-plane in $\mathbb{R}^{2k}$, $X^\perp$, the orthogonal complement is also a k-plane in $\mathbb{R}^{2k}$. Thus $X \mapsto X^\perp$ gives a smooth fixed point free involution on $G_{2k,k}$ which is easily seen to commute with each T_j, $j = 1, 2, \ldots, n$. The result follows by Lemma 3.1.

(*b*) Suppose $k \neq n/2$. Let λ be any subset of $\{1, 2, \ldots, n\}$ consisting of k elements, that is, $|\lambda| = k$. We shall write elements of λ in increasing order. Let $e_\lambda = \{e_i : i \in \lambda\}$. Thus for each such λ there correspond a stationary point of $G_{n,k}$ which is the k-plane E_λ spanned by the vectors in e_λ. Let $\gamma_{n,k}$ be the canonical k-plane bundle over $G_{n,k}$. Then the tangent bundle $\tau G_{n,k}$ has the following description [5], $\tau G_{n,k} \cong \gamma_{n,k} \otimes \gamma_{n,k}^\perp$. Thus the tangent space at any point $X \in G_{n,k}$ is $X \otimes X^\perp$, where $X^\perp$ is the orthogonal complement of X in $\mathbb{R}^n$. Let $X_\lambda := T_{E_\lambda} G_{n,k}$ denote the tangent space at the fixed point corresponding to λ. Then the standard basis of the tangent space X_λ is given by $k(n-k)$ vectors $\{e_{i_r j} = e_{i_r} \otimes e_j\}_{r=1,2,\ldots,k}$, where $i_1 < i_2 < \cdots < i_k$ are elements of λ and $j \in \{1, 2, \ldots, n\} - \lambda$. Note that the action of $(\mathbb{Z}_2)^n$ on X_λ is given by the pairwise commuting actions of the involutions T_α, $\alpha = 1, 2, \ldots, n$, thus,

$$T_\alpha(e_{i_r j}) = \begin{cases} e_{i_r j} & \text{if } \alpha \neq i_r, j \\ -e_{i_r j} & \text{if } \alpha = i_r \text{ or } j. \end{cases}$$

These give the representation class $X(\lambda)$ of $(\mathbb{Z}_2)^n$ on X_λ. Let $\omega \subset \{1, 2, \ldots, n\}$ be given by $\omega = \{1, 2, \ldots, k\}$. We claim that the representation class $X(\omega)$ never occurs at any other

stationary point; in other words, if $\lambda \neq \omega$ then the representation of $(\mathbb{Z}_2)^n$ at X_λ is not equivalent to the representation at X_ω. Suppose, $\lambda \neq \omega$. We can choose $\alpha \in \omega$ such that $\alpha \notin \lambda$. Now a basis at X_ω is given by $\{e_{ij}\}$, $i \in \{1,2,\ldots,k\}$ and $j \in \{k+1,\ldots,n\}$, where the span of e_{ij} is a $(\mathbb{Z}_2)^n$-module for any $i \leqslant k$ and $j > k$. Thus the action of T_α on X_ω has (-1)-eigen space of dimension $n-k$, whereas the action of T_α on X_λ has (-1)-eigen space of dimension k. If there exists a $(\mathbb{Z}_2)^n$-isomorphism between X_ω and X_λ, then we must have $k = n-k$, which is impossible as $k \neq n/2$. Thus $X(\lambda)$ is distinct from $X(\omega)$, as claimed. Hence the element $\sum_\lambda X(\lambda) \in S_{k(n-k)}((\mathbb{Z}_2)^n)$ is not zero. It follows from § 2 that $[G_{n,k}, \phi]_2 \neq 0$. ■

Remark 3.2 1. The proof of part (b) actually shows that the representation classes $X(\lambda)$ and $X(\mu)$ are distinct if $\lambda \neq \mu$, $\lambda, \mu \subset \{1,2,\ldots,n\}$, as we have not made any use of the special choice ω.

2. Note that $\nu(n) > \nu(k)$ is a necessary condition for $(G_{n,k}, \phi)$ to bound equivariantly. For if $[G_{n,k}, \phi_2] = 0$ in $Z_{k(n-k)}((\mathbb{Z}_2)^n)$ then $[G_{n,k}]_2 = 0$ in $MO_{k(n-k)}$, hence $\nu(n) > \nu(k)$ by Theorem 1.1 of [8]. Moreover, note that the above theorem produces elements in the kernel of the homomorphism ε, for $[G_{n,k}, \phi]_2$ belongs to kernel of ε whenever $\nu(k) < \nu(n)$ and $k \neq n/2$.

3. In the case (a), that is when $n = 2k$, if $\lambda \subset \{1,2,\ldots,n\}$ such that $|\lambda| = k$, then $\lambda' = \{1,2,\ldots,n\} - \lambda$ has cardinality k. In this case, one can check alternatively, that $X(\lambda) = X(\lambda')$, so that each representation class $X(\lambda)$ occurs twice. As a result, $\sum_\lambda X(\lambda) = 0$. It follows from Stong's theorem that $[G_{n,k}, \phi]_2 = 0$.

Next, we consider certain natural S^1 action on complex Grassmann manifolds $\mathbb{C}G_{n,k}$. For $w \in S^1$, let $\phi_w : \mathbb{C}^n \to \mathbb{C}^n$ denote the unitary map defined by

$$\phi_w(z_1, z_2, \ldots, z_n) = (wz_1, w^2 z_2, \ldots, w^n z_n).$$

This induces an action of S^1 on the complex Grassmann manifold $\mathbb{C}G_{n,k}$ of k-dimensional complex subspaces of $\mathbb{C}^n$. Let $e_1, e_2, \ldots, e_n$ denote the standard basis of $\mathbb{C}^n$. This action of S^1 on $\mathbb{C}G_{n,k}$ has finite stationary point set and the stationary points are given by $\{\langle e_{i_1}, e_{i_2}, \ldots, e_{i_k}\rangle : 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}$, where $\langle e_{i_1}, e_{i_2}, \ldots, e_{i_k}\rangle$ is the space spanned by $\{e_{i_1}, \ldots, e_{i_k}\}$ (cf. [9], § 4). We denote this action by ϕ. We now prove

Theorem 3.3. a) *If k or $n-k$ is even then $(\mathbb{C}G_{n,k}, \phi)$ does not bound equivariantly.*
b) *If n is even and k is odd then $(\mathbb{C}G_{n,k}, \phi)$ bounds equivariantly.*

Proof. (*a*) In [7] it was proved that if k or $n-k$ is even then the signature of $\mathbb{C}G_{n,k}$ is non-zero and $[\mathbb{C}G_{n,k}]$ generates an infinite cyclic group of $MSO_{2k(n-k)}$. It follows immediately that $[\mathbb{C}G_{n,k}, \phi] \neq 0$ in $\mathscr{F}_{2k(n-k)}(S^1)$. Alternatively, one can check that $\sum_\lambda X(\lambda) \neq 0$ in $\tilde{R}_*(S^1)$, just as in 1.1, and get the result, as $\tilde{\eta} : \mathscr{F}_*(S^1) \to \tilde{R}_*(S^1)$ is a homomorphism. Here, $X(\lambda)$ denote the oriented representation class at $X_\lambda = T_{E_\lambda}\mathbb{C}G_{n,k}$.

(*b*) Let k be odd and first assume that $k = n/2$. In this case $X \mapsto X^\perp$ gives a smooth involution of $\mathbb{C}G_{n,k}$, without fixed point. This commutes with the given action of S^1, as this action preserves innerproduct. We claim that this involution is orientation reversing. To see this, note that $H^2(\mathbb{C}G_{2k,k}; \mathbb{Z})$ is generated by the first Chern class $c_1(\gamma_{2k,k})$ of the canonical k-plane bundle over $\mathbb{C}G_{2k,k}$. Let $\theta : H^*(\mathbb{C}G_{2k,k}; \mathbb{Z}) \to$

$H^*(\mathbb{C}G_{2k,k};\mathbb{Z})$ denote the isomorphism induced by $\perp:\mathbb{C}G_{2k,k}\to\mathbb{C}G_{2k,k}$. Note that the involution $\perp$ is covered by the bundle map which sends $\gamma_{2k,k}$ to $\gamma_{2k,k}^{\perp}$ and hence $\theta(c_1(\gamma_{2k,k}))=-c_1(\gamma_{2k,k})$. Now, $c_1^{k^2}(\gamma_{2k,k})\in H^{2k^2}(\mathbb{C}G_{2k,k};\mathbb{Z})$ is a non-zero element, therefore there exists a unique $\alpha\in\mathbb{Z}-\{0\}$ such that $c_1^{k^2}(\gamma_{2k,k})=\alpha\cdot u$, where u is a generator of $H^{2k^2}(\mathbb{C}G_{2k,k};\mathbb{Z})$. But as k is odd and $\theta(c_1(\gamma_{2k,k}))=-c_1(\gamma_{2k,k})$, we have $\theta(c_1^{k^2}(\gamma_{2k,k}))=(-1)c_1^{k^2}(\gamma_{2k,k})$. This implies $\theta(u)=-u$. Hence the involution $\perp$ is orientation reversing. The result follows from Lemma 3.1.

Next consider the case k is odd, n is even and $k\neq n/2$. Let $n=2m$. We regard $\mathbb{C}^n$ as an m-dimensional right $\mathbb{H}$-space, where $\mathbb{H}$ is the division ring of quaternions. If (z_1,z_2) is a pair of complex numbers then it can be considered as a quaternion z_1+z_2j. Since $jz=\bar{z}j$ for any complex number z, we have $j(z_1+z_2j)=-\bar{z}_2+\bar{z}_1j$. Write elements of $\mathbb{C}^n$ as $(z_1,z_2,\ldots,z_n)$ with respect to the basis $e_1,e_{2m},e_2,e_{2m-1},\ldots,e_m,e_{m+1}$ and consider $\mathbb{C}^n$ as the m-tuple of quaternions $\mathbb{H}^m$ with basis $e_1+e_{2m}j$, $e_2+e_{2m-1}j,\ldots,e_m+e_{m+1}j$. Then we can define a map $j:\mathbb{C}^n\to\mathbb{C}^n$ by $j(z_1,z_2,\ldots,z_n)=(-\bar{z}_2,\bar{z}_1,\ldots,-\bar{z}_n,\bar{z}_{n-1})$. Note that j is conjugate linear and hence if X is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ then $j(X)$ is again a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$. Moreover, we have $j^2=-id$. Thus j induces an involution J on $\mathbb{C}G_{n,k}$, clearly J is a smooth involution on $\mathbb{C}G_{n,k}$. We claim that J is a fixed point free involution. For suppose, $J(X)=X$, $X\in\mathbb{C}G_{n,k}$. Then X is a left $\mathbb{H}$-space and $J^2=id$, so $\dim_{\mathbb{C}}X=k$ must be even, as $\dim_{\mathbb{H}}X=(1/2)\dim_{\mathbb{C}}X$; which is a contradiction as k is odd by our assumption. Next, we claim that the action ϕ on $\mathbb{C}G_{n,k}$ commutes with J. To see this, note that for each e_i and $w\in S^1$, $\phi_w(e_i)=w^ie_i$. Thus if $(z_1,z_2,\ldots,z_{2m})$ is the coordinate of a point in $\mathbb{C}^n$ with respect to the basis $e_1,e_{2m},e_2,\ldots,e_m,e_{m+1}$, then

$$w^{2m+1}j\phi_w(z_1,z_2,\ldots,z_{2m})=\phi_w j(z_1,z_2,\ldots,z_{2m}).$$

Hence the induced maps J and ϕ_w on $\mathbb{C}G_{n,k}$ commutes with each other for each $w\in S^1$. Next, we show that the involution J is orientation reversing. Since $\mathbb{C}G_{n,k}$ is path-connected, it is enough to check it at one point. Note that the orientation of $\mathbb{C}G_{n,k}$ as a real manifold is given by the orientation of each k-plane in $\mathbb{C}^n$ considered as an oriented real vector subspace of $\mathbb{R}^{2n}$, with the standard orientation on $\mathbb{R}^{2n}$. The oriented real basis of $\langle e_{i_1},e_{i_2},\ldots,e_{i_k}\rangle$ is $\{e_{i_1},\ldots,e_{i_k},\sqrt{-1}e_{i_1},\ldots,\sqrt{-1}e_{i_k}\}$, where $1\leqslant i_1<i_2<\cdots<i_k\leqslant n$. Moreover note that $j(e_r)=e_{n+1-r}$; hence $J(\langle e_{i_1},\ldots,e_{i_k}\rangle)=\langle e_{n+1-i_1},\ldots,e_{n+1-i_k}\rangle$. From this one can show that J is orientation reversing, as k is odd. The result now follows again by Lemma 3.1. ∎

It is proved in [9] that $\mathbb{C}G_{n,k}$ is an oriented boundary if n is even and k is odd. Thus the above action does not give any non-trivial element in the kernel of $\tilde{\varepsilon}$. However, we can perturb the above action of S^1 on $\mathbb{C}G_{n,k}$ in a suitable way to generate infinitely many elements in the kernel of $\tilde{\varepsilon}$. Before we do that, let us have a close look at the representation class of $(\mathbb{C}G_{n,k},\phi)$ in the case n is even and k is odd. Since $\tilde{\eta}$ is a homomorphism, it is clear from the above theorem that $\tilde{\eta}([\mathbb{C}G_{n,k},\phi])=0$. Let us establish this, alternatively, by analysing the tangential representations at stationary points. This description will be useful in proving the next theorem. The tangent bundle of $\mathbb{C}G_{n,k}$ has the following description [5]: $\tau\mathbb{C}G_{n,k}\cong\bar{\gamma}_{n,k}\otimes\gamma_{n,k}^{\perp}$, where $\gamma_{n,k}$ is the canonical k-plane bundle, $\gamma_{n,k}^{\perp}$ its orthogonal complement and $\bar{\gamma}_{n,k}=\mathrm{Hom}_{\mathbb{C}}(\gamma_{n,k},\mathbb{C})$ is its conjugate. Let $\lambda=\{r_1,r_2,\ldots,r_k\}\subset\{1,2,\ldots,n\}$ and E_λ be the stationary point corresponding to λ. Let X_λ be the tangent space at E_λ. Then $X_\lambda=\langle\bar{e}_{r_1},\ldots,\bar{e}_{r_k}\rangle\otimes\langle e_j:j\neq r_1,r_2,\ldots,r_k\rangle$, where $\{\bar{e}_{r_1},\ldots,\bar{e}_{r_k}\}$ is a basis of $\mathrm{Hom}_{\mathbb{C}}(E_\lambda;\mathbb{C})$. Note that for each $w\in S^1$, $\phi_w(e_j)=w^je_j$ and the induced action on $\bar{e}_j$ is $\phi_w(\bar{e}_j)=w^{-j}\bar{e}_j$. Note that a natural

complex basis of X_λ is given by $\bar{e}_i \otimes e_j$, $i \in \lambda$, $j \notin \lambda$, written in dictionary ordering with respect to the subscripts. In fact, $\{\bar{e}_i \otimes e_j, i \in \lambda, j \notin \lambda\}$, forms a basis of eigen vectors for $\phi_w : X_\lambda \to X_\lambda, w \in S^1$. Clearly, the complex representation of S^1 at X_λ is the sum of 1-dimensional irreducible complex representations of S^1 with corresponding eigen values w^{j-r}. Note that since n is even and k is odd, the number of stationary points is even, moreover if $\lambda = \{r_1, r_2, \ldots, r_k\}$, then $\lambda' = \{n+1-r_1, \ldots, n+1-r_k\}$ is distinct from λ. It is now easy to check that, the assignment

$$\bar{e}_r \otimes e_j \mapsto \bar{e}_{n+1-r} \otimes e_{n+1-j}, \quad r \in \lambda, j \notin \lambda,$$

extends to a conjugate linear isomorphism between X_λ and $X_{\lambda'}$, which preserves the group action. Since $\dim_{\mathbb{C}} X_\lambda$ is odd, it follows that there is an orientation reversing S^1-equivariant isomorphism $X_\lambda \cong X_{\lambda'}$. Consequently, according to our definition of $\tilde{R}_*(G)$, $[X_\lambda] + [X_{\lambda'}] = 0$. Since $\lambda \subset \{1, \ldots, n\}$ is arbitrary, it follows that $\tilde{\eta}(\mathbb{C}G_{n,k}, \phi) = 0$.

Next, we consider a different action of S^1 as follows. We choose distinct integers $v_1, v_2, \ldots, v_n$ such that $|v_i - v_j| \neq |v_k - v_l|$ for any $i \neq j$, $k \neq l$ and $\{i, j\} \neq \{k, l\}$. For each $w \in S^1$ define $\psi_w : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\psi_w(z_1, z_2, \ldots, z_n) = (w^{v_1} z_1, w^{v_2} z_2, \ldots, w^{v_n} z_n).$$

As before, this induces an action ψ of S^1 on $\mathbb{C}G_{n,k}$. We claim that this action of S^1 has finite number of stationary points of $\mathbb{C}G_{n,k}$. Since $\psi_w(e_i) = w^{v_i} e_i$, it is clear that for any $\lambda = \{r_1, r_2, \ldots, r_k\} \subset \{1, 2, \ldots, n\}$, $E_\lambda = \langle e_{r_1}, \ldots, e_{r_k} \rangle$ is a stationary point. We shall show that these are the only stationary points of this action. Let X be a k-dimensional subspace of $\mathbb{C}^n$ such that $\psi_w(X) = X$ for all $w \in S^1$. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for X. Write each v_i as a linear combination $\sum_{a_i e_i}$ of the canonical basis vectors. Let $\lambda = \{i_1, i_2, \ldots, i_l\} \subset \{1, 2, \ldots, n\}$ be such that e_{i_r}, $i_r \in \lambda$, appears in the representation of v_j as above for at least one j. Clearly, $l = |\lambda| \geqslant k$. If we show that e_{i_r} belongs to X for each $i_r \in \lambda$, then it will follow that $l = k$ and $X = \langle e_{i_1}, \ldots, e_{i_k} \rangle$. So let $v = v_i$ and $v = \sum_{r=1}^{l} a_r e_{i_r}$, we may assume without any loss of generality that $a_r \neq 0$ for each $r = 1, 2, \ldots, l$. Since $\psi_w(X) = X$ for all $w \in S^1$ and $v \in X$, $\psi_w(v) = \sum_{r=1}^{l} a_r w^{v_{i_r}} e_{i_r} \in X$. We may choose $w_1, w_2, \ldots, w_l \in S^1$ such that $\det W \neq 0$, where W is the $l \times l$ matrix, $W = (w_s^{v_{i_r}})$. In fact, $\det W =$ Vandermonde determinant $\times$ a certain Schur function and we can choose $w_1, w_2, \ldots, w_l$, algebraically independent over Q (the field of rationals) so that the Schur function is never zero, (cf. [6]). Set $u_j = \psi_{w_j}(v) = \sum_{r=1}^{l} a_r w_j^{v_{i_r}} e_{i_r} \in X$, $1 \leqslant j \leqslant l$. Then we have an $l \times l$ matrix $(\alpha_{rs}) = (a_r w_s^{v_{i_r}})$. Clearly, $\det(\alpha_{rs}) = a_1 \cdot a_2 \cdots a_l \det W \neq 0$. Since $\det(\alpha_{rs}) \neq 0$, it is now straightforward to check that for each $i_r \in \lambda$, there exist $\beta_1, \beta_2, \ldots, \beta_l$, not all zero, such that $e_{i_r} = \sum_{j=1}^{l} \beta_j u_j$. Thus $e_{i_r} \in X$. Therefore, the action ψ on $\mathbb{C}G_{n,k}$ has finite stationary point set. We now prove with ψ as above,

Theorem 3.4. *$[\mathbb{C}G_{n,k}, \psi] = 0$ in $\mathscr{F}_{2k(n-k)}(S^1)$ if $n = 2k$ and k is odd and $[\mathbb{C}G_{n,k}, \psi] \neq 0$ in $\mathscr{F}_{2k(n-k)}(S^1)$ otherwise.*

Proof. If k or $n-k$ is even or if $n = 2k$ and k is odd, then the proof is same as the corresponding cases of 3.3. So we assume that n is even, k is odd and $k \neq n/2$. Since $\tilde{\eta} : \mathscr{F}_*(S^1) \to \tilde{R}_*(S^1)$ is a homomorphism, it is enough to prove that $\tilde{\eta}(\mathbb{C}G_{n,k}, \psi) \neq 0$. Let $\lambda \subset \{1, 2, \ldots, n\}$ and E_λ be the corresponding stationary point. Then from the discussion following Theorem 3.3, it is clear that the complex representation at X_λ is decomposed into irreducible 1-dimensional complex representations characterized by the corresponding eigen values $w^{v_j - v_i}$, $i \in \lambda, j \notin \lambda$. If $\lambda' \subset \{1, 2, \ldots, n\}$ is distinct from λ then we can always

choose $p \in \lambda'$, $q \notin \lambda'$ such that for every $i \in \lambda, j \notin \lambda$, $\{p,q\} \neq \{i,j\}$. By our choice $|v_q - v_p| \neq |v_j - v_i|$. Therefore, unlike the previous action, there does not exist an S^1-equivariant orientation reversing isomorphism between the $\mathbb{R}S^1$-modules X_λ and $X_{\lambda'}$. As a result, there will be no cancellation. Hence $\tilde{\eta}(\mathbb{C}G_{n,k}, \psi) \neq 0$. ∎

Remark 3.5. By Theorem 3.1(ii) of [9], $[\mathbb{C}G_{n,k}] = 0$ if n is even and k is odd. Therefore Theorem 3.4 implies that $[\mathbb{C}G_{n,k}, \psi]$ belongs to kernel of $\tilde{\varepsilon}$ whenever n is even, k is odd and $k \neq n/2$. It is also interesting to note that in the case when n is even, k is odd and $k \neq n/2$, we can choose integers $v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n$, in an arbitrary way, satisfying the mentioned condition so that $[\mathbb{C}G_{n,k}, \psi] \neq [\mathbb{C}G_{n,k}, \psi']$, where ψ' is same as ψ, replacing v_i by v'_i. Thus we have infinitely many nontrivial elements in the kernel of $\tilde{\varepsilon}$.

For any $n \geq 3$ we can choose a sequence $\{v^r\}$ of finite sequences $v^r = \{v^r_1, v^r_2, \ldots, v^r_n\}$ of length n so as to satisfy $|v^r_i - v^r_j| \neq |v^s_p - v^s_q|$, for $i \neq j$, $p \neq q$ and $\{i,j\} \neq \{p,q\}$ and for any r and s (including the case $r = s$) and $|v^r_i - v^r_j| \neq |v^s_i - v^s_j|$ for any $r, s, r \neq s$, and $i, j, i \neq j$. For instance, choose natural numbers $p_1, p_2, \ldots, p_r, \ldots, p_1 > 1$, $p_r > p^n_{r-1}$ for $r \geq 2$, and set $v^r = \{p_r, p^2_r, \ldots, p^n_r\}$. Now for $k < n$, let ψ_r denote the action of S^1 on $\mathbb{C}G_{n,k}$ defined by v^r as above. We exclude the case when $n = 2k$ and k is odd. Then for any such choice of $\{v^r\}$, $[\mathbb{C}G_{n,k}, \psi_r] \neq [\mathbb{C}G_{n,k}, \psi_s]$ for $r \neq s$, as mentioned in the above remark and moreover, any finite number of these classes $[\mathbb{C}G_{n,k}, \psi_r], r \geq 1$, are linearly independent over $\mathbb{Z}$. This can be seen easily by applying the homomorphism $\tilde{\eta}$ and comparing the monomials in $\mathbb{Z}[B_0]$ (cf. § 2 and note that all irreducible real representations of S^1 are 2-dimensional). In particular, we can take $k = 1$ and $d > 1$ and consider $\mathbb{C}G_{d+1,1}$ so that $\dim_{\mathbb{R}} \mathbb{C}G_{d+1,1} = 2d$. This yields,

Theorem 3.6. *For any $d > 1$, rank $\mathscr{F}_{2d}(S^1)$ is not finite.* ∎

4. Action on flag manifolds

Let $G(n_1, n_2, \ldots, n_s), n = n_1 + n_2 + \cdots + n_s, s \geq 3$ denote the real flag manifold of all flags $(A_1, A_2, \ldots, A_s)$ where A_i is a left vector subspace of $\mathbb{R}^n$, $A_i \perp A_j$, for $i \neq j$, $\dim_{\mathbb{R}} A_i = n_i$, $1 \leq i, j \leq s$, $G(n_1, n_2, \ldots, n_s)$ is a smooth manifold of dimension $\sum_{1 \leq i < j \leq s} n_i n_j$. Alternatively, it can be described as the homogeneous space $O(n)/O(n_1) \times \cdots \times O(n_s)$. The group $(\mathbb{Z}_2)^n$ acts on $G(n_1, n_2, \ldots, n_s)$ by pairwise commuting involutions T_α, $\alpha = 1, 2, \ldots, n$, having finite stationary point set. This action is induced from the actions of T_αs on $\mathbb{R}^n$ as described in the last section. The number of stationary points of the action of $(\mathbb{Z}_2)^n$ on $G(n_1, n_2, \ldots, n_s)$ is $n!/n_1! \cdots n_s!$. We denote this action by $(G(n_1, n_2, \ldots, n_s), \phi)$.

Proof of Theorem 1.2. Suppose $n_i = n_j$ for some $i \neq j$. In this case there exists an obvious smooth fixed point free involution which interchanges the ith and the jth component of each flag in $G(n_1, n_2, \ldots, n_s)$ which is easily seen to commute with each T_α. Hence by Lemma 3.1 $[G(n_1, n_2, \ldots, n_s), \phi]_2 = 0$.

Next suppose that $n_i \neq n_j$ for $i \neq j$. We may without loss of generality always write n_i in increasing order.

$$\left.\begin{array}{l} \text{Let } \lambda = (\lambda^1, \lambda^2, \ldots, \lambda^s) \text{ be a partition of } \{1, 2, \ldots, n\}, \\ \qquad \text{where the subset } \lambda^i \text{ has cardinality } n_i. \\ \text{We shall write elements of } \lambda^i \text{ in increasing order.} \end{array}\right\} \quad (1)$$

Let $e_{\lambda^i} = \{e_k : k \in \lambda^i\}$. Then the fixed points of $G(n_1, n_2, \ldots, n_s)$ are

$$\{(\langle e_{\lambda^1}\rangle, \langle e_{\lambda^2}\rangle, \ldots, \langle e_{\lambda^s}\rangle) : \text{for all partition } \lambda = (\lambda^1, \lambda^2, \ldots, \lambda^s) \text{ as stated in (1)}\},$$

where $\langle e_{\lambda^i}\rangle = E_{\lambda^i}$ (say) is the space spanned by e_{λ^i}. Thus for each λ as stated in (1) there exists a fixed point of $(G(n_1, n_2, \ldots, n_s), \phi)$ and as before, we shall denote by X_λ the tangent space to $G(n_1, n_2, \ldots n_s)$ at the stationary point corresponding to λ. Then by [5]

$$X_\lambda = \oplus_{1 \leqslant i < j \leqslant s} E_{\lambda^i} \otimes E_{\lambda^j} \tag{2}$$

A basis of this is given by $\{e_{\lambda^i \lambda^j}, 1 \leqslant i < j \leqslant s\}$, where $e_{\lambda^i \lambda^j} = \{e_k \otimes e_l : k \in \lambda^i, l \in \lambda^j\}$. The representation of $(\mathbb{Z}_2)^n$ on X_λ is given by its action on the basis element:

$$T_\alpha(e_k \otimes e_l) = \begin{cases} -e_k \otimes e_l & \text{if } \alpha = k \text{ or } l \\ e_k \otimes e_l & \text{otherwise} \end{cases} \tag{3}$$

Let us now consider the partition $\omega = (\omega^1, \omega^2, \ldots, \omega^s)$, where

$$\omega^1 = \{1, 2, \ldots, n_1\},$$
$$\omega^2 = \{n_1 + 1, \ldots, n_1 + n_2\}, \ldots,$$
$$\omega^s = \{n_1 + n_2 + \cdots + n_{s-1} + 1, \ldots n_1 + \cdots + n_s\}.$$

Then

$$TE_\omega = \oplus_{1 \leqslant i < j \leqslant s} E_{\omega^i} \otimes E_{\omega^j} \tag{4}$$

We claim that if $\lambda \neq \omega$ then $X(\lambda)$ is distinct from $X(\omega)$, where $X(\lambda)$ is the representation class of $(\mathbb{Z}_2)^n$ at $X(\lambda)$. To see this, suppose $\lambda \neq \omega$. Then $\omega^i \neq \lambda^i$ for some i. Choose $\alpha \in \omega^i$ such that $\alpha \notin \lambda^i$. Let $\alpha \in \lambda^j$, $i \neq j$. Then from (3) and (4) it follows that the action of T_α on X_ω has (-1)-eigen space of dimension $n_1 + \cdots + n_{i-1} + n_{i+1} + \cdots + n_s = n - n_i$, whereas from (2) and (3) it follows that the action of T_α on X_λ has (-1)-eigen space of dimension $n_1 + \cdots + n_{j-1} + n_{j+1} + \cdots + n_s = n - n_j$. If there exist an equivariant linear isomorphism $X_\lambda \cong X_\omega$, then we must have $n - n_i = n - n_j$, that is $n_i = n_j$ for $i \neq j$, which is impossible. Thus the representation class $X(\omega)$ does not occur at any other stationary point. In other words $\sum X(\lambda) \in S_d((\mathbb{Z}_2)^n)$ is non-zero, where $d = \sum_{1 \leqslant i < j \leqslant s} n_i n_j$ is the dimension of $G(n_1, \ldots, n_s)$. Hence $[G(n_1, \ldots, n_s), \phi]_2 \neq 0$. This completes the proof. ■

Remark 4.1. (*a*) In [9] it was proved (Theorem 2.2(a)) that $[G(n_1, \ldots, n_s)]_2 = 0$ if $n_i = n_j$ for some $i \neq j$, $1 \leqslant i, j \leqslant s$, or for some $\nu(n_i) < \nu(n)$, where $\nu(n)$ is as in § 3. Thus Theorem 1.2 implies that $[G(n_1, n_2, \ldots, n_s), \phi]_2$ is a nontrivial element of kernel of ε if $n_i \neq n_j$ for $i \neq j$ and $\nu(n_i) < \nu(n)$ for some i.

(*b*) To get a complete answer to the question 'Which flag manifolds bound?' it would be enough to determine whether $[G(n_1, \ldots, n_s), \phi]_2$ belongs to kernel of ε or not, in the case when n is odd and n_is are distinct.

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Local behaviour of the first derivative of a deficient cubic spline interpolator

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Abstract. Considering a given function $f \in C^4$ and its unique deficient cubic spline interpolant, which match the given function and its derivative at mid point between the successive mesh point, we have obtained in the present paper asymptotically precise estimate for $s' - f'$.

Keywords. Local behaviour; deficient cubic spline; mid point interpolation; precise estimate.

1. Introduction

Let $P: 0 = x_0 < x_1, < \cdots < x_n = 1$ denote a partition of $[0, 1]$ with equidistant mesh points so that $h = x_i - x_{i-1} = 1/n$. Let Π_m be the set of all real algebraic polynomials of degree not greater than m. For a function s defined over $[0, 1]$ we denote the restriction of s over $[x_{i-1}, x_i]$ by s_i. The class of periodic deficient cubic splines over $[0, 1]$ with mesh P is defined by

$$S(3, P) = \{s : s_i \in \Pi_3, s \in C^1[0, 1], s^{(j)}(0) = s^{(j)}(1), \quad j = 0, 1\}.$$

Considering a nondecreasing function g on $[0, 1]$ such that $g(x + h) - g(x) = H(\text{const.}) = \int_0^h \mathrm{d}g$, $x \in [0, 1 - h]$, Rana and Purohit [4] have proved the following for deficient cubic splines:

Theorem 1. *Let $f \in C^1[0, 1]$. Then there exists a unique 1-periodic spline $s \in S(3, P)$ which satisfies the following interpolatory conditions,*

$$\int_{x_{i-1}}^{x_i} (f(x) - s(x)) \mathrm{d}g = 0, \quad i = 1, 2, \ldots, n, \tag{1.1}$$

$$s'(\theta_i) = f'(\theta_i), \quad \theta_i = (x_i + x_{i-1})/2, \quad i = 1, 2, \ldots, n. \tag{1.2}$$

It is interesting to observe that condition (1.1) reduces to different interpolatory conditions by suitable choice of $g(x)$. Thus, if g is a step function with a single jump of one at $h/2$ then condition (1.1) reduces to the interpolatory condition,

$$s(\theta_i) = f(\theta_i), \quad i = 1, 2, \ldots, n. \tag{1.3}$$

Considering a function $f \in C^4$ and its unique spline interpolant s matching at the mesh points, Rosenblatt [5] has obtained asymptotically precise estimate for $s' - f'$. For further results concerning asymptotically precise estimate for cubic spline interpolant reference may be made to Dikshit and Rana [3]. Similar to the result of Rosenblatt [5], we obtain in the present paper a precise estimate for $s' - f'$ concerning the deficient cubic spline interpolating the given function and its derivative at mid points between the successive mesh points. It may be worthwhile to mention that Boneva, Kendall and

Stefanov [2] have shown the use of derivative of a cubic spline interpolator for smoothing of histograms.

Without any loss of generality, we consider for the rest of this paper that the deficient cubic spline s under consideration satisfies the condition $s'(0) = 0$. Thus, we have from the proof of Theorem 1 that the system of equations for determining the first derivative $m_i = s'(x_i)$ of the deficient cubic spline interpolant s is written as,

$$-(m_{i+1} - 10m_i + m_{i-1})/2 = F_i, \quad i = 1, 2, \ldots, n-1 \tag{1.4}$$

where $F_i = 12\{f(\theta_{i+1}) - f(\theta_i)\}/h - 4\{f'(\theta_{i+1}) + f'(\theta_i)\}$.

2. Estimation of the inverse of the coefficient matrix

Ahlberg, Nilson and Walsh [1] have estimated precisely the inverse of the coefficient matrix appearing in the studies concerning cubic spline interpolant matching at the mesh points. Following Ahlberg *et al* we propose to obtain here a precise estimate for the inverse of the coefficient matrix in (1.4). It may be mentioned that this method permits the immediate application to the spline to standard problem of numerical analysis (see [1], p. 34). For this we introduce the following square matrix of order n.

$$D_n(a,b) = \begin{bmatrix} 2b & a & 0 & \cdots & 0 & 0 & 0 \\ a & 2b & a & \cdots & 0 & 0 & 0 \\ 0 & a & 2b & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a & 2b & a \\ 0 & 0 & 0 & \cdots & 0 & a & 2b \end{bmatrix}$$

where a and b are given real numbers such that $b^2 \geqslant a^2$. By using the induction hypothesis it may be seen easily that $|D_n|$ satisfies the following difference equation,

$$|D_n(a,b)| = 2b|D_{n-1}(a,b)| - a^2|D_{n-2}(a,b)| \tag{2.1}$$

with $|D_{-1}(a,b)| = 0, |D_0(a,b)| = 1$ and $|D_1(a,b)| = 2b$ and for $\alpha = (b^2 - a^2)^{1/2}$,

$$2\alpha|D_n(a,b)| = (b+\alpha)^{n+1} - (b-\alpha)^{n+1}, \quad b^2 > a^2$$

$$|D_n(a,b)| = (n+1)b^n, \quad \text{otherwise.} \tag{2.2}$$

Further, it may be observed that the system of eq. (1.4) may be written as

$$AM = F \tag{2.3}$$

where the coefficient matrix A is a square matrix of order $n-1$, M and F are the transposes of the matrices $[m_1, m_2, \ldots, m_{n-1}]$ and $[F_1, F_2, \ldots, F_{n-1}]$ respectively. In order to determine the inverse of the coefficient matrix A we first observe that for $a = -1/2$,

$$\beta^{-n}(2b + r)|D_n(a,b)| = 2b(1 - r^{2n}) + r(1 - r^{2n-2})/2 \tag{2.4}$$

where $-r = (2\beta)^{-1} = 2[b - (b^2 - 1/4)^{1/2}]$.

Taking $2b=5$ and $a=-1/2$ in $|D_n(a,b)|$, we observe that the coefficient matrix A satisfies the following difference equation,

$$4|A| = 20|D_{n-2}(-1/2, 5/2)| - |D_{n-3}(-1/2, 5/2)|. \tag{2.5}$$

Thus, using (2.4) in (2.5) we have

$$(5+r)\beta^{2-n}|A| = (5+r/2)^2 - r^{2n-6}(5r+1/2)^2. \tag{2.6}$$

We get the elements $a_{i,j}$ of A^{-1} from the cofactors of the transpose matrix. Thus, for $0<i\leqslant j\leqslant n-2$ or $i=j=0$ (cf. [1, pp. 35–38])

$$|A|a_{i,j} = (\beta\cdot r)^{j-i} D_i(-1/2, 5/2) D_{n-j-2}(-1/2, 5/2) \tag{2.7}$$

and

$$|A|a_{0,j} = (\beta\cdot r)^j D_{n-j-2}(-1/2, 5/2) \quad \text{for } 0<j\leqslant n-2. \tag{2.8}$$

Thus, in view of (2.4) and (2.5), we have for $0<i\leqslant j<n-2$

$$(5+r)(1-r^{2n})a_{i,j} = r^{j-i}(1-r^{2i+2})(1-r^{2n-2j-2}),$$

$$(5+r/2)(1-r^{2n})a_{i,n-2} = r^{n-2-i}(1-r^{2i+2}), \quad \text{for } 0<i\leqslant n-2,$$

$$(5+r/2)(1-r^{2n})a_{0,j} = r^j(1-r^{2n-2-2j}), \quad \text{for } 0<j<n-2,$$

and

$$(5+r/2)^2(1-r^{2n})a_{0,n-2} = r^{n-2}(5+r).$$

From the above expression, we observe that A^{-1} is symmetric. Now considering a fixed value x such that $0<x<1$, we see that for fixed $\varepsilon>0$ and $\varepsilon<i/n$, $j/n<1-\varepsilon$ the elements $a_{i,j}$ of A^{-1} may be approximated asymptotically by $r^{|j-i|}/(5+r)$.

We thus complete the proof of the following:

Theorem 2. *The coefficient matrix A of (2.3) is invertible and if $A^{-1}=(a_{i,j})$, then $a_{i,j}$ can just be approximated asymptotically by $r^{|j-i|}/(5+r)$ and the row max norm of its inverse; that is,*

$$\|A^{-1}\| \leqslant \frac{(1+r)}{(1-r)(5+r)}, \tag{2.9}$$

where $r=2\sqrt{6}-5$.

Remark 1. It is worthwhile to mention that the estimate (2.9) is sharper than that obtained in terms of the infimum of the excess of the positive value of the leading diagonal element over the sum of the positive values of other elements in each row. For adopting the latter approach, we observe from (2.3) that $\|A^{-1}\| \leqslant 0.25$ whereas (2.9) shows that the $\|A^{-1}\|$ does not exceed $1/6$.

Since A is invertible, it follows from the proof of Theorem 1 or more precisely (2.3), that there exists a unique spline $s\in S(3, P)$ satisfying the interpolatory conditions (1.2) and (1.3).

3. Error bounds

Considering a 1-periodic function $f\in C^4$ in this section of the paper we shall estimate the precise bounds of the function $e'=s'-f'$ where s is the deficient cubic spline

interpolant of a 1-periodic function f which satisfies the interpolatory conditions (1.2), (1.3). Considering the interval $[x_{i-1}, x_i]$, we see that, since s' is quadratic, hence in the interval $[x_{i-1}, x_i]$, we may write

$$h^2 s'(x) = h(x - x_{i-1})m_i + h(x_i - x)m_{i-1} + (x - x_{i-1})(x_i - x)c_i \tag{3.1}$$

where the constant c_i is to be determined. Using the interpolatory condition (1.2), we notice that,

$$4f'(\theta_i) = 2(m_i + m_{i-1}) + c_i. \tag{3.2}$$

Now applying (3.2) in (3.1), we get

$$\begin{aligned} h^2 s'(x) = {} & (x - x_{i-1})[h - 2(x_i - x)]m_i + (x_i - x)[h - 2(x - x_{i-1})]m_{i-1} \\ & + 4(x - x_{i-1})(x_i - x)f'(\theta_i). \end{aligned} \tag{3.3}$$

Thus, replacing now m_i by $e'(x_i)$ in (3.3), we have

$$\begin{aligned} h^2 s'(x) = {} & (x - x_{i-1})[h - 2(x_i - x)]e'(x_i) \\ & + (x_i - x)[h - 2(x - x_{i-1})]e'(x_{i-1}) + R_i(f) \end{aligned} \tag{3.4}$$

where $R_i(f) = (x - x_{i-1})[h - 2(x_i - x)]f'(x_i) + (x_i - x)[h - 2(x - x_{i-1})]f'(x_{i-1}) + 4(x - x_{i-1})(x_i - x)f'(\theta_i)$.

Now using the fact that $f \in C^4$, we see by Taylor's theorem that $R_i(f)$ may be expressed as a linear combination of the values of the fourth derivative $f^{(4)}$ of f. Thus,

$$R_i(f) = h^2 f'(x) + f^{(4)}(x)(x - x_{i-1})(x_i - x)(2x - x_i - x_{i-1})h^2/12 + 0(h^5) \tag{3.5}$$

where x is an appropriate point in (x_{i-1}, x_i) which is not necessarily the same at each occurrence. Rewriting (2.3) as,

$$A(e'(x_i)) = (F_i) - A(f'(x_i)) = (H_i), \tag{3.6}$$

say, we first estimate (H_i). Thus, applying Taylor's theorem again to the right hand side of (3.6), we get

$$(H_i) = -h^3 f^{(4)}(x)/6 + 0(h^3) \tag{3.7}$$

Recalling eq. (3.6) and noticing that $A^{-1} = (a_{i,j})$, we have

$$(e'(x_i)) = \left(\sum_{|k-i| \geq m} + \sum_{|k-i| < m} (a_{i,k} H_k) \right) = (R_1) + (R_2)$$

say, where m is a sufficiently large but fixed positive integer. We shall estimate R_1 and R_2 separately. Suppose that x is a fixed point in $(0, 1)$ and let $x_i = [nx]/n$ where $[nx]$ denotes the largest integer less than or equal to nx. Then it is clear that as $n \to \infty$, $i \cong nx$ and $n - i \cong n(1 - x)$. Now assuming that $f^{(4)}$ is monotonic, we get from Theorem 2

$$|(R_1)| \leqslant d_1 (1/6)^m h^3 \tag{3.8}$$

where d_1 is some positive constant.

Next, we see that the points x_k for the values of k occurring in R_2 satisfy

$$|x_k - x| = 0(h). \tag{3.9}$$

Thus, using the continuity of $f^{(4)}$ and applying the result of Theorem 2 alongwith (3.7), we have

$$\left|(R_2) - \sum_{|k-i|<m} \frac{r^{|k-i|}}{(5+r)}(-h^3 f^{(4)}(x)/6)\right| = 0(h^3). \tag{3.10}$$

Combining the estimates of (R_1) and (R_2) and noticing that m is arbitrary, we prove the following:

Theorem 3. *Let $s \in S(3, P)$ be the deficient cubic spline interpolant of a 1-periodic function f satisfying the interpolatory conditions* (1.2) *and* (1.3). *Let $f^{(4)}$ exist and be a nonnegative monotonic continuous function. Then for any fixed point x such that $0 < x < 1$,*

$$\begin{aligned} s'(x) - f'(x) = f^{(4)}(x)[3(x - x_{i-1})(x_i - x)(2x - x_i - x_{i-1}) \\ + 4h(x_i - x)(x - x_{i-1}) - h^3]/36 + 0(h^3) \end{aligned} \tag{3.11}$$

as $n \to \infty$.

Remark 2. It may be interesting to investigate the similar precise estimate for deficient cubic spline in the case of nonuniform mesh.

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On the partial sums, Cesáro and de la Valleé Poussin means of convex and starlike functions of order 1/2

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Abstract. In this paper we study certain properties of partial sums, cesáro and de la valleé Poussin means of convex and starlike functions.

Keywords. Partial sums; Cesáro; de la Valleé Poussin means.

1. Introduction

Let S denote the class of functions $f(z) = z + a_2 z^2 + \cdots$ which are regular and univalent in the unit disc $E = \{z/|z| < 1\}$. Denote by S_t and K the usual subclasses of S consisting of functions which map E onto starlike (with respect to origin) and convex domains, respectively. Let $S_t(1/2) \subset S_t$ be the class of functions which are starlike of order 1/2. It is known that $K \subset S_t(1/2)$.

For a given function $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$ and $n \in N$, let $s_n(z, f) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$,

$$v_n(z, f) = \frac{n}{n+1} z + \frac{n(n-1)}{(n+1)(n+2)} a_2 z^2 + \cdots + \frac{n(n-1)(n-2)\ldots 3\cdot 2\cdot 1}{(n+1)(n+2)\ldots(2n)} a_n z^n,$$

$$\sigma_n^{(1)}(z, f) = z + \frac{(n-1)}{n} a_2 z^2 + \frac{(n-2)}{n} a_3 z^3 + \cdots + \frac{1}{n} a_n z^n,$$

and

$$\sigma_n^{(2)}(z, f) = z + \frac{n(n-1)}{n(n+1)} a_2 z^2 + \frac{(n-1)(n-2)}{n(n+1)} a_3 z^3 + \cdots + \frac{2\cdot 1}{n(n+1)} a_n z^n$$

denote, respectively, the nth partial sum, the nth de la Valleé Poussin mean, the nth Cesáro mean of first order and the nth Cesáro mean of second order of f.

A function f is said to be subordinate to a function F (in symbols $f(z) \prec F(z)$) in $|z| < r$ if F is univalent in $|z| < r$, $f(0) = F(0)$ and $f(|z| < r) \subset F(|z| < r)$.

For every $f \in K$ the following results are well-known:

(i) $z/2 = s_1(z, f)/2 = \sigma_1^{(1)}(z, f)/2 = \sigma_1^{(2)}(z, f)/2 \prec f(z)$ in E [2];
(ii) (4/9) $s_2(z, f) \prec f(z)$ in E [10];
(iii) (2/3) $\sigma_2^{(1)}(z, f) \prec f(z)$ in E [10];
(iv) $v_n(z, f) \prec f(z)$ in E.

The fascinating result (iv) is due to Pólya and Schoenberg [6] (see also Robertson [7]).

In the present paper, we establish the analogue of the Pólya–Schoenberg theorem for a certain transformation of the nth partial sum, $s_n(z, f)$, and the nth Cesáro mean of first order, $\sigma_n^{(1)}(z, f)$, of $f \in K$. We also prove that for every $f \in S_t(1/2)$ and for every positive

integer n, $\mathrm{Re}(v_n(z,f)/\sigma_n^{(2)}(z,f))>0$, $z\in E$. An alternative and simple proof of a well-known result of Basgöze, Frank and Keogh [1] pertaining to subordination of the partial sums of convex functions is also given.

2. Preliminaries

We shall need the following definitions and results.

DEFINITION 2.1

A sequence $\{b_n\}_1^\infty$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)=z+\sum_{n=2}^\infty a_n z^n$ is regular, univalent and convex in E, we have

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z), \quad (a_1=1)$$

in E.

DEFINITION 2.2

A sequence $\{c_n\}_0^\infty$ of non-negative numbers is said to be a convex null sequence if $c_n\to 0$ as $n\to\infty$ and

$$c_0-c_1\geqslant c_1-c_2\geqslant\cdots\geqslant c_n-c_{n+1}\geqslant\cdots\geqslant 0.$$

Lemma 2.1. (*Wilf* [11]). *A sequence* $\{b_n\}_1^\infty$ *of complex numbers is a subordinating factor sequence if and only if* $\mathrm{Re}[1+2\sum_{n=1}^\infty b_n z^n]>0$, $z\in E$.

Lemma 2.2. *For all* θ, $0\leqslant\theta\leqslant\pi$,

$$\frac{1}{2}+\sum_{k=1}^{n}\frac{\cos k\theta}{k+1}\geqslant 0.$$

Lemma 2.2 is due to Rogosinski and Szegö [8].

Lemma 2.3. (*Fejér* [4]). *Let* $\{c_n\}_0^\infty$ *be a convex null sequence. Then the function*

$$q(z)=\frac{c_0}{2}+\sum_{n=1}^{\infty} c_n z^n$$

is analytic in E *and* $\mathrm{Re}\,q(z)>0$, $z\in E$

Lemma 2.4. *Let*

$$g_n(z)=\frac{(n+1)}{2}+nz+(n-1)z^2+\cdots+z^n.$$

Then $\mathrm{Re}\,g_n(z)>0$ *in* E.

Proof. In view of the minimum principle for harmonic functions, we have

$$\min_{z\in E}\mathrm{Re}\,g_n(z)=\min_{0\leqslant\theta\leqslant 2\pi}\mathrm{Re}\,g_n(e^{i\theta})$$

$$= \min_{0\leqslant\theta\leqslant 2\pi} \operatorname{Re}\left[\frac{n+1}{2} + \sum_{k=1}^{n}(n-k+1)\cos k\theta + i\sum_{k=1}^{n}(n-k+1)\sin k\theta\right]$$

$$= \min_{0\leqslant\theta\leqslant 2\pi} \operatorname{Re}\left[\frac{\sin^2[(n+1)\theta/2]}{2\sin^2(\theta/2)} + i\frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2(\theta/2)}\right], \quad \text{(p3, [5])}$$

$$> 0.$$

Lemma 2.5. *Let f and g be starlike of order* 1/2. *Then for each function F analytic in E and satisfying*

$$\operatorname{Re} F(z) > 0 \quad (z\in E),$$

we have

$$Re\frac{f(z)*F(z)g(z)}{f(z)*g(z)} > 0 \quad (z\in E).$$

Lemma 2.5 is due to Ruscheweyh and Sheil-Small [9].

3. Theorems and their proofs

Theorem 3.1. *Let $f\in K$ and let $s_n(z,f)$, $n\in N$, denote its nth partial sum. Then*

$$S_n(z,f) = \frac{1}{2}\int_0^z s_n(t,f)\,dt \prec f(z)$$

in E for every $n = 1, 2, 3\ldots$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in K. Then

$$S_n(z,f) = \frac{1}{2}z + \frac{a_2}{3}z^2 + \frac{a_3}{4}z^3 + \cdots + \frac{a_n}{n+1}z^n.$$

In view of the Definition 2.1, the desired conclusion will follow if and only if the sequence $\langle 1/2, 1/3, \ldots, 1/(n+1), 0, 0\ldots\rangle$ is a subordinating factor sequence. By Lemma 2.1, this will be the case if and only if

$$\operatorname{Re}\left(1 + 2\sum_{k=1}^{n}\frac{z^k}{k+1}\right) > 0, \quad z\in E. \tag{3.1}$$

Putting $z = re^{i\theta}$, $0\leqslant r<1$, $-\pi\leqslant\theta\leqslant\pi$ and making use of the minimum principle for harmonic functions along with Lemma 2.2, we have

$$\operatorname{Re}\left(1 + 2\sum_{k=1}^{n}\frac{z^k}{k+1}\right) = 1 + 2\sum_{k=1}^{n}\frac{r^k\cos k\theta}{k+1} > 2\min_{0\leqslant\theta\leqslant\pi}\left(\frac{1}{2} + \sum_{k=1}^{n}\frac{\cos k\theta}{k+1}\right) > 0,$$

showing that the inequality (3.1) holds and, therefore, the proof of our theorem is complete.

Taking $n = 1$, we obtain the following well-known result (also cited in the Introduction).

COROLLARY 3.1

$(1/2)z \prec f(z)$ *in* E, *for all* $f\in K$.

Theorem 3.2. *For all elements f of K and for all positive integers n, we have*

$$(n/(n+1))\,\sigma_n^{(1)}(z,f) \prec f(z)$$

in E. This result is sharp for every n.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be any element of K. Since

$$\frac{n}{n+1}\sigma_n^{(1)}(z,f) = \frac{n}{n+1}z + \frac{n-1}{n+1}a_2 z^2 + \frac{n-2}{n+1}a_3 z^3 + \cdots + \frac{1}{n+1}a_n z^n,$$

in the light of Definition 2.1, the assertion $(n/(n+1))\,\sigma_n^{(1)}(z,f) \prec f(z)$ in E will hold if and only if the sequence $\langle n/(n+1), (n-1)/(n+1), \ldots, 1/(n+1), 0, 0, \ldots \rangle$ is a subordinating factor sequence. By Lemma 2.1, we see that this is equivalent to

$$\operatorname{Re}\left[1 + \frac{2}{n+1}(nz + (n-1)z^2 + (n-2)z^3 + \cdots + z^n)\right] > 0, \quad z \in E,$$

or

$$\operatorname{Re}\left[\frac{n+1}{2} + nz + (n-1)z^2 + (n-2)z^3 + \cdots + z^n\right] > 0, \quad z \in E,$$

which is true in view of Lemma 2.4. To establish the claim regarding sharpness we consider the function $h(z) = z/(1-z)$ which is a member of K. For any positive real number ρ, we have

$$\rho\sigma_n^{(1)}(e^{i\theta}, h) = \frac{\rho}{n}\left[-\frac{(n+1)}{2} + \sum_{k=1}^{n}(n-k+1)\cos k\theta + i\sum_{k=1}^{n}(n-k+1)\sin k\theta\right]$$

$$= \frac{\rho}{n}\left[-\frac{(n+1)}{2} + \frac{\sin^2[(n+1)\theta/2]}{2\sin^2(\theta/2)} + i\frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2(\theta/2)}\right].$$

Now let $\theta = \theta_0 = 2\pi/(n+1)$. Then

$$\operatorname{Re}\rho\sigma_n^{(1)}(e^{i\theta_0}, h) = -\frac{\rho}{2}\frac{(n+1)}{n}.$$

Now, if $\rho > n/(n+1)$, then it follows that $\operatorname{Re}\rho\sigma_n^{(1)}(z,h) < -1/2$ and hence (since h maps E onto the right half plane $\operatorname{Re} w > -1/2$) we conclude that $\rho\sigma_n^{(1)}(z,h)$ will not be subordinate to h in E.

Taking $n = 2$, we obtain the following result of Singh and Singh [10].

COROLLARY 3.2

$(2/3)\sigma_2^{(1)}(z,f) \prec f(z)$ in E, for every $f \in K$.

In the next theorem we present a simple and interesting proof of a well-known result which was established by Basgöze, Frank and Keogh [1] in 1970.

Theorem 3.3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$ and let $s_n(z,f)$ denote its nth partial sum. Then*

$$s_n(z/2, f) \prec f(z)$$

in E for every $n = 1, 2, 3, \ldots$. The constant $1/2$ cannot be replaced by a larger one.

Proof. Since $s_n(z/2, f) = (1/2)z + (1/2^2)a_2 z^2 + (1/2^3)a_3 z^3 + \cdots + (1/2^n)a_n z^n$, the conclusion $s_n(z/2, f) \prec f(z)$ in E will follow if and only if the sequence $\langle 1/2, 1/2^2, \ldots, 1/2^n, 0, 0, \ldots\rangle$ is a subordinating factor sequence. In view of Lemma 2.1, this will be the case if and only if

$$\operatorname{Re}\left[1 + 2\sum_{k=1}^{n} \frac{z^k}{2^k}\right] > 0, \quad z \in E. \tag{3.2}$$

It is readily seen that the sequence $\{c_k\}_0^\infty$ defined by $c_0 = 1$, $c_k = 1/2^k$, $k = 1, 2, 3, \ldots, n$ and $c_k = 0$ if $k = n+1, n+2, \ldots$, is a convex null sequence. Thus using Lemma 2.3 we get

$$\operatorname{Re}\left(\frac{1}{2} + \sum_{k=1}^{n} \frac{z^k}{2^k}\right) > 0, \quad z \in E,$$

which in turn shows that the inequality (3.2) holds. The function $h(z) = z/(1-z) \in K$, which maps E onto the half plane $\operatorname{Re} w > -1/2$, shows that the constant $1/2$ cannot be replaced by any larger number. This completes the proof of our theorem.

Egerváry [3] has shown that

$$\sigma_n^{(2)}(z, z/(1-z)) = \frac{1}{n(n+1)}[(n+1)nz + n(n-1)z^2 + (n-1)(n-2)z^3 + \cdots + 2 \cdot 1 \cdot z^n]$$

is a member of $S_t(1/2)$. Using this fact and the well-known result of Ruscheweyh and Sheil-Small (Theorem 3.1, [9]) we conclude that for every $f \in S_t(1/2)$

$$\sigma_n^{(2)}(z, f) = f(z) * \sigma_n^{(2)}(z, z/(1-z))$$

is a member of $S_t(1/2)$.

Theorem 3.4. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be any member of $S_t(1/2)$. Then for every positive integer n, we have*

$$\operatorname{Re} \frac{v_n(z, f)}{\sigma_n^{(2)}(z, f)} > 0, \quad z \in E.$$

Proof. Consider the function F_n defined by

$$F_n(z) = (1-z)\left[\frac{n}{n+1} + \frac{n}{n+2}z + \frac{n^2}{(n+2)(n+3)}z^2 + \frac{n^2(n-1)}{(n+2)(n+3)(n+4)}z^3 + \frac{n^2(n-1)(n-2)}{(n+2)(n+3)(n+4)(n+5)}z^4 + \cdots + \frac{n^2(n-1), \ldots, 3}{(n+1)(n+2), \ldots, (2n)}z^n\right]. \tag{3.3}$$

Obviously F_n is regular in E (in fact it is an entire function), and we can write it in the form

$$F_n(z) = \frac{n}{n+1} - \frac{n}{n+1}\left(1 - \frac{n+1}{n+2}\right)z - \frac{n}{n+2}\left(1 - \frac{n}{n+3}\right)z^2$$

$$-\frac{n^2}{(n+2)(n+3)}\left(1-\frac{n-1}{n+4}\right)z^3$$
$$-\frac{n^2(n-1)(n-2),\ldots,4}{(n+2)(n+3),\ldots,(2n-1)}\left(1-\frac{3}{2n}\right)z^{n-1}$$
$$-\frac{n(n-1)(n-2),\ldots,3}{(n+2)(n+3),\ldots,(2n)}z^n.$$

In view of (3.3) and (3.4) it is now easy to see that in E we have

$$\operatorname{Re} F_n(z) \geqslant F_n(|z|) > F(1) = 0.$$

In Lemma 2.5 taking $f(z) = \sigma_n^{(2)}(z, f)$, $g(z) = z/(1-z)$ and $F(z) = F_n(z)$ we get

$$\operatorname{Re}\frac{\sigma_n^{(2)}(z,f)*z/(1-z)F(z)}{\sigma_n^{(2)}(z,f)*z/(1-z)} = \operatorname{Re}\frac{v_n(z,f)}{\sigma_n^{(2)}(z,f)} > 0, \quad z \in E.$$

This completes the proof.

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Uniqueness of the uniform norm and adjoining identity in Banach algebras

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Abstract. Let A_e be the algebra obtained by adjoining identity to a non-unital Banach algebra $(A, \|\cdot\|)$. Unlike the case for a C^*-norm on a Banach $*$-algebra, A_e admits exactly one uniform norm (not necessarily complete) if so does A. This is used to show that the spectral extension property carries over from A to A_e. Norms on A_e that extend the given complete norm $\|\cdot\|$ on A are investigated. The operator seminorm $\|\cdot\|_{op}$ on A_e defined by $\|\cdot\|$ is a norm (resp. a complete norm) iff A has trivial left annihilator (resp. $\|\cdot\|_{op}$ restricted to A is equivalent to $\|\cdot\|$).

Keywords. Adjoining identity to a Banach algebra; unique uniform norm property; spectral extension property; regular norm; weakly regular Banach algebra.

1. Introduction

Let $A_e = A + \mathbb{C}1$ be the algebra obtained by adjoining identity to a non-unital Banach algebra $(A, \|\cdot\|)$ [8]. There are two natural problems associated with this elementary unitification construction: (1) which are (all) algebra norms $|\cdot|$ on A_e that are closely related with (e.g. extending) $\|\cdot\|$ on A? (2) Which properties of the Banach algebra $(A, \|\cdot\|)$ are shared by the normed algebra $(A_e, |\cdot|)$? In the present paper, it is shown that A has unique uniform norm (not necessarily complete) (resp. spectral extension property [9]) iff A_e has the same. This is interesting in view of the fact that for a Banach $*$-algebra $(A, \|\cdot\|)$ with a unique C^*-norm, A_e can admit more than one C^*-norm [1, Example 4.4, p. 850]. This holds in spite of apparent similarity between the defining properties $\|x^2\| = \|x\|^2$ and $\|x^*x\| = \|x\|^2$ of uniform norms and C^*-norms respectively. This main result, together with a couple of corollaries, is formulated and proved in § 3. Their proofs require some properties of norms on A that are regular [5]. There are two standard constructs of norms on A_e, viz. the *l^1-norm* $\|x + \lambda 1\|_1 = \|x\| + |\lambda|$ and the *operator norm* $\|x + \lambda 1\|_{op} = \sup\{\|xy + \lambda y\| : \|y\| \leqslant 1,\ y \in A\}$. In general, $\|\cdot\|_{op}$ need neither be a norm nor be complete [6, Example 4.2]. Also, in general, $\|\cdot\|_{op|A} \neq \|\cdot\|$. It is easy to see that if p is any algebra seminorm on A_e such that $p_{|A} = \|\cdot\|$, then $\|a + \lambda 1\|_{op} \leqslant p(a + \lambda 1) \leqslant p(1)\|a + \lambda 1\|_1$. The norm $\|\cdot\|$ on A is *regular* (resp. *weakly regular*) if the restriction of $\|\cdot\|_{op}$ on A $\|\cdot\|_{op|A} = \|\cdot\|$ (resp. $\|\cdot\|_{op|A}$ is equivalent to $\|\cdot\|$). These are essentially non-unital phenomena, for if A is unital (resp. having a bai (e_i)), then any norm $|\cdot|$ on A with $|1| \leqslant 1$ (or $|e_i| \leqslant 1$) is regular [5]. It is shown in § 2 that $\|\cdot\|_{op}$ is a norm on A_e iff the left annihilator $\mathrm{lan}(A) = \{0\}$; and in this case, $\|\cdot\|_{op}$ is complete iff $\|\cdot\|$ is weakly regular iff $\|\cdot\|_1$ is equivalent to $\|\cdot\|_{op}$ on A_e.

Throughout, A is a non-unital algebra. By a *norm* on A, we mean an algebra norm; i.e. a norm satisfying $\|xy\| \leqslant \|x\|\,\|y\|$ for all x, y. A *uniform norm* on A (resp. a *C^*-norm* on a $*$-algebra) is a norm satisfying the square property $\|x^2\| = \|x\|^2$ (resp. the C^*-property $\|x^*x\| = \|x\|^2$) for all x.

2. Weakly regular norms

Let $(A, \|\cdot\|)$ be a normed algebra. The following shows that if $\|\cdot\|_{op}$ is a norm on A_e, then $|\cdot|_{op}$ is also a norm on A_e for all norms $|\cdot|$ on A. The *left annihilator* of A is $\text{lan}(A) = \{x \in A : xA = \{0\}\}$.

PROPOSITION 2.1

The seminorm $\|\cdot\|_{op}$ *is a norm on* A_e *iff* $\text{lan}(A) = \{0\}$.

Proof. Let $\|\cdot\|_{op}$ be a norm on A_e. Let $a \in \text{lan}(A)$. Then $ax = 0$ $(x \in A)$, hence $\|a\|_{op} = \sup\{\|ax\| : \|x\| \leqslant 1, x \in A\} = 0$, so that $a = 0$. Hence $\text{lan}(A) = \{0\}$. Conversely, assume that $\text{lan}(A) = \{0\}$. Let $\|a + \lambda 1\|_{op} = 0$. Then $ax + \lambda x = 0$ for all $x \in A$. Suppose $\lambda \neq 0$. Then $-\lambda^{-1} ax = x$ $(x \in A)$. Define $L_e(x) = ex (x \in A)$, where $e = -\lambda^{-1} a$. Then L_e is an identity operator on A. Then, for $x \in A$, $L_x L_e = L_e L_x$, i.e. $xey = L_x L_e(y) = L_e L_x(y) = exy$ $(y \in A)$, i.e. $(xe - ex)y = 0$ $(y \in A)$. Hence, $xe = ex = x$. Thus A has an identity which is a contradiction. Thus $\lambda = 0$. This implies $ax = 0$ for all $x \in A$, hence $a = 0$. This completes the proof.

PROPOSITION 2.2

(a) Let $|\cdot|$ *be a uniform norm on* A. *Then* $|\cdot|$ *is regular and* $|\cdot|_{op}$ *is a uniform norm on* A_e.
(b) Let A *be a* $*$*-algebra. Let* $|\cdot|$ *be a* C*-*norm on* A. *Then* $|\cdot|$ *is regular and* $|\cdot|_{op}$ *is a* C*-*norm on* A_e.

Note that if a Banach algebra admits a uniform norm, then it is commutative and semisimple. In the above, the proof of (a) is similar to that of (b) in [4, Lemma 19, p. 67]. In the following, the proof of (1) implies (2) is along the lines of [7, Theorem 1]; whereas that of the remaining part is simple.

PROPOSITION 2.3

Let $(A, \|\cdot\|)$ *be a Banach algebra. Then the following are equivalent.*

(1) $\|\cdot\|$ *is weakly regular (so that* $\|a\|_{op} \leqslant \|a\| \leqslant m\|a\|_{op}$ $(a \in A)$, *for some* $m > 0$).
(2) $\|a + \lambda 1\|_{op} \leqslant \|a + \lambda 1\|_1 \leqslant 2(2 + m)(\exp 1)\|a + \lambda 1\|_{op}$ $(a + \lambda 1 \in A_e)$
(3) $\|\cdot\|_{op}$ *is a complete norm on* A_e.

If $\|\cdot\|$ *is regular, then* $m = 1$ *so that* $\|a + \lambda 1\|_{op} \leqslant \|a + \lambda 1\|_1 \leqslant 6(\exp 1)\|a + \lambda 1\|_{op}$ *for all* $a + \lambda 1 \in A_e$ [7, *Theorem* 1].

3. Uniqueness of uniform norm and unitification

A Banach algebra $(A, \|\cdot\|)$ has *unique uniform norm property* (UUNP) if A admits exactly one (not necessarily complete) uniform norm. The uniform algebra $C(X)$ has UUNP, whereas the disc algebra does not have. In [2] and [3], Banach algebras with UUNP have been investigated. Such an A is necessarily commutative, semisimple and the spectral radius $r(= r_A(\cdot))$ is the unique uniform norm. We denote the Hausdorff completion of (A, r) by $U(A)$. The spectral radius on $U(A)$ is the complete uniform norm on $U(A)$. A norm $|\cdot|$ on A is *functionally continuous* (FC) if every multiplicative linear functional on A is $|\cdot|$-continuous. A subset F of the Gelfand space of A is a set of uniqueness for A if $|x|_F = \sup\{|f(x)| : f \in F\}$ defines a norm on A.

Theorem 3.1. *A Banach algebra $(A, \|\cdot\|)$ has UUNP iff A_e has UUNP.*

We shall need the following. The proofs are straightforward. For details we refer to [3].

Lemma A. *Let $|\cdot|$ be an FC norm on any commutative algebra A. Let B be the completion of $(A, |\cdot|)$. Then the Gelfand space $\Delta(A)$ (resp. Silove boundary ∂A) is homeomorphic to $\Delta(B)$ (resp. ∂B).*

Lemma B. *Let A be a semisimple commutative Banach algebra. Then the following are equivalent.*

(1) *A has UUNP.*

(2) *$U(A)$ has UUNP; and any closed set F in $\Delta(U(A))$ which is a set of uniqueness for A, is also a set of uniqueness for $U(A)$.*

(3) *$U(A)$ has UUNP; and for a non-zero closed ideal I of $U(A)$ with $I = k(h(I))$ (kernel of hull of I), $I \cap A$ is non-zero.*

Lemma C. *Let A be a Banach algebra with UUNP, and I be a closed ideal such that $I = k(h(I))$. Then I has UUNP.*

Proof of Theorem 3.1. Assume that A has UUNP.

Case 1. Let $\|\cdot\|$ have the square property. By Proposition 2.2 (a) and Proposition 2.3, $(A_e, \|\cdot\|_{op})$ is a Banach algebra, $\|\cdot\|_{op}$ has square property and $\|\cdot\|_{op}$ is equivalent to $\|\cdot\|_1$. Let $|\cdot|$ be any uniform norm on A_e, then $|1\cdot 1|_A$ is a uniform norm on A. Since A has UUNP, $|1\cdot 1|_A = \|\cdot\|$. Hence $\|\cdot\|_{op} \leqslant |\cdot| \leqslant \|\cdot\|_1 \leqslant 6(\exp 1)\|\cdot\|_{op}$ on A_e. Thus $\|\cdot\|_{op}$ and $|\cdot|$ are equivalent uniform norms on A_e. Since equivalent uniform norms are equal, $\|\cdot\|_{op} = |\cdot|$ on A_e. Thus A_e has UUNP.

Case 2. In the general case, note that $U(A)$ is an ideal of $U(A_e)$ and, by Lemma A, the Gelfand space $\Delta(U(A_e))$ is homeomorphic to the one point compactifications of each of $\Delta(A)$ and $\Delta(U(A))$. Define $K = \{x \in U(A_e) : xU(A) = \{0\}\}$. We prove that $K = \{0\}$. Let $x \in K$. Then its Gelfand transform $\hat{x} : \Delta(U(A_e)) \to \mathbb{C}$ is continuous. Since $x \in K$, $xy = 0$ $(y \in U(A))$. We prove that $\hat{x}$ is zero on $\Delta(U(A_e))$. Since $\Delta(U(A))$ is dense in $\Delta(U(A_e))$, it is enough to prove that $\hat{x}$ is zero on $\Delta(U(A))$. Suppose there exists $\phi \in \Delta(U(A))$ such that $\phi(x) \neq 0$. Since ϕ is non-zero, there exists y in $U(A)$ such that $\phi(y)$ is non-zero. This implies $\phi(xy) \neq 0$, hence $xy \neq 0$ which is a contradiction. Thus $K = \{0\}$. By Lemma B, it is enough to prove that $U(A_e)$ has UUNP; and for every non-zero closed ideal I of $U(A_e)$ with $I = k(h(I))$, $A_e \cap I$ is non-zero. Let I be a non-zero closed ideal of $U(A_e)$ such that $I = k(h(I))$. We prove that $I \cap A_e \neq \{0\}$. Let $J = I \cap U(A)$. Then, first, we prove that $J = k(h(J))$ in $U(A)$. Clearly $J \subseteq k(h(J))$. Let $x \in U(A)$ such that $x \notin J$. Then $x \notin I$, hence there exists $\phi \in h(I) \subseteq \Delta(U(A_e))$ such that $\phi(x) \neq 0$. Then $\psi = \phi|_{U(A)}$ is zero on J and $\psi(x) \neq 0$. Thus $x \notin k(h(J))$, and so $J = k(h(J))$. From $K = \{0\}$, $I \neq \{0\}$ and $IU(A) \subseteq J$, it follows that $J \neq \{0\}$. Since A has UUNP and J is a non-zero closed ideal of $U(A)$ such that $J = k(h(J))$, $A \cap I = A \cap J \neq \{0\}$ by Lemma B. Hence $I \cap A_e \neq \{0\}$. Finally, we show that $U(A_e)$ has UUNP. Note that, by Proposition 2.2 (a) and Proposition 2.3, the operator norm on $U(A)_e$ is a complete uniform norm; and is the spectral radius $r_{U(A)_e}$ itself. Further, $U(A)_e$, is clearly isometrically isomorphic to $U(A_e)$ via the map $T : U(A)_e \to U(A_e)$, $T(a + \lambda 1) = a + \lambda e$, where e is the identity of $U(A_e)$. By Lemma C,

$U(A)$ has UUNP, hence by the isomorphism T and by Case 1, $U(A_e)$ has UUNP. Conversely, if A_e have UUNP, then, A being a closed ideal of A_e satisfying $A = k(h(A))$ in A_e, A has UUNP by Lemma C. This completes the proof.

Following [1], a Banach $*$-algebra B has unique C^*-norm (i.e. B has UC^*NP) if B admits exactly one C^*-norm (not necessarily complete). In spite of the apparent similarity between the square property and the C^*-property of norms the above result differs from the corresponding situation in B, viz. UC^*NP for B need not imply UC^*NP for B_e [1, Example 4.4, p. 850]. In fact, by [1, Theorem 4.1, p. 849], for a non-unital B with UC^*NP, B_e has UC^*NP iff the enveloping C^*-algebra $C^*(B)$ is non-unital. Like $C^*(B)$ for B, the uniform Banach algebra $U(A)$ is universal for A in an appropriate sense. Unlike the case of B, it happens that A is unital iff $U(A)$ is unital. This explains why the above result for A differs from the corresponding result for B.

A Banach algebra $(A, \|\cdot\|)$ has *the spectral extension property* (SEP) [9] (i.e. A is a *permanent Q-algebra* [10]), if for every Banach algebra B such that A is algebraically embedded in B, $r_A(x) = r_B(x)$ for all $x \in A$; equivalently, every norm $|\cdot|$ on A satisfies $r_A(x) \leqslant |x|$ for all $x \in A$ [9, Proposition 1].

COROLLARY 3.2

Let $(A, \|\cdot\|)$ be a semisimple commutative Banach algebra. Then A has SEP iff A_e has SEP.

Proof. Let A have SEP. Then, by [2, Proposition 2.1] and Theorem 3.1, A_e has UUNP. By [2, Proposition 2.6], it is enough to prove that A_e has (P)-property; i.e. every non-zero closed ideal I of A_e has an element $a + \lambda 1$ such that $r_1(a + \lambda 1) > 0$, where $r_1(a + \lambda 1) = \inf\{|a + \lambda 1| : |\cdot| \text{ is a norm on } A_e\}$, called the permanent radius of $a + \lambda 1$ in A_e [9]. Let I be a non-zero closed ideal of A_e. Then $J = I \cap A$ is a non-zero closed ideal of A by [8, Theorem 1.1.6, p. 11]. Since A has SEP, by [2, Proposition 2.6], it has (P)-property, hence there exists $a \in J$ such that the permanent radius, say $r_2(a)$, of a in A is positive. Then clearly $r_1(a) \geqslant r_2(a) > 0$. Thus A_e has (P)-property. Conversely, assume that A_e has SEP. Let $|\cdot|$ be any norm on A. Then, since A is semisimple, Proposition 2.1 implies the operator norm $|\cdot|_{\text{op}}$ is a norm on A_e. Since A_e has SEP, $r_A(a) = r_{A_e}(a) \leqslant |a|_{\text{op}} \leqslant |a|$ $(a \in A)$. Thus $r_A(a) \leqslant |a|$ for all a in A and for any norm $|\cdot|$ on A. Hence, A has SEP. This completes the proof.

By [9, Corollary 2], a regular Banach algebra has SEP. In understanding the relation between UUNP and SEP, a weaker notion of regularity has been found useful in [2], viz. a semisimple commutative Banach algebra $(A, \|\cdot\|)$ is *weakly regular* if for any proper closed subset F of the Gelfand space $\Delta(A)$ of A, there exists a non-zero element a in A such that $\hat{a}|F = 0$.

COROLLARY 3.3

Let $(A, \|\cdot\|)$ be a semisimple commutative Banach algebra. Then A is weakly regular iff A_e is weakly regular.

Proof. Let A be weakly regular. Then, by [2, Corollary 2.4(II)], A has UUNP and $\Delta(A) = \partial A$, the Silov boundary of A. By Theorem 3.1, A_e has UUNP. Note that $\Delta(A) = \partial A \subseteq \partial A_e$, $\Delta(A)$ is dense in $\Delta(A_e)$ and ∂A_e is closed. These imply $\partial A_e = \Delta(A_e)$. Hence, again by [2, Corollary 2.4 (II)], A_e is weakly regular. Conversely, assume that A_e

is weakly regular. The proof of Lemma C will work for the following statement; If A is weakly regular and I is a closed ideal of A such that $I = k(h(I))$, then I is also weakly regular. Since A is a closed ideal of A_e with $k(h(A)) = A$, A is weakly regular.

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Weakly prime sets for function spaces

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Abstract. We define and study weakly prime sets for a function space and show that it coincides with the known concept of weakly prime sets for function algebras and spaces of affine functions.

Keywords. Weakly prime set; function space; function algebra; space of affine functions.

1. Introduction

A *function space* A on a compact Hausdorff space X is a closed subspace of the space $C(X)$ of all continuous, complex-valued functions on X separating points and containing constants. If A is an algebra, it is called a function algebra. The Bishop and Šilov decompositions play an important role in characterizing function algebras. Later on these decompositions were studied for function spaces [6]. Ellis [3] defined and studied these decompositions for the spaces of affine functions on a compact convex set.

For a function algebra, certain decompositions finer than the Bishop and Šilov decompositions have been defined and studied [6]. One such decomposition of weakly prime sets, was defined and discussed by Ellis [4] for function algebras as well as for spaces of affine functions. Here we generalize this concept for a function space, study its properties and show that it coincides with the corresponding definitions of Ellis.

We also give examples of function spaces whose family of maximal weakly prime sets differ from the corresponding families of its induced algebras.

2. Function space

Let X be a compact Hausdorff space. Throughout this paper we assume that A is a function space on X. For a closed subset E of X, we define

$$N(A|_E) = \{f \in C(E): \quad fg \in A|_E \quad \text{for all } g \in A|_E\}.$$

For the concepts like peak set, p-set, etc. related to a function space and for the various properties of a decomposition for a function space, we refer to [2], [5] and [7].

DEFINITION 2.1

A closed subset E of X is called a *weakly prime set* for A if $E = G \cup H$, with G and H generalized peak sets for $N(A|_E)$, then either $G = E$ or $H = E$.

The function space A is called *weakly prime* if X is a weakly prime set for A.

Remarks 2.2. (i) If A is an algebra, then $N(A|_E) = A|_E$ and hence Definition 2.1 coincides with the definition for a function algebra given by Ellis [4].

(ii) It can be shown that each weakly prime set is contained in a maximal weakly prime set for A.

The collection of all maximal weakly prime sets for A is denoted by $\mathscr{P}(A)$.

(iii) It is easy to check that $\mathscr{P}(A)$ is finer than the Bishop decompositions for A and hence $\mathscr{P}(C(X)) = \{\{x\} : x \in X\}$.

(iv) It can be easily verified that A is weakly prime if and only if $N(A)$ is weakly prime. Further, for a closed subset E of X, $N(A)_{|E} \subset N(A_{|E})$ and so, $\mathscr{P}(A)$ is weaker than $\mathscr{P}(N(A))$. But, in general, $\mathscr{P}(A) \neq \mathscr{P}(N(A))$ (see Example 2.6(i)).

As in case of a function algebra, we shall show that here also every member of $\mathscr{P}(A)$ is a p-set and $\mathscr{P}(A)$ has the (GA)-property [5] for A.

We shall need the following lemma.

Lemma 2.3. *If E is a p-set for A and $F \subset E$ is a generalized peak set for $N(A_{|E})$, then F is a p-set for A.*

Proof. Let $\mu \in A^{\perp}$ and $\varepsilon > 0$ be given. Then there is an open set U in X such that $|\mu|(U - F) < \varepsilon$. Clearly, $E \cap U$ is open in E and $F \subset E \cap U$. Since F is a generalized peak set for $N(A_{|E})$, there is a peak set T for $N(A_{|E})$ such that $F \subset T \subset E \cap U$. Let $f \in N(A_{|E})$ be a peaking function for T. Define h on E by $h = 1$ on T and $h = 0$ on $E \setminus T$. Then f^n converges pointwise and boundedly to h on E.

Now let $g \in A$. Then

$$\int_T g \, d\mu = \int_E gh \, d\mu = \lim \int_E g f^n d\mu.$$

But $\int_E g f^n d\mu = 0$, as $f^n g_{|E} \in A_{|E}$ and E is a p-set for A. Thus $\int_T g \, d\mu = 0$.

Now

$$\left| \int_F g \, d\mu \right| = \left| \int_F g \, d\mu - \int_T g \, d\mu \right| \leqslant \|g\| (|\mu|(T - F))$$

$$\leqslant \|g\| (|\mu|(U - F))$$

$$< \varepsilon \|g\|.$$

Since ε is arbitrary, $|\int_F g \, d\mu| = 0$ or F is a p-set for A.

PROPOSITION 2.4

A maximal weakly prime set for A is a p-set for A.

Proof. Let E be a maximal weakly prime set for A and let F denote the smallest p-set for A which contains E. We shall show that F is a weakly prime set for A.

Let F_1 and F_2 be generalized peak sets for $N(A_{|F})$ with $F_1 \cup F_2 = F$. Then $E = (F_1 \cap E) \cup (F_2 \cap E)$ and since $N(A_{|F})_{|E} \subset N(A_{|E})$, $F_1 \cap E$ and $F_2 \cap E$ are generalized peak sets for $N(A_{|E})$. Since E is a weakly prime set for A, either $F_1 \cap E = E$ or $F_2 \cap E = E$, i.e., either $E \subset F_1$ or $E \subset F_2$. If $E \subset F_1$, then $E \subset F_1 \subset F$ where F is a p-set for A and F_1 is a generalized peak set for $N(A_{|F})$. So, by Lemma 2.3, F_1 is a p-set for A and hence $F_1 = F$. Similarly, if $E \subset F_2$, then $F_2 = F$. Thus F is a weakly prime set for A and by the maximality of E, we have $E = F$.

Next, we show that the family $\mathscr{P}(A)$ characterizes a function space A in the sense that it has the (D)-property for A [5.7], i.e. if $f \in C(X)$ and $f_{|E} \in (A_{|E})^{-}$ for every $E \in \mathscr{P}(A)$, then

$f \in A$. Actually the Bishop's theorem can be restated as "The Bishop decomposition has the (D)-property for A". In fact the Bishop decomposition has a stronger property than the (D)-property, namely the (GA)-property.

By (GA)-property for a family $\mathcal{F}$ of closed subsets of X for A [7] we mean that for each $\mu \in b(A^\perp)^e$, $\operatorname{supp}\mu \subset F$ for some $F \in \mathcal{F}$, where $b(A^\perp)^e$ denotes the set of extreme points of the unit ball of $A^\perp$.

We shall show that $\mathcal{P}(A)$ has the (GA)-property for A.

Theorem 2.5. *$\mathcal{P}(A)$ has the (GA)-property for A.*

Proof. Let $\mu \in b(A^\perp)^e$, the set of extreme points of the unit ball in $A^\perp$ and let $S = \operatorname{supp}\mu$. It is enough to show that S is a weakly prime set for A.

Let G and H be generalized peak sets for $N(A_{|S})$ with $S = G \cup H$. Let $\mu_1 = \mu_{|G}$, $\mu_2 = \mu - \mu_1$ and $g \in A$. Since $\mu = \mu_S \in A^\perp$ and G is a generalized peak set for $N(A_{|S})$, by Lemma 2.3, we get $\int_G g\, d\mu = 0$. Thus $\mu_1 \in A^\perp$ and hence $\mu_2 \in A^\perp$. Also, $\|\mu\| = 1 = \|\mu_1\| + \|\mu_2\|$. Hence $\mu_1 = \mu$ or $\mu_2 = \mu$, as $\mu \in b(A^\perp)^e$, i.e., $G = S$ or $H = S$. Thus S is a weakly prime set for A.

Examples 2.6. (i) Let X be the union of a line segment F and a sequence of disjoint solid rectangles $\{F_n; n = 1, 2, \ldots\}$ converging to F. Let A be the set of all f in $C(X)$ such that $f_{|F_n}$ is a polynomial of degree atmost n. Then A is a function space on X and as in [7] it can be checked that $\mathcal{P}(A) = \{F_n | n \in \mathbb{N}\} \cup \{\{x\};\ x \in X\}$. Note that, here $N(A) = \{f \in C(X): f_{|F_n}$ is constant, for each $n \in \mathbb{N}\}$ and hence $\mathcal{P}(N(A)) = \{F_n: n \in \mathbb{N}\} \cup \{F\}$. Therefore, $\mathcal{P}(A) \neq \mathcal{P}(N(A))$.

(ii) Let T denote the unit circle in $\mathbb{C}$ and $A(T)$ denote the disc algebra on T. Let $\Phi \in A(T)$ be such that $\Phi \neq 0$ on T. Define $A = \{\Phi^{-1} f: f \in A(T)\}$. Then A is a function space on T and $N(A) = A(T)$ [8]. It is clear that $N(A)$ is weakly prime and hence by Remark 2.2 (iv), A is also weakly prime, i.e., $\mathcal{P}(A) = \{T\}$. Since $A(T)$ is a maximal function algebra on T and $A(T) \subsetneq A$, the algebra generated by A will be $C(T)$. But $\mathcal{P}(C(T)) = \{\{x\}: x \in T\}$ by Remark 2.2 (iii) while $\mathcal{P}(A) = \{T\}$.

3. Space of affine functions

Let K be a compact convex subset of a locally convex Hausdorff space and let $A(K)$ denote the Banach space of all real-valued continuous affine functions on K with the supremum norm. The set of extreme points of K will be denoted by ∂K.

Ellis [4] has defined weakly prime sets for $A(K)$ with the help of concepts of convexity. Now $A(K)$ can also be looked upon as a function space on K. So we can discuss $\mathcal{P}(A(K))$ for $A(K)$. But, since the functions in $A(K)$ are determined by their values on ∂K, we shall consider the space $A(K)_{|\partial K}$. In fact, weakly prime sets defined by Ellis, are also subsets of ∂K. In this section, we shall prove that $\mathcal{P}(A(K)_{|\partial K})$ coincides with the family of maximal weakly prime sets as defined by Ellis.

For the definitions and results regarding compact convex sets and space of affine functions, we refer to [1] and [2].

Let us recall the definition due to Ellis [4].

DEFINITION 3.1

A subset E of ∂K is called a *weakly prime set for $A(K)$* if $E = \partial G$ for some closed face G of

K and if every proper facially closed subset of G has empty interior in the facial topology of G.

Equivalently, for a closed face G of K, ∂G is weakly prime if whenever $G = \mathrm{Co}(H_1 \cup H_2)$ for some closed split faces H_1 and H_2 of G, then either $H_1 = G$ or $H_2 = G$.

If ∂K is a weakly prime set, then $A(K)$ is called weakly prime.

We shall denote the family of maximal weakly prime sets for $A(K)$ according to Definition 3.1 by $\mathscr{P}_E(A(K))$.

The following proposition can be easily proved.

PROPOSITION 3.2

$$\mathrm{Ce}(A(K))_{|\partial k} = N(A(K)_{|\partial K}),$$

where $\mathrm{Ce}(A(K)) = \{f \in A(K)\colon fg_{|\partial K} \in A(K)_{|\partial K}$ *for every* $g \in A(K)\}$, *the centre of* $A(K)$.

Since $\mathrm{Ce}(A(K))_{|\partial k}$ is the set of facially continuous functions on ∂K [2, Theorem 1.4, p. 105], we immediately get the following result.

COROLLARY 3.3

A subset E *of* ∂K *is a facially closed subset of* ∂K *if and only if* E *is a generalized peak set for* $N(A(K)_{|\partial K})$.

PROPOSITION 3.4

If $E \in \mathscr{P}(A(K)_{|\partial K})$, *then* $\overline{\mathrm{Co}}E$, *the closed convex hull of* E, *is a closed split face of* K.

Proof. Let F be the smallest closed split face of K containing $\overline{\mathrm{Co}}E$. Then $E \subset \overline{\mathrm{Co}}E \cap \partial K \subset F \cap \partial K = \partial F$, as F is a face. It is enough to show that ∂F is a weakly prime set for $A(K)_{|\partial K}$.

Let H_1 and H_2 be generalized peak sets for $N((A(K)_{|\partial K})_{|\partial F})$ with $\partial F = H_1 \cup H_2$. Then $E = E \cap \partial F = (H_1 \cap E) \cup (H_2 \cap E)$ and $H_1 \cap E$, $H_2 \cap E$ are generalized peak sets for $N((A(K)_{|\partial K})_{|E})$. Since E is a weakly prime set for $A(K)_{|\partial k}$, either $H_1 \cap E = E$ or $H_2 \cap E = E$. Thus, either $E \subset H_1$ or $E \subset H_2$.

Now, since F is a closed split face of K, $(A(K)_{|\partial K})_{|\partial F} = A(K)_{|\partial F} = A(F)_{|\partial F}$. So, H_1 and H_2 are generalized peak sets for $N(A(K)_{|\partial K})$ and hence by Corollary 3.3, H_1 and H_2 are facially closed subsets of ∂F, i.e., $H_1 = \partial G_1$ and $H_2 = \partial G_2$ for some closed split faces G_1 and G_2 of F. Since F is a closed split face of K, G_1 and G_2 are closed split faces of K. Now $E \subset H_1 \Rightarrow \overline{\mathrm{Co}}E \subset \overline{\mathrm{Co}}H_1 = G_1$. Thus we get $\overline{\mathrm{Co}}E \subset G_1 \subset F$ and hence $G_1 = F$, as F is the smallest closed split face containing $\overline{\mathrm{Co}}E$, i.e., $H_1 = \partial F$. Similarly, if $E \subset H_2$, then we get $H_2 = \partial F$. So ∂F is a weakly prime set for $A(K)_{|\partial K}$.

If $\overline{\mathrm{Co}}H$ is a closed face of K for $H \subset \partial K$, then $\partial(\overline{\mathrm{Co}}H) = H$ and hence we get the following result.

COROLLARY 3.5

If $E \in \mathscr{P}(A(K)_{|\partial K})$, *then* E *is facially closed.*

Now we prove the main result.

Theorem 3.6. $\mathscr{P}(A(K)_{|\partial K}) = \mathscr{P}_E(A(K))$.

Proof. Let $F \in \mathscr{P}_E(A(K))$. We want to show that F is a weakly prime set for $A(K)_{|\partial K}$.

Let H_1 and H_2 be generalized peak sets for $N((A(K)_{|\hat{\partial}K})_{|F})$ with $H_1 \cup H_2 = F$. Since $F \in \mathscr{P}_E(A(K))$, F is facially closed [4], i.e., $F = \partial G$ for some closed split face G of K. Hence $A(K)_{|G} = A(G)$ and so $(A(K)_{|\hat{\partial}K})_{|F} = A(G)_{|\hat{\partial}G}$. Thus H_1 and H_2 are generalized peak sets for $N(A(G)_{|\hat{\partial}G})$. So by Corollary 3.3, H_1 and H_2 are facially closed subsets of G. Also, by definition, $F = \partial G$, where G is a closed face of K and $F = H_1 \cup H_2$. Since F is a weakly prime set for $A(K)$, either $H_1 = F$ or $H_2 = F$. Hence F is a weakly prime set for $A(K)_{|\hat{\partial}K}$.

Conversely, let $F \in \mathscr{P}(A(K)_{|\partial K})$. Then by Proposition 3.4, $\overline{\text{co}}F$ is a closed split face of K. Let $\overline{\text{co}}F = G$. Then $F = \partial G$ and $A(G)_{|\partial G} = (A(K)_{|\hat{\partial}K})_{|F}$. Suppose $F = H_1 \cup H_2$, where H_1 and H_2 are facially closed in G. Then by Corollary 3.3, H_1 and H_2 are generalized peak sets for $N((A(K)_{|\partial K})_{|F})$. Since F is a weakly prime set for $A(K)_{|\partial K}$, either $H_1 = F$ or $H_2 = F$. Hence F is a weakly prime set for $A(K)$. Consequently, $\mathscr{P}(A(K)_{|\partial K}) = \mathscr{P}_E(A(K))$.

COROLLARY 3.7

$A(K)$ is weakly prime if and only if $A(K)_{|\hat{\partial}K}$ is weakly prime.

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Oscillation of higher order delay differential equations

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Abstract. A sufficient condition was obtained for oscillation of all solutions of the *odd-order* delay differential equation

$$x^{(n)}(t) + \sum_{i=1}^{m} p_i(t)x(t-\sigma_i) = 0, \qquad (*)$$

where $p_i(t)$ are non-negative real valued continuous function in $[T, \infty]$ for some $T \geqslant 0$ and $\sigma_i \in (0, \infty)$ $(i = 1, 2, \ldots, m)$. In particular, for $p_i(t) = p_i \in (0, \infty)$ and $n > 1$ the result reduces to

$$\frac{1}{m}\left(\sum_{i=1}^{m} (p_i\sigma_i^n)^{1/2}\right)^2 > (n-2)!\frac{(n)^n}{e},$$

implies that every solution of $(*)$ oscillates. This result supplements for $n > 1$ to a similar result proved by Ladas *et al* [*J. Diff. Equn.*, **42** (1982) 134–152] which was proved for the case $n = 1$.

Keywords. Odd order; delay equation; oscillation of all solutions.

1. Introduction

This paper was motivated by certain results of the paper [7] and [8] due to Ladas *et al.* In [7] authors proved that all solutions of the *odd-order* delay differential equation

$$x^{(n)}(t) + \sum_{i=1}^{m} p_i x(t-\sigma_i) = 0, \qquad (1)$$

oscillates (i.e., every solution $x(t)$ has zeros for arbitrarily large t) if and only if the associated characteristic equation

$$\lambda^n + \sum_{i=1}^{m} p_i e^{-\lambda\sigma_i} = 0 \qquad (2)$$

has no real roots, where p_i and $\sigma_i \in (0, \infty)$ for $i = 1, 2, \ldots, m$. Further, it was proved that (2) has no real roots if and only if

$$(p_1)^{1/n}\sigma_1 > \frac{n}{e}.$$

In the literature, it was observed that the *odd-order* differential equations of the form

$$x^{(n)}(t) + \sum_{i=1}^{m} p_i(t)x(t-\sigma_i) = 0, \qquad (3)$$

where $p_i \in C([T, \infty), (0, \infty))$, $T \geqslant 0$ and $\sigma_i \in (0, \infty)$, is least studied. In this connection, we may refer, in particular, to [4], [5], [9] and the references therein. For $n = 1$, (3) is almost well-studied. In this case there are several results associated with its characteristic

equation (see [3] and [7]) as well as conditions on coefficients and deviating arguments which ensures that every solution of (3) oscillates. In [8], authors proved that if $p_i \in C([T, \infty), (0, \infty))$, $\sigma_i \in (0, \infty)$ $(i = 1, 2, \ldots, m)$ and $n = 1$ then

$$\liminf_{t\to\infty} \int_{t-\sigma_i/2}^{t} p_i(s)\mathrm{d}s > 0 \quad (i = 1, 2, \ldots, m) \tag{4}$$

and

$$\frac{1}{m}\sum_{i=1}^{m}\left(\liminf_{t\to\infty}\int_{t-\sigma_i}^{t} p_i(s)\mathrm{d}s\right) + \frac{2}{m}\sum_{\substack{i<j\\ i,j=1}}^{m}\left[\left(\liminf_{t\to\infty}\int_{t-\sigma_j}^{t} p_i(s)\mathrm{d}s\right)\left(\liminf_{t\to\infty}\int_{t-\sigma_i}^{t} p_j(s)\mathrm{d}s\right)\right] > \frac{1}{e} \tag{5}$$

then every solution of (3) oscillates. If $p_i(t) = p_i \in (0, \infty)(i = 1, 2, \ldots, m)$ then the above result becomes

$$\frac{1}{m}\left(\sum_{i=1}^{m}(p_i\sigma_i)^{1/2}\right)^2 > \frac{1}{e}$$

implies that every solution of (3) oscillates. In this paper an attempt has been made to obtain a similar result which shows that every solution of (3) oscillates. Our result fails to hold when $n = 1$. Indeed, when $p_i(t) = p_i \in (0, \infty)$, the main result of this paper shows that if

$$\frac{1}{m}\left(\sum_{i=1}^{m}(p_i\sigma_i^n)^{1/2}\right)^2 > (n^n(n-2)!)\frac{1}{e}$$

then every solution of (3) oscillates. Although our result does not generalize the result of Ladas *et al* [8], but certainly supplements for higher order equations.

2. Main results

In the beginning of this section we prove a lemma for its use in the sequel.

Lemma 1. *Let* $f \in C^{(n)}([T, \infty), (0, \infty))$, $T \geqslant 0$ *such that* $f^{(n)}(t) \leqslant 0$, $t \geqslant T$. *If* n *is odd and* $\sigma \in (0, \infty)$ *then there exists* $T_0 \geqslant T$ *such that*

$$\frac{f(t-\sigma)}{f^{(n-1)}(t)} \geqslant \frac{\sigma^{n-1}}{(n-1)!}, \quad t \geqslant T_0. \tag{6}$$

Proof. Since $f(t) \geqslant 0$ and $f^{(n)}(t) \leqslant 0$ for $t \geqslant T$, there exists $T_1 \geqslant T$ and $0 \leqslant k \leqslant n-1$ such that

$$f^{(j)}(t) > 0 \quad \text{for } j \leqslant k$$

and

$$f^{(j)}(t) f^{(j+1)}(t) \leqslant 0 \quad \text{for } k \leqslant j \leqslant n-1.$$

Expanding $f(t)$ by Taylor's theorem, there exists $x \in (t-\sigma, t)$ such that

$$f(t) = \sum_{j=0}^{k-1}\frac{\sigma^j}{j!}f^{(j)}(t-\sigma) + \frac{\sigma^k}{k!}f^{(k)}(x) \geqslant \frac{\sigma^k}{k!}f^{(k)}(t), \quad t \geqslant T_1 + \sigma. \tag{7}$$

Similarly, expanding $f^{(k)}(t)$ by Taylor's theorem we get

$$f^{(k)}(t-\sigma) \geqslant \frac{(-\sigma)^{n-k-1}}{(n-k-1)!} f^{(n-1)}(t), \quad t \geqslant T_1 + \sigma. \tag{8}$$

Replacing t by $t-\sigma$ in the inequality (7) we get

$$f(t-\sigma) \geqslant \frac{\sigma^k}{k!} f^{(k)}(t-\sigma), \quad t \geqslant T_1 + 2\sigma. \tag{9}$$

Further, using (8) in (9) along with the fact that $k!(n-k-1)! \leqslant (n-1)!$ and setting $T_0 = T_1 + 2\sigma$ we have our proposed inequality.

This completes the proof of the lemma.

Theorem 1. *Suppose that* $p_i \in C([T, \infty), (0, \infty))$, $T > 0$ *and* $\sigma_i \in (0, \infty)$ $(i = 1, 2, 3, \ldots, m)$. *Further if*

$$\liminf_{t \to \infty} \int_{t-\omega\sigma_i}^{t} p_i(s)\,ds > 0 \quad (i = 1, 2, \ldots, m)$$

and (10)

$$\frac{1}{m} \sum_{i=1}^{m} \sigma_i^{n-1} p_{ii} + \frac{2}{m} \sum_{\substack{i<j \\ i,j=1}}^{m} (p_{ij} p_{ji} (\sigma_i \sigma_j)^{n-1})^{1/2} > (n-1)! \frac{(n)^{n-1}}{e},$$

where

$$p_{ij} = \int_{t-\omega\sigma_j}^{t} p_i(s)\,ds$$

and

$$\omega = \left(\frac{n-1}{n}\right),$$

then every solution of (3) *oscillates.*

Proof. On the contrary, suppose that $x(t) > 0$ for $t \geqslant t_0$. Dividing (3) throughout by $x^{(n-1)}(t)$ we get

$$\frac{x^{(n)}(t)}{x^{(n-1)}(t)} + \sum_{i=1}^{m} p_i(t) \frac{x(t-\sigma_i)}{x^{(n-1)}(t)} = 0, \tag{11}$$

that is,

$$\frac{x^{(n)}(t)}{x^{(n-1)}(t)} + \sum_{i=1}^{m} p_i(t) \frac{x(t-\sigma_i)}{x^{(n-1)}(t-\omega\sigma_i)} \frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t)} = 0. \tag{12}$$

By Lemma 1, there exists $t_1 \geqslant t_0$ such that

$$\frac{x(t-\sigma_i)}{x^{(n-1)}(t-\omega\sigma_i)} = \frac{x(t-\omega\sigma_i-\sigma_i/n)}{x^{(n-1)}(t-\omega\sigma_i)} \geqslant \frac{(\sigma_i/n)^{n-1}}{(n-1)!}, \quad t \geqslant t_1,$$

and the use of this inequality in (12) results

$$\frac{x^{(n)}(t)}{x^{(n-1)}(t)} + \sum_{i=1}^{m} K_i p_i(t) \frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t)} \leqslant 0, \tag{13}$$

where

$$K_i = \frac{1}{(n-1)!} \left(\frac{\sigma_i}{n}\right)^{n-1}.$$

Integrating both sides of (13) from $t-\omega\sigma_k$ to t we get

$$\log\left(\frac{x^{(n-1)}(t-\omega\sigma_k)}{x^{(n-1)}(t)}\right) \geqslant \sum_{i=1}^{m} K_i \int_{t-\omega\sigma_k}^{t} p_i(s)\frac{x^{(n-1)}(s-\omega\sigma_i)}{x^{(n-1)}(s)}\mathrm{d}s.$$

Setting

$$\alpha_i = \liminf_{t\to\infty} \frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t)}$$

and

$$p_{ik} = \liminf_{t\to\infty} \int_{t-\omega\sigma_k}^{t} p_i(s)\mathrm{d}s \quad i,j=1,2,\ldots,m$$

we see that

$$\log(\alpha_k) \geqslant \sum_{i=1}^{m} K_i\alpha_i p_{ik}.$$

Suppose that $\alpha_k < \infty$ for $k=1,2,3,\ldots,m$. In this case, dividing both sides of the above inequality by α_k and using the fact that

$$\frac{\log(\alpha_k)}{\alpha_k} \leqslant \frac{1}{e} \quad \text{for } \alpha_k \geqslant 1,$$

and $\alpha_k \geqslant 1$ (since $x^{(n-1)}(t)$ is positive decreasing) it follows that

$$\frac{1}{e} \geqslant \sum_{i=1}^{m} K_i\frac{\alpha_i}{\alpha_k}p_{ik}, \quad k=1,2,\ldots,m.$$

Summing the above inequality for $k=1,2,\ldots,m$ we obtain

$$\frac{m}{e} \geqslant \sum_{k=1}^{m}\sum_{i=1}^{m} K_i\frac{\alpha_i}{\alpha_k}p_{ik}.$$

that is,

$$\begin{aligned}\frac{m}{e} \geqslant\ & K_1\frac{\alpha_1}{\alpha_1}p_{11} + K_2\frac{\alpha_2}{\alpha_1}p_{21} + \cdots + K_m\frac{\alpha_m}{\alpha_1}p_{m1} \\ & + K_1\frac{\alpha_1}{\alpha_2}p_{12} + K_2\frac{\alpha_2}{\alpha_2}p_{22} + \cdots + K_m\frac{\alpha_m}{\alpha_2}p_{m2} \\ & + \cdots\cdots\cdots \\ & + \cdots\cdots\cdots \\ & + K_1\frac{\alpha_1}{\alpha_m}p_{1m} + K_2\frac{\alpha_2}{\alpha_m}p_{2m} + \cdots + K_m\frac{\alpha_m}{\alpha_m}p_{mm}.\end{aligned}$$

Rearranging the right hand side elements of the above inequality first along the diagonal then above and below the diagonal respectively, we get

$$\frac{m}{e} \geqslant \sum_{i=1}^{m} K_i\frac{\alpha_i}{\alpha_i}p_{ii} + \sum_{\substack{i>j\\ i,j=1}}^{m} K_i\frac{\alpha_i}{\alpha_j}p_{ij} + \sum_{\substack{i<j\\ i,j=1}}^{m} K_i\frac{\alpha_i}{\alpha_j}p_{ij}. \tag{14}$$

that is,

$$\frac{m}{e} \geqslant \sum_{i=1}^{m} K_i p_{ii} + \sum_{\substack{i<j\\ i,j=1}}^{m}\left(K_i p_{ij}\frac{\alpha_i}{\alpha_j} + K_j\frac{\alpha_j}{\alpha_i}p_{ji}\right). \tag{15}$$

Since the arithmetic mean is greater than the geometric mean

$$K_i p_{ij}\frac{\alpha_i}{\alpha_j} + K_j p_{ji}\frac{\alpha_j}{\alpha_i} \geqslant 2\sqrt{(p_{ij}p_{ji})(K_iK_j)}. \tag{16}$$

In view of (16), (15) reduces to

$$\frac{m}{e} \geqslant \sum_{i=1}^{m} K_i p_{ii} + 2 \sum_{\substack{i<j \\ i,j=1}}^{m} ((p_{ij}p_{ji})(K_iK_j))^{1/2}.$$

Putting the value of K_i and K_j in the above inequality we obtain

$$\frac{1}{m}\sum_{i=1}^{m} \sigma_i^{n-1} p_{ii} + \frac{2}{m} \sum_{\substack{i<j \\ i,j=1}}^{m} ((p_{ij}p_{ji})(\sigma_i\sigma_j)^{n-1})^{1/2} \leqslant (n-1)!\frac{n^{n-1}}{e},$$

which is a contradiction to our assumption.

Next, assume that $\alpha_i = \infty$ for some $i = 1, 2, \ldots, m$. That is,

$$\liminf_{t\to\infty} \frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t)} = \infty, \tag{17}$$

for some $i = 1, 2, \ldots, m$. From (3) it follows that

$$x^{(n)}(t) + p_i(t)x(t-\sigma_i) \leqslant 0,$$

for the value of i for which (17) holds. From the inequality above (13) it follows that

$$\frac{x(t-\sigma_i)}{x^{(n-1)}(t-\omega\sigma_i)} \geqslant \frac{(\sigma_i/n)^{n-1}}{(n-1)!}. \tag{18}$$

From (17) and (18) it follows that

$$x^{(n)}(t) + p_i(t)\frac{(\sigma_i/n)^{n-1}}{(n-1)!}x^{(n-1)}(t-\omega\sigma_i) \leqslant 0. \tag{19}$$

Integrating both sides of (19) from $t-\omega\sigma_i/2$ to t and using the fact that $x^{(n-1)}(t) > 0$ and decreasing we get

$$x^{(n-1)}(t) - x^{(n-1)}(t-\omega\sigma_i/2) + \frac{(\sigma_i/n)^{n-1}}{(n-1)!}\int_{t-\omega\sigma_i/2}^{t} p(s)\mathrm{d}s \leqslant 0. \tag{20}$$

Dividing both sides of (20) first by $x^{(n-1)}(t)$ and then by $x^{(n-1)}(t-\omega\sigma_i/2)$ we have the following inequalities respectively:

$$1 - \frac{x^{(n-1)}(t-\omega\sigma_i/2)}{x^{(n-1)}(t)} + \frac{(\sigma_i/n)^{n-1}}{(n-1)!}\frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t)}\int_{t-\omega\sigma_i/2}^{t} p_i(s)\mathrm{d}s < 0 \tag{21}$$

and

$$\frac{x^{(n-1)}(t)}{x^{(n-1)}(t-\omega\sigma_i/2)} - 1 + \frac{(\sigma_i/n)^{n-1}}{(n-1)!}\frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t-\omega\sigma_i/2)}\int_{t-\omega\sigma_i/2}^{t} p_i(s)\mathrm{d}s < 0. \tag{22}$$

In view of (10), (17) and (21) we obtain

$$\lim_{t\to\infty} \frac{x^{(n-1)}(t-\omega\sigma_i/2)}{x^{(n-1)}(t)} = \infty. \tag{23}$$

Using (23) in (22) along with (10) we see that

$$\lim_{t\to\infty}\frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t-\omega\sigma_i/2)}<\infty,$$

Replacing t by $t+\omega\sigma_i/2$ in the above inequality we get

$$\lim_{t\to\infty}\frac{x^{(n-1)}(t-\omega\sigma_i/2)}{x^{(n-1)}(t)}<\infty,$$

which is a contradiction to (23).

This completes the proof of the theorem.

COROLLARY 1.

If $p_i(t)=p_i\in(0,\infty)$ *and* $\sigma_i\in(0,\infty)$ *then*

$$\frac{1}{m}\left(\sum_{i=1}^{m}(p_i\sigma_i^n)^{1/2}\right)^2>(n^n(n-2)!)\frac{1}{e}$$

implies that every solution of (3) *oscillates.*

Proof. In this particular case

$$p_{ij}=\left(\frac{n-1}{n}\right)p_i\sigma_j$$

and hence (10) reduces to

$$\frac{1}{m}\left(\frac{n-1}{n}\right)\left\{\sum_{i=1}^{m}\sigma_i^n p_i+2\sum_{\substack{i<j\\ i,j=1}}^{m}(p_ip_j\sigma_i^n\sigma_j^n)^{1/2}\right\}>(n-1)!\frac{n^{n-1}}{e},$$

that is, (24) holds. Hence the proof follows from Theorem 1.

Example 1. The equation

$$x^{(3)}(t)+\left(4+\frac{1}{t}\right)x(t-1)+\left(16+\frac{1}{t^2}\right)x(t-2)=0,$$

satisfies the hypotheses of Theorem 1 and hence every solution of it oscillates. But Theorem 5.2 of [8] is not applicable to it.

Example 2. Consider the equation

$$x'(t)+\left(4+\frac{1}{t}\right)x(t-1)+\left(16+\frac{1}{t^2}\right)x(t-2)=0.$$

By Theorem 5.2 of [8], every solution of it oscillates. But Theorem 1 of this paper is not applicable to this equation. This is due to the fact that Theorem 1 holds only for $n>1$ and is an odd integer.

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Nontrivial solution of a quasilinear elliptic equation with critical growth in $\mathbb{R}^n$

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Abstract. Suppose $\Delta_n u = \operatorname{div}(|\nabla u|^{n-2}\nabla u)$ denotes the n-Laplacian. We prove the existence of a nontrivial solution for the problem

$$\begin{cases} -\Delta_n u + |u|^{n-2}u = f(x,u)u^{n-2} & \text{in } \mathbb{R}^n \\ u \in W^{1,n}(\mathbb{R}^n) \end{cases}$$

where $f(x,t) = o(t)$ as $t \to 0$ and $|f(x,t)| \leqslant C\exp(\alpha_n |t|^{n/(n-1)})$ for some constant $C > 0$ and for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ with $\alpha_n = n\omega_n^{1/(n-1)}$, ω_n = surface measure of S^{n-1}.

Keywords. Elliptic equation; critical growth; Palais–Smale condition; concentration compactness; mountain pass lemma.

1. Introduction

Suppose $\Delta_n u = \operatorname{div}(|\nabla u|^{n-2}\nabla u)$ denotes the n-Laplacian. We look for a solution of the problem

$$\begin{cases} -\Delta_n u + |u|^{n-2}u = f(x,u)u^{n-2} & \text{in } \mathbb{R}^n \\ u \in W^{1,n}(\mathbb{R}^n) \end{cases} \tag{1.1}$$

where $f(x,t) = o(t)$ as $t \to 0$ and $|f(x,t)| \leqslant C\exp(\alpha_n |t|^{n/(n-1)})$ for some constant $C > 0$ and for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ with $\alpha_n = n\omega_n^{1/(n-1)}$, ω_n = surface measure of S^{n-1}.

In the case where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, and $f(x,t) = h(x,t)\exp(\alpha_n |t|^{n/(n-1)})$ with $h(x,t)$ a lower order term in t, the problem (1.1) with Dirichlet boundary condition has been considered by Adimurthi [1] and with Neumann boundary condition by the author [9]. In case of $n = 2$, D M Cao [5] has shown the existence of a nontrivial solution for the problem (1.1). In this paper, applying the concentration-compactness principle of P L Lions [6, 7], we show that the functional associated with (1.1) satisfies (Palais–Smale)$_c$ (in short (PS)$_c$) condition for all $c \in (0, J)$ for some $J > 0$ (for definition of J see § 3). Then we show the existence of a nontrivial solution for (1.1) by using Mountain Pass lemma as given in [4] and constructing a critical point of the functional with critical value in $(0, J)$. The main difficulty here is to show that whenever a Palais–Smale sequence $u_m \xrightarrow{m} u$ weakly in $W^{1,n}(\mathbb{R}^n)$,

$$|\nabla u_m|^{n-2}\nabla u_m \xrightarrow{m} |\nabla u|^{n-2}\nabla u \quad \text{weakly in } (L^{n/(n-1)}(\mathbb{R}^n))^n$$

We need the following assumptions on the nonlinearity $f(x,t) \in C(\mathbb{R}^n \times \mathbb{R})$:

(f_1) $|f(x,t)| \leqslant C\exp(\alpha_n |t| n/(n-1))$ for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, where $C > 0$ is some constant.

(f_2) $f(x,t)t^{n-1} = f(x,-t)(-t)^{n-1}$ for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$; $\dfrac{f(x,t)}{t}$ is nondecreasing with respect

to t, for $t>0$;

$$\lim_{t\to 0}\frac{f(x,t)}{t}=0 \quad \text{uniformly with respect to } x\in\mathbb{R}^n;$$

$$\lim_{t\to\infty}\frac{f(x,t)}{t}=\infty \quad \text{uniformly with respect to } x\in\mathbb{R}^n.$$

(f_3) There exists $\theta\in\left(0,\frac{1}{n}\right)$ such that

$$F(x,t)\leqslant \theta t^{n-1}f(x,t) \quad \text{for} \quad x\in\mathbb{R}^n, t\in\mathbb{R},$$

where $F(x,t)=\int_0^t f(x,s)s^{n-2}\mathrm{d}s$

(f_4) $\exists \bar{f}(t)$ such that $\lim_{|x|\to\infty} f(x,t)=\bar{f}(t)$ uniformly for t bounded, more precisely,

$$|f(x,t)-\bar{f}(t)|\leqslant \varepsilon(R)|t|^{n-1} \quad \text{for} \quad |x|\geqslant R,$$

where $\varepsilon(R)\to 0$ as $R\to\infty$.

(f_5) $\exists p>n$ such that $f(x,t)\geqslant \bar{f}(t)\geqslant C_p t^{p-n+1}>(p/n)S_p^p(1-n\theta)^{1-(p/n)}t^{p-n+1}$ for $x\in\mathbb{R}^n$, $t\in\mathbb{R}^+$, where

$$S_p=\inf_{\substack{u\in W^{1,n}(R^n)\\ u\neq 0}}\frac{[\int_{\mathbb{R}_n}(|\nabla u|^n+|u|^n)\mathrm{d}x]^{1/n}}{(\int_{\mathbb{R}_n}|u|^p\mathrm{d}x)^{1/p}}.$$

For $u\in W^{1,n}(\mathbb{R}^n)$ let

$$I^\infty(u)=\frac{1}{n}\int_{\mathbb{R}^n}(|\nabla u|^n+|u|^n)\mathrm{d}x-\int_{\mathbb{R}^n}\bar{F}(u)\mathrm{d}x, \tag{1.2}$$

where $\bar{F}(t)=\int_0^t \bar{f}(s)s^{n-2}\mathrm{d}s$. The main results in this paper are as follows.

Theorem 1.1. *Suppose $f(x,t)\equiv\bar{f}(t)$ does not depend on x and satisfies (f_1)–(f_3) and (f_5). Then (1.1) has a nontrivial solution u_0. Moreover $I^\infty(u_0)<(1/n)-\theta$.*

Theorem 1.2. *Suppose $f(x,t)$ satisfies (f_1)–(f_5) and $f(x,t)\not\equiv\bar{f}(t)$ for fixed t with respect to $x\in\mathbb{R}^n$. Then (1.1) has a nontrivial solution.*

We remark that the (PS) condition is not needed for the proof of Theorem (1.1).

2. Preliminaries and notations

We shall denote $\int\cdots$ to mean $\int_{\mathbb{R}^n}\cdots\mathrm{d}x$. Define

$$\|u\|=\left(\int(|\nabla u|^n+|u|^n)\right)^{1/n} \quad \text{for} \quad u\in W^{1,n}(\mathbb{R}^n) \tag{2.1}$$

$$|u|_q=\left(\int|u|^q\right)^{1/q} \quad \text{for} \quad u\in L^q(\mathbb{R}^n). \tag{2.2}$$

The variational functional associated with (1.1) is

$$I(u)=\frac{1}{n}\|u\|^n-\int F(x,u) \tag{2.3}$$

where $F(x,t)=\int_0^t f(x,s)s^{n-2}\mathrm{d}s$

Let $I^\infty(u)$ be as in (1.2),

$$M^\infty = \left\{ u \in W^{1,n}(\mathbb{R}^n) \setminus \{0\} \mid \|u\|^n = \int u^{n-1} \bar{f}(u) \right\} \tag{2.4}$$

$$C^\infty = \begin{cases} \inf \{ I^\infty(u) \mid u \in M^\infty \}, & \text{if } M^\infty \neq \phi \\ \infty, & \text{if } M^\infty = \phi, \end{cases} \tag{2.5}$$

$$I_0^\infty = \inf \left\{ \int |\nabla u|^n \mid u \in W^{1,n}(\mathbb{R})^n \setminus \{0\}, \int \bar{F}(u) = \frac{1}{n} \int |u|^n \right\}.$$

Remark. If $\bar{f}(t)$ satisfies (f_1) and (f_2) then $I_0^\infty > 0$.

Proof. Suppose, on the contrary, $I_0^\infty = 0$. Then there exists a sequence $\{u_m\}$ in $W^{1,n}(\mathbb{R}^n)$ such that

$$\int \bar{F}(u_m) = \frac{1}{n} \int |u_m|^n$$

$$\int |\nabla u_m|^n \xrightarrow[m]{} 0.$$

Then by (f_1), (f_2) and Lemma 2.3 (to be proved)

$$\left| \int \bar{F}(u_m) \right| \leqslant \frac{1}{2n} \int |u_m|^n + C_1 \int |u_m|^{n-1} (\exp(\alpha_n |u_m|^{n/(n-1)}) - 1 - \alpha_n |u_m|^{n/(n-1)})$$

$$\leqslant \frac{1}{2n} |u_m|_n^n + C_2 |\nabla u_m|_n |u_m|_n^n,$$

where $C_1, C_2 > 0$ are some constants. Thus

$$\frac{1}{2n} |u_m|_n^n \leqslant C_2 |\nabla u_m|_n |u_m|_n^n,$$

and therefore $|\nabla u_m|_n \geqslant 1/2nC_2$, a contradiction which proves the remark. □

Remark. If $\bar{f}(t)$ satisfies (f_1)–(f_3) then $C^\infty > 0$.

Proof. Suppose $M^\infty \neq \phi$. For $u \in M^\infty$, using (f_3) we get

$$I^\infty(u) = \frac{1}{n} \int (|\nabla u|^n + |u|^n) - \int \bar{F}(u)$$

$$= \frac{1}{n} \int (|\nabla u|^n + |u|^n) - \int \theta \bar{f}(u) u^{n-1}$$

$$= \left(\frac{1}{n} - \theta \right) \|u\|^n.$$

Since $\bar{f}(t)$ satisfies (f_1) and (f_2) we have

$$\bar{f}(t) t^{n-1} \leqslant \frac{1}{n} |t|^n + C_1 |t|^{n-1} (\exp(\alpha_n |t|^{n/(n-1)}) - 1 - \alpha_n |t|^{n/(n-1)})$$

and therefore as in the above Remark we obtain

$$\inf\left\{\int |\nabla u|^n : u \in M^\infty\right\} \geqslant C_2$$

for some positive constant C_2. Therefore by above estimate $C^\infty \geqslant ((1/n) - \theta)C_2$. This proves the Remark. □

Similar to the imbedding of Moser [8] we have

Lemma 2.1. *Suppose* $u \in W^{1,n}(\mathbb{R}^n)$, $|\nabla u|_n^n \leqslant r < 1$, $|u|_n \leqslant M < \infty$. *Then*

$$\int \left[\exp(\alpha_n |u|^{n/(n-1)}) - \sum_{m=0}^{n-2} \frac{\alpha_n^m |u|^{nm/(n-1)}}{m!}\right] \leqslant C(M, r), \tag{2.6}$$

where $C(M, r) > 0$ *is a constant independent of* u.

Proof. As in Moser [8] we use the method of symmetrization. Let u^* be the symmetrization of u. Then u^* is a radial, nonnegative and nonincreasing function. Further,

$$\int |u^*|^p = \int |u|^p, \quad 1 < p < \infty \tag{2.7}$$

$$\int G(u) = \int G(u^*), \tag{2.8}$$

$$\int |\nabla u^*|^n \leqslant \int |\nabla u|^n, \tag{2.9}$$

where $G(u)$ is the integrand on the l.h.s. of (2.6). We have

$$\int G(u) = \int G(u^*) = \int_{|x|<s} G(u^*) + \int_{|x|\geqslant s} G(u^*) \tag{2.10}$$

where $s > 0$ is a number to be determined.

First we estimate the second integral in (2.10). By the radial Lemma A. IV in [3] we have

$$|u^*(x)| \leqslant \left(\frac{n}{\omega_n}\right)^{1/n} |u^*|_n |x|^{-1} \quad \text{for} \quad x \neq 0. \tag{2.11}$$

Thus

$$\begin{aligned}
\int_{|x|\geqslant s} G(u) &= \frac{\alpha_n^{n-1}|u^*|_n^n}{(n-1)!} + \int_{|x|\geqslant s} \left(\sum_{m=n}^{\infty} \frac{\alpha_n^m |u^*|_{n-1}^{nm}}{m!}\right) \\
&\leqslant \frac{\alpha_n^{n-1}|u^*|_n^n}{(n-1)!} + \sum_{m=n}^{\infty} \frac{1}{m!} \alpha_n^m \left(\frac{n}{\omega_n}\right)^{m/(n-1)} |u^*|_n^{nm/(n-1)} \int_{|x|\geqslant s} \frac{1}{|x|^{nm/(n-1)}} \, dx \\
&\leqslant \frac{\alpha_n^{n-1}|u^*|_n^n}{(n-1)!} + \frac{\omega_n}{s^{-n}} \left(\frac{n-1}{n}\right) \sum_{m=n}^{\infty} \frac{1}{m!} \left(\frac{n|u^*|_n}{s}\right)^{nm/(n-1)} \\
&\leqslant C(M) \quad \text{if} \quad s > n|u^*|_n.
\end{aligned} \tag{2.12}$$

To estimate the first integral in (2.10), let us put $|x|^n = s^n e^{-t}$, $v(t) = n^{(n-1)/n} u^*(x)$. Then

$$\int_0^\infty \dot{v}^n(t)\mathrm{d}t = \int_{|x|<s} |\nabla u^*|^n, \tag{2.13}$$

$$\int_0^\infty \exp(|v(t)|^{n/(n-1)} - t)\mathrm{d}t = \frac{n}{\omega_n r^n} \int_{|x|<s} \exp(\alpha_n |u^*|^{n/(n-1)})\mathrm{d}x, \tag{2.14}$$

where $\dot{v} = \mathrm{d}v/\mathrm{d}t$. By Hölder inequality we have

$$\begin{aligned} v(t) &= v(0) + \int_0^t \dot{v}(s)\,\mathrm{d}s \\ &\leqslant v(0) + \left(\int_0^t |\dot{v}(s)|^n\right)^{1/n} t^{(n-1)/n} \\ &\leqslant n^{(n-1)/n} \omega_n^{1/n} u^*(sx_0) + |\nabla u^*|_n t^{(n-1)/n} \end{aligned} \tag{2.15}$$

where x_0 is some unit vector in $\mathbb{R}^n$. Now

$$\begin{aligned} \int_{|x|<s} G(u^*) &< \int_{|x|<s} \exp(\alpha_n |u^*|^{n/(n-1)}) \\ &= \frac{\omega_n s^n}{n} \int_0^\infty \exp(|v(t)|^{n/(n-1)} - t)\mathrm{d}t \\ &\leqslant \frac{\omega_n s^n}{n} 2^{n/(n-1)} \exp(\alpha_n |u^*(sx_0)|^{n/(n-1)}) \int_0^\infty \exp(|\nabla u^*|_n^{n/(n-1)} t - t)\mathrm{d}t \\ &\leqslant \frac{\omega_n s^n}{n} 2^{n/(n-1)} \exp(\alpha_n |u^*(sx_0)|^{n/(n-1)}). \end{aligned} \tag{2.16}$$

Combining (2.11), (2.12) and (2.16) we have (2.6). □

Lemma 2.2. *There exists $\beta = \beta(n) > 0$ such that for all $u \in W^{1,n}(\mathbb{R}^n)$ with $|\nabla u|_n^n < 1/\alpha_n \beta e$, we have*

$$\int |u|^{n-1} (\exp(\alpha_n |u|^{n/(n-1)}) - 1 - \alpha_n |u|^{n/(n-1)}) \leqslant C |u|_n^n |\nabla u|_n$$

where $C > 0$ is a constant independent of u.

Proof. By the result of Talenti [10] (or of Aubin [2]) we know that if $t, s > 1$, $t < n$ and $1/s = 1/t - 1/n$, then all $\varphi \in W^{1,t}(\mathbb{R}^n)$ satisfy

$$|\varphi|_s \leqslant K(n,t) |\nabla \varphi|_t, \tag{2.17}$$

with

$$K(n,t) = \frac{t-1}{n-t} \left[\frac{n-t}{n(t-1)}\right]^{1/t} \left[\frac{\Gamma(n+1)}{\Gamma(n/t)\Gamma(n+1-n/t)\omega_n}\right]^{1/n}.$$

Let us set $\varphi = |u|^\nu$, where $\nu = ((n-1)^2 + nm)/(nm - n + 1)$, $m \geqslant 2$. Then $|\nabla \varphi| = \nu |u|^{\nu-1} |\nabla u|$. Taking $s = (n/(n-1))m - 1$, $t = n - (n^2(n-1))/((n-1)^2 + nm)$ and using

Hölder inequality we get

$$\int |u|^{nm/(n-1)+n-1} \leqslant (Kv)^{nm/(n-1)-1} \left(\int |u|^{t(v-1)} |\nabla u|^t \right)^{s/t}$$

$$\leqslant (Kv)^{nm/(n-1)-1} |u|_n^n |\nabla u|_n^{nm/(n-1)-1}. \tag{2.18}$$

Now

$$K(n,t) = \left(\frac{1}{n}\right)^{1/t} (t-1)^{(t-1)/t} \left[\frac{\Gamma(n+1)}{\Gamma(n/t)\Gamma(n+1-(n/t))\omega_n} \right]^{1/n} (n-t)^{(1-t)/t}$$

$$\leqslant C(n)(n-t)^{(1-t)/t}$$

$$= C(n) \left[\frac{(n-1)^2 + nm}{n^2(n-1)} \right]^{(t-1)/t}$$

where $C(n)$ is a constant dependent on n. Since $t < n$ we get $(t-1)/t < (n-1)/n$. Also for all $m \geqslant 2, n \geqslant 2$ we have $((n-1)^2 + nm)/n^2(n-1) < m$ and $v \leqslant C_1(n)$, a constant dependent on n. Hence we get $(Kv)^{nm/(n-1)-1} \leqslant C(\beta m)^m$ for some $\beta = \beta(n) > 0$ and $C = C(n) > 0$. Therefore by (2.18)

$$\int |u|^{n-1} (\exp(\alpha_n |u|^{n/(n-1)}) - 1 - \alpha_n |u|^{n/(n-1)})$$

$$= \int \left(\sum_{m=2}^{\infty} \frac{1}{m!} \alpha_n^m |u|^{nm/(n-1)+n-1} \right)$$

$$\leqslant C \sum_{m=2}^{\infty} \frac{1}{m!} (m\alpha_n \beta)^m |\nabla u|_n^{nm/(n-1)-1} |u|_n^n.$$

For $m \geqslant 2$ we have $mn/(n-1) - 2 \geqslant m/(n-1)$. Thus for $|\nabla u|_n^{1/(n-1)} < 1/\alpha_n \beta e$ we have

$$\int |u|^{n-1} (\exp(\alpha_n |u|^{n/(n-1)}) - 1 - \alpha_n |u|^{n/(n-1)})$$

$$\leqslant C \sum_{m=2}^{\infty} \frac{1}{m!} (m\alpha_n \beta)^m (|\nabla u|_n^{1/(n-1)})^m |\nabla u|_n |u|_n^n$$

$$\leqslant C |\nabla u|_n |u|_n^n.$$

where we have used the same C to denote various constants. □

Lemma 2.3. *Let $\bar{f}(t)$ satisfy (f_1) and (f_2). Suppose there exists $u_0 \in W^{1,n}(\mathbb{R}^n) \setminus \{0\}$ such that $\int \bar{F}(u_0) \geqslant (1/n) \int |u_0|^n$ and $|\nabla u_0|_n^n < 1$. Then I_0^∞ is achieved. Moreover $I_0^\infty \leqslant |\nabla u_0|_n^n$.*

Proof. Using (f_2) and the hypothesis that $\int \bar{F}(u_0) \geqslant (1/n)|u_0|_n^n$ it is easy to see that there exists $t_0 \in (0,1]$ such that $\int \bar{F}(t_0 u_0) = (1/n) \int |t_0 u_0|^n$. Thus $I_0^\infty \leqslant \int |\nabla u_0|^n$. Let $\{u_m\}$ be a minimizing sequence for I_0^∞. Without loss of generality we can assume that $\int |\nabla u_m|^n \leqslant r < 1$. Denote by u_m^* the symmetrization of u_m. Then u_m^* is a radial, nonincreasing function. Furthermore,

$$\int F(u_m^*) = \int F(u_m)$$

$$\int |u_m^*|^n = \int |u_m|^n$$

$$\int |\nabla u_m^*|^n \leqslant \int |\nabla u_m|^n.$$

Thus $\{u_m^*\}$ is still a minimizing sequence of I_0^∞. We denote it simply by $\{u_m\}$ in what follows. Without loss of generality we can assume that $|u_m|_n = 1$. Thus $\{u_m\}$ is bounded in $W^{1,n}(\mathbb{R}^n)$ and so there exists $u \in W^{1,n}(\mathbb{R}^n)$ such that for a subsequence

$$u_m \to u \quad \text{weakly in } W^{1,n}(\mathbb{R}^n)$$

$$u_m \to u \quad \text{a.e. in } \mathbb{R}^n.$$

We want to use a compactness lemma of Strauss (see Theorem A.I. of [3]). Set $P(t) = \bar{F}(t)$,

$$Q(t) = \exp\left(\frac{2\alpha_n}{1+r}|t|^{n/(n-1)}\right) - \sum_{j=0}^{n-2} \frac{1}{j!}\left(\frac{2\alpha_n}{1+r}\right)^j |t|^{nj/(n-1)} + |t|^n.$$

Then using (f_1) we get $\lim_{|t|\to\infty} P(t)/Q(t) = 0$ and using (f_2), $\lim_{|t|\to 0} P(t)/Q(t) = 0$. Again by radial lemma A.IV of [3] we get (2.11) and so as $|x| \to \infty$, $u_m(x) \to 0$ uniformly in m. Further by Lemma 2.1,

$$\sup_m \int Q(u_m) \leqslant C.$$

Thus all the conditions of Strauss' lemma are satisfied and we get

$$\lim_{m\to\infty} \int \bar{F}(u_m) = \int \bar{F}(u).$$

Since $\int F(u_m) = \frac{1}{2}$ we get $u \not\equiv 0$. Now

$$\frac{1}{n}\int |u|^n \leqslant \frac{1}{n} \liminf_{m\to\infty} \int |u_m|^n = \liminf_{m\to\infty} \int \bar{F}(u_m) = \int \bar{F}(u).$$

If $(1/n)\int |u|^n = \int \bar{F}(u)$, then

$$I_0^\infty \leqslant \int |\nabla u|^n \leqslant \liminf_{m\to\infty} \int |\nabla u_m|^n = I_0^\infty,$$

and so I_0^∞ is achieved by u. If on the other hand $(1/n)\int |\nabla u|^n < \int \bar{F}(u)$, then there exists $t \in (0,1)$ such that $(1/n)\int |tu|^n = \int \bar{F}(tu)$. Hence

$$I_0^\infty \leqslant t^n \int |\nabla u|^n < \int |\nabla u|^n \leqslant \liminf_{m\to\infty} \int |\nabla u_m|^n = I_0^\infty,$$

a contradiction which proves the lemma □

Lemma 2.4. *Suppose* $\{u_m\} \subset W^{1,n}(\mathbb{R}^n)$ *satisfies* $|\nabla u_m|_n < 1$, $|u_m|_n^n < M$ *and*

$$\lim_{m\to\infty} \sup_{y\in\mathbb{R}^n} \int_{y+B_R} (|\nabla u_m|^n + |u_m|^n)\mathrm{d}x = 0 \quad \textit{for some } R > 0,$$

where $B_R = \{x \in \mathbb{R}^n; |x| < R\}$. *Then*

$$\lim_{m\to\infty} \int F(x, u_m) = 0$$

$$\lim_{m\to\infty} \int f(x, u_m) u_m^{n-1} = 0. \tag{2.19}$$

Proof. Let $\xi \in C_0^\infty(\mathbb{R}^n)$ be such that $\xi \equiv 1$ for $|x| < R/2$; $\xi \equiv 0$ for $|x| > R$ and $|\nabla \xi| < 4n/R$. Let $\xi_y = \xi(\cdot - y)$. Then

$$\begin{aligned}\int_{y+B_R} |\nabla(\xi_y u_m)|^n &\leqslant 2^n \int_{y+B_R} [|\nabla \xi_y|^n |u_m|^n + |\xi_y|^n |\nabla u_m|^n] \mathrm{d}x \\ &\leqslant 2^n \left[1 + \left(\frac{4n}{R}\right)^n\right] \int_{y+B_R} (|\nabla u_m|^n + |u_m|^n) \mathrm{d}x. \end{aligned} \tag{2.20}$$

In view of (f_1) and (f_2), given $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}&\int_{y+B_{R2}} |F(x, u_m)| \mathrm{d}x \\ &\leqslant C_\varepsilon \int_{y+B_{R2}} |u_m|^{n-1} (\exp(\alpha_n |u_m|^{n/(n-1)}) - 1 - \alpha_n |u_m|^{n/(n-1)}) \mathrm{d}x \\ &\quad + \varepsilon \int_{y+B_{R2}} |u_m|^n \mathrm{d}x \\ &\leqslant C_\varepsilon \int_{y+B_R} |u_m|^{n-1} (\exp(\alpha_n |\xi_y u_m|^{n/(n-1)}) - 1 - \alpha_n |\xi_y u_m|^{n/(n-1)}) \mathrm{d}x \\ &\quad + \varepsilon \int_{y+B_{R/2}} |u_m|^n \mathrm{d}x.\end{aligned}$$

For m large enough $|\nabla(\xi_y u_m)|_n^n < 1/\alpha_n \beta e$ and hence Lemma 2.3 gives

$$\int_{y+B_{R/2}} |F(x, u_m)| \mathrm{d}x \leqslant \tilde{C}_\varepsilon |\nabla(\xi_y u_m)|_n \int_{y+B_R} |u_m|^n \mathrm{d}x + \varepsilon \int_{y+B_R} |u_m|^n \mathrm{d}x.$$

We cover $\mathbb{R}^n$ by balls $B_{R/2}(x_i)$ in such a way that any point of $\mathbb{R}^n$ is contained in at most k balls $B_R(x_i)$ of radius R. For large m we have

$$\begin{aligned}\int_{\mathbb{R}^n} |F(x, u_m)| \mathrm{d}x &\leqslant k\tilde{C}_\varepsilon \sup_{y\in\mathbb{R}^n} \left[\int_{y+B_R} (|\nabla u_m|^n + |u_m|^n) \mathrm{d}x\right]^{1/n} \left[\int_{\mathbb{R}^n} |u_m|^n \mathrm{d}x\right] \\ &\quad + k\varepsilon \int_{\mathbb{R}^n} |u_m^{n-1} f(x, u_m)|. \end{aligned} \tag{2.21}$$

Making $m \to \infty$ and then $\varepsilon \to 0+$ in (2.21) we obtain

$$\lim_{m\to\infty} \int F(x, u_m) = 0. \tag{2.22}$$

Similarly we have

$$\lim_{m\to\infty}\int f(x,u_m)u_m^{n-1}=0. \tag{2.23}$$

□

3. Proof of the main results

First we prove the following

Lemma 3.1. *Let* C^∞ *be as in* (2.5) *and*

$$J=\min\left(C^\infty,\frac{1}{n}-\theta\right).$$

Suppose $f(x,t)$ *satisfies* (f_1)–(f_4). *Then* $I(u)$ *satisfies* $(\mathrm{PS})_c$ *condition for* $c\in(0,J)$.

Proof. Let $\{u_m\}$ be a $(\mathrm{PS})_c$ sequence in $W^{1,n}(\mathbb{R}^n)$. That is,

$$I(u_m)\xrightarrow[m]{} C\in(0,J)$$

$$I'(u_m)\xrightarrow[m]{}0 \quad \text{in } (W^{1,n}(\mathbb{R}^n))^*.$$

Then

$$\frac{1}{n}\int(|\nabla u_m|^n+|u_m|^n)-F(x,u_m)=c+o(1) \tag{3.1}$$

$$\int(|\nabla u_m|^{n-2}\nabla u_m\cdot\nabla\varphi+|u_m|^{n-2}u_m\varphi)-\int f(x,u_m)u_m^{n-2}\varphi=\langle\xi_m,\varphi\rangle \tag{3.2}$$

where $o(1)$ denotes the quantities that tend to 0 as $m\to\infty$ and $\xi_m\xrightarrow[m]{}0$ in $(W^{1,n}(\mathbb{R}^n))^*$. Taking $\varphi=u_m$ in (3.2) we obtain

$$\int(|\nabla u_m|^n+|u_m|^n)-\int f(x,u_m)u_m^{n-1}=\langle\xi_m,u_m\rangle. \tag{3.3}$$

Claim 1: $\|u_m\|^n\leqslant nc/(1-n\theta)+o(1)$

From (3.1) and (3.3) we get

$$\int[f(x,u_m)u_m^{n-1}-F(x,u_m)]\leqslant nc+o(1)+|\langle\xi_m,u_m\rangle|.$$

Thus using (f_3),

$$(1-n\theta)\int f(x,u_m)u_m^{n-1}\leqslant nc+o(1)+|\langle\xi_m,u_m\rangle|$$

and hence in view of (3.3) $\{u_m\}$ is bounded. Further (3.3) gives

$$\|u_m\|^n\leqslant\frac{nc}{1-n\theta}+o(1) \quad \text{as } m\to\infty$$

as desired.

Thus for m large enough

$$|\nabla u_m|_n^n \leqslant r \tag{3.4}$$

where $r \in (0,1)$ is some fixed number, and there exists $u \in W^{1,n}(\mathbb{R}^n)$ such that for a subsequence

$$\begin{cases} u_m \to u & \text{weakly in } W^{1,n}(\mathbb{R}^n) \\ u_m \to u & \text{a.e. in } \mathbb{R}^n \\ (|\nabla u_m|^n + |u_m|^n)\mathrm{d}x \to \mathrm{d}\mu & \text{in measure} \\ |\nabla u_m|^{n-2}\nabla u_m \to T & \text{weakly in } (L^{n/(n-1)}(\mathbb{R}^n))^n. \end{cases} \tag{3.5}$$

Without loss of generality we may assume

$$\|u_m\|^n \to l \geqslant 0$$

$$|\nabla u_m|_n^n \leqslant r < 1 \quad \text{for all } m \geqslant 1.$$

Claim 2: $l > 0$.

If not, suppose $l = 0$. Then by (f_1), (f_2) and Lemma 2.3

$$\left| \int f(x, u_m) u_m^{n-1} \right|$$

$$\leqslant C \int |u_m|^{n-1} (\exp(\alpha_n |u_m|^{n/(n-1)}) - 1 - \alpha_n |u_m|^{n/(n-1)}) + C \int |u_m|^n$$

$$\leqslant C \int |u_m|^n \xrightarrow[m]{} 0.$$

Similarly $|\int F(x, u_m)| \xrightarrow[m]{} 0$.

So $I(u_m) = (1/n)\|u_m\|^n - \int F(x, u_m) \xrightarrow[m]{} 0$, which contradicts the fact that $I(u_m) \xrightarrow[m]{} c \neq 0$. Hence $l > 0$.

We want to apply concentration-compactness principle of P L Lions [6, 7] to the sequence $\{\rho_m\}$ where $\rho_m = |\nabla u_m|^n + |u_m|^n$. Applying Lemma 1.1 of [6] we conclude that for a subsequence one of the three possibilities holds: (a) vanishing, (b) dischotomy (c) compactness. We use contradiction argument to show that only (c) compactness occurs.

Step 1: Vanishing does not occur.
Suppose instead that

$$\lim_{m \to \infty} \sup_{y \in \mathbb{R}^n} \int_{y + B_R} (|\nabla u_m|^n + |u_m|^n) = 0 \quad \text{for all } R > 0.$$

Then Lemma 2.4 yields

$$\int F(x, u_m) \xrightarrow[m]{} 0, \quad \int f(x, u_m) u_m^{n-1} \xrightarrow[m]{} 0.$$

This implies, in view of (3.3), that $\|u_m\|^n \xrightarrow[m]{} 0$, which is not possible since $l > 0$. So vanishing does not occur.

Step 2: Dichotomy does not occur.
Suppose dichotomy occurs. Let $Q_m(t) = \sup_{y\in\mathbb{R}^n}\int_{y+B_t}\rho_m(x)\,dx$ denote the concentration function of ρ_m. Then $\{Q_m\}$ is a sequence of nondecreasing nonnegative uniformly bounded functions on $\mathbb{R}^+$. As in [6], by extracting a subsequence we can assume that there exists $Q(t)$ such that $Q_m(t) \xrightarrow[m]{} Q(t)$ and since dichotomy occurs, $\lim_{t\to\infty} Q(t) = \alpha\in(0, l)$. For any $\varepsilon>0$, $\varepsilon < 1/(2n)^n\alpha_n\beta e$, we can choose $t_0>0$ such that $Q(t)\geqslant\alpha-(\varepsilon/4)$ if $t\geqslant t_0$. Then for m large enough $\alpha-(\varepsilon/4)\leqslant Q_m(t)\leqslant\alpha+\varepsilon/4$ if $t\geqslant t_0$. Furthermore, there exists $\{y_m\}\subset\mathbb{R}^n$ such that

$$\int_{y_m+B_t}\rho_m\in\left(\alpha-\frac{\varepsilon}{4},\alpha+\frac{\varepsilon}{4}\right) \tag{3.6}$$

for $t\geqslant t_0$ and m large enough. Also we can find $t_m\xrightarrow[m]{}\infty$ such that

$$\int_{y_m+B_{2t_m+2}}\rho_m\in\left(\alpha-\frac{\varepsilon}{2},\alpha+\frac{\varepsilon}{2}\right). \tag{3.7}$$

Let $\varphi,\psi\in C^\infty(\mathbb{R}^n)$ be bounded cut-off functions such that $0\leqslant\varphi,\psi\leqslant 1$, $\psi\equiv 0$ if $|x|\geqslant 3$, $\psi\equiv 1$ if $|x|\leqslant 2$; $\varphi\equiv 1$ if $|x|\geqslant 2$; $\varphi\equiv 0$ if $|x|\leqslant 1$. Set $\psi_m=\psi((\cdot-y_m)/t_1)$ and $\varphi_m=\varphi((\cdot-y_m)/t_m)$, where $t_1>t_0$. Denote $v_m=\psi_m u_m$ and $w_m=\varphi_m u_m$.

Claim 3: $I(u_m)\geqslant I(v_m)+I(w_m)-C\varepsilon$.

By computation we deduce

$$\left|\int(\psi_m^n|\nabla u_m|^n-|\nabla v_m|^n)\right|\leqslant\frac{C}{t_1}.$$

Choosing t_1 large enough we have

$$\left|\int(\psi_m^n|\nabla u_m|^n-|\nabla v_m|^n)\right|<\varepsilon. \tag{3.8}$$

With m large enough so that $t_m>3t_1$, using (f_1), (f_2) we get

$$\begin{aligned}
&\left|\int[\psi_m^n u_m^{n-1}f(x,u_m)-v_m^{n-1}f(x,v_m)]\right| \\
&\quad=\left|\int_{2t_1<|x-y_m|<3t_1}[\psi_m^n u_m^{n-1}f(x,u_m)-v_m^{n-1}f(x,v_m)]\right| \\
&\quad=C\int_{2t_1<|x-y_m|<3t_1}|u_m|^{n-1}(\exp(\alpha_n|u_m|^{n/(n-1)})-1-\alpha_n|u_m|^{n/(n-1)}) \\
&\qquad+C\int_{2t_1<|x-y_m|<3t_1}|u_m|^n.
\end{aligned} \tag{3.9}$$

Let $\eta_m\in C_0^\infty(\mathbb{R}^n)$ be such that $0\leqslant\eta_m\leqslant 1$, $\eta_m\equiv 0$ if $|x-y_m|<t_1$ or $|x-y_m|>2t_m+2$; $\eta_m\equiv 1$ if $2t_1\leqslant|x-y_m|\leqslant 2t_m$ and $|\nabla\eta_m|\leqslant n$. Then using (3.6) and (3.7) we get

$$\begin{aligned}
\int|\nabla(\eta_m u_m)|^n&\leqslant 2^n\int[|\nabla\eta_m|^n|u_m|^n+|\nabla u_m|^n|\eta_m|^n] \\
&\leqslant(2n)^n\int_{t_1\leqslant|x-y_m|\leqslant 2t_m+2}[|u_m|^n+|\nabla u_m|^n]\leqslant\frac{1}{\alpha_n\beta e}.
\end{aligned}$$

Hence applying Lemma 2.3 we get

$$\left|\int [\psi_m^n u_m^{n-1} f(x,u_m) - v_m^{n-1} f(x,v_m)]\right|$$

$$\leqslant C \int_{2t_1 \leqslant |x-y_m|} |\eta_m u_m|^{n-1} (\exp(\alpha_n |\eta_m u_m|^{n/(n-1)}) - 1 - \alpha_n |\eta_m u_m|^{n/(n-1)})$$

$$+ C \int_{2t_1 \leqslant |x-y_m| \leqslant 3t_1} |u_m|^n$$

$$\leqslant C \int_{t_1 \leqslant |x-y_m| \leqslant 2t_m+2} |\eta_m u_m|^n + C \int_{2t_1 \leqslant |x-y_m| \leqslant 3t_1} |u_m|^n < C\varepsilon. \tag{3.10}$$

Similarly we have for m large,

$$\left|\int [\varphi_m^n |\nabla u_m|^n - |\nabla w_m|^n]\right| < C\varepsilon \tag{3.11}$$

$$\left|\int [\varphi_m^n u_m^{n-1} f(x,u_m) - w_m^{n-1} f(x,w_m)]\right| < C\varepsilon. \tag{3.12}$$

Combining (3.6), (3.8) with (3.11) we get

$$\left|\int [|\nabla u_m|^n - |\nabla v_m|^n - |\nabla w_m|^n]\right| \leqslant C\varepsilon \tag{3.13}$$

$$\left|\int [|u_m|^n - |v_m|^n - |w_m|^n]\right| < C\varepsilon. \tag{3.14}$$

Also using (f_1) and (f_2) we get

$$\left|\int [F(x,u_m) - F(x,v_m) - F(x,w_m)]\right|$$

$$\leqslant C \int_{2t_1 \leqslant |x-y_m| \leqslant 2t_m} |u_m|^{n-1} (\exp(\alpha_n |u_m|^{n/(n-1)}) - 1 - \alpha_n |u_m|^{n/(n-1)})$$

$$+ C \int_{2t_1 \leqslant |x-y_m| \leqslant 2t_m} |u_m|^n,$$

and as in the proof of (3.10) we obtain

$$\left|\int [F(x,u_m) - F(x,v_m) - F(x,w_m)]\right| \leqslant C\varepsilon. \tag{3.15}$$

From (3.13), (3.14) and (3.15) we have

$$I(u_m) \geqslant I(v_m) + I(w_m) - C\varepsilon \tag{3.16}$$

and this proves the claim.

We will now consider two cases, Case 1: $\{y_m\}$ is bounded, and Case 2: $\{y_m\}$ is unbounded.

Case 1: $\{y_m\}$ is bounded.

aim 4: $I(w_m) \geqslant I^\infty(w_m) - O(\varepsilon) - o(1)$ as $\varepsilon \to 0+$, $m \to \infty$.

We have

$$I(w_m) = I^\infty(w_m) - \int [F(x, w_m) - \bar{F}(w_m)]$$

d for $\delta > 0$,

$$\left| \int [F(x, w_m) - \bar{F}(w_m)] \right|$$

$$= \left| \left(\int_{\substack{|x-y_m| \geqslant t_m \\ |w_m| \leqslant \delta}} + \int_{\substack{|x-y_m| \geqslant t_m \\ \delta \leqslant |w_m| \leqslant 1/\delta}} + \int_{\substack{|x-y_m| \geqslant t_m \\ |w_m| \geqslant 1/\delta}} \right) (F(x, \dot{w}_m) - \bar{F}(w_m)) \right|.$$

(f_4),

$$\left| \int_{\substack{|x-y_m| \geqslant t_m \\ |w_m| \leqslant \delta}} (F(x, w_m) - \bar{F}(w_m)) \right| \leqslant \frac{\varepsilon(t_m)}{n} |u_m|_n^n \leqslant C\varepsilon(t_m) \tag{3.17}$$

here $\varepsilon(t_m) \to 0$ as $t_m \to \infty$. Here we have used the assumption that $\{y_m\}$ is bounded. so when $|x|$ is large enough, for $|w_m| \leqslant 1/\delta$,

$$|F(x, w_m) - \bar{F}(w_m)| \leqslant \varepsilon(R), \quad |x| \geqslant R.$$

hus

$$\left| \int_{\substack{|x-y_m| \geqslant t_m \\ \delta < |w_m| \leqslant 1/\delta}} (F(x, w_m) - \bar{F}(w_m)) \right| \leqslant \varepsilon(t_m) \operatorname{meas} \left\{ \delta \leqslant |w_m| \leqslant \frac{1}{\delta} \right\}$$

$$\leqslant \frac{1}{\delta^n} \varepsilon(t_m) |w_m|_n^n \leqslant C\delta^{-n} \varepsilon(t_m). \tag{3.18}$$

(f_1) and (f_5), for $t > 0$ we have

$$\lim_{t \to \infty} \frac{F(x, t)}{\exp((2\alpha_n/(1+r))|t|^{n/(n-1)}) - 1} = 0 \quad \text{uniformly in } x \text{ and}$$

$$\lim_{t \to \infty} \frac{\bar{F}(t)}{\exp((2\alpha_n/(1+r))|t|^{n/(n-1)}) - 1} = 0.$$

ence by (3.4) and Lemma 2.1

$$\left| \int_{|w_m| > 1/\delta} F(x, w_m) \right| = \left| \int_{|w_m| > 1/\delta} \frac{F(x, w_m)}{\exp((2\alpha_n/(1+r))|w_m|^{n/(n-1)}) - 1} \right.$$

$$\left. \times \left(\exp((2\alpha_n/(1+r))|w_m|^{n/(n-1)} - 1 \right) \right|$$

$$\leqslant O(\delta) \tag{3.19}$$

$\delta \to 0$. Similarly

$$\left| \int_{|w_m| > 1/\delta} \bar{F}(w_m) \right| \leqslant O(\delta). \tag{3.20}$$

Thus (3.17)–(3.20) imply

$$\left| \int [F(x, w_m) - \bar{F}(w_m)] \right| \underset{m}{\longrightarrow} 0. \tag{3.21}$$

Therefore we get, as desired

$$I(w_m) \geqslant I^\infty(w_m) - O(\varepsilon) - o(1) \tag{3.22}$$

where $O(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $o(1) \to 0$ as $m \to \infty$.

Claim 5: $I(w_m) > C^\infty - O(\varepsilon) - o(1)$.

We have

$$\langle I'(w_m), w_m \rangle = \langle I^{\infty\prime}(w_m), w_m \rangle + \int (w_m^{n-1} \bar{f}(w_m) - w_m^{n-1} f(x, w_m)).$$

Arguing as in the proof of (3.21) we can prove that

$$\int (u_m^{n-1} \bar{f}(u_m) - u_m^{n-1} f(x, u_m)) = o(1). \tag{3.23}$$

Also by using (3.3), (3.6), (3.11) and (3.12) we can get

$$\begin{aligned} \langle I'(w_m), w_m \rangle &= \langle I'(u_m), \varphi_m^n u_m \rangle + O(\varepsilon) \\ &= o(1) + O(\varepsilon). \end{aligned}$$

Hence

$$\langle I^{\infty\prime}(w_m), w_m \rangle = o(1) + O(\varepsilon). \tag{3.24}$$

With $\tilde{w}_m(x) = w_m(\sigma x)$ we have

$$\begin{aligned} &\int (|\nabla \tilde{w}_m|^n + |\tilde{w}_m|^n - \tilde{w}_m^{n-1} \bar{f}(\tilde{w}_m)) \\ &= \int |\nabla w_m|^n + \sigma^{-n} \int (|w_m|^n - w_m^{n-1} \bar{f}(w_m)) \\ &= (1 - \sigma^{-n}) \int |\nabla w_m|^n + \sigma^{-n} \langle I^{\infty\prime}(w_m), w_m \rangle. \end{aligned} \tag{3.25}$$

We want to choose σ_m close to 1 in such a way that $\tilde{w}_m \in M^\infty$. First we show that $|\nabla w_m|_n^n$ has a lower bound $A > 0$ independent of ε small enough and independent of m. If not, then there is a sequence $\delta_k \underset{k}{\longrightarrow} 0$ such that

$$\lim_{m \to \infty} |\nabla w_m(\delta_k)|_n^n = \bar{\mu}(\delta_k), \bar{\mu}(\delta_k) \to 0 \quad \text{as } \delta_k \to 0,$$

where $w_m(\delta_k)$ is a subsequence selected by the above process for each δ_k. Now, by dichotomy we have

$$|\nabla w_m(\delta_k)|_n^n + |w_m(\delta_k)|_n^n \geqslant l - \alpha - \delta_k. \tag{3.26}$$

On the other hand using (3.24), (f_1), (f_2) and Lemma 2.3

$$|\nabla w_m(\delta_k)|_n^n + |w_m(\delta_k)|_n^n = O(\delta_k) + o(1) + \int w_m^{n-1}(\delta_k) \bar{f}(w_m(\delta_k))$$

$$\leqslant O(\delta_k) + o(1) + \tfrac{1}{2}|w_m(\delta_k)|_n^n$$

$$+ C\int w_m^{n-1}(\delta_k)[\exp(\alpha_n|w_m(\delta_k)|^{n/(n-1)})$$

$$- 1 - \alpha_n|w_m(\delta_k)|^{n/(n-1)}]$$

$$\leqslant O(\delta_k) + o(1) + \tfrac{1}{2}|w_m(\delta_k)|_n^n + C|\nabla w_m(\delta_k)|_n|w_m(\delta_k)|_n^n.$$

Thus $|\nabla w_m(\delta_k)|_n^n + |w_m(\delta_k)|_n^n \leqslant O(\delta_k) + o(1) + C|\bar{\mu}(\delta_k)|$, which contradicts (3.26). Therefore

$$\lim_{m\to\infty}\int|\nabla w_m|^n \geqslant A > 0 \quad \text{for } \varepsilon \text{ small enough.}$$

Now choosing $\sigma_m^n = 1 - \langle I'_\infty(w_m), w_m\rangle|\nabla w_m|_n^{-n}$ we see from (3.25) that $\tilde{w}_m \in M^\infty$, which implies that $M^\infty \neq \phi$ if dichotomy occurs. Also from (3.24) it follows that

$$\sigma_m^n = 1 + O(\varepsilon) + o(1).$$

Again, in view of (3.11) we can assume that $|\nabla w_m|_n^n < (1+r)/2$ for ε small enough, and therefore $\int \bar{F}(w_m)$ is bounded. Hence

$$I^\infty(w_m) = I^\infty(\tilde{w}_m) - \frac{1-\sigma_m^n}{2\sigma_m}|w_m|_n^n + \frac{1-\sigma_m^n}{2\sigma_m}\int\bar{F}(w_m)$$

$$\geqslant I^\infty(\tilde{w}_m) - O(\varepsilon) - o(1)$$

$$\geqslant C^\infty - O(\varepsilon) - o(1).$$

Thus, in view of (3.22), we obtain (3.27)

$$I(w_m) \geqslant C^\infty - O(\varepsilon) - o(1)$$

and this proves the claim.

Now as in (3.24) we obtain (3.28)

$$\langle I'(v_m), v_m\rangle = O(\varepsilon) + o(1).$$

So, in view of (f_3) we have

$$I(v_m) = \langle I'(v_m), v_m\rangle + \frac{1}{n}\int(v_m^{n-1}f(x, v_m) - F(x, v_m))$$

$$\geqslant\left(\frac{1}{n} - \theta\right)\int v_m^{n-1}f(x, v_m) + O(\varepsilon) + o(1)$$

$$=\left(\frac{1}{n} - \theta\right)\int(|\nabla v_m|^n + |v_m|^n) + O(\varepsilon) + o(1)$$

$$\geqslant\left(\frac{1}{n} - \theta\right)\alpha + O(\varepsilon) + o(1).$$

Therefore (3.16) and (3.27) imply

$$I(u_m) \geqslant C^\infty + \left(\frac{1}{n} - \theta\right)\alpha - O(\varepsilon) - o(1).$$

Letting $m \to \infty$ and then $\varepsilon \to 0$ we get $c \geq C^{\infty} + ((1/n) - \theta)\alpha$, a contradiction. This completes the case of bounded $\{y_m\}$.

Case 2: $\{y_m\}$ is unbounded.

In this case we change the role of $\{v_m\}$ and $\{w_m\}$ and then we can still get a contradiction as above.

Thus we have ruled out dichotomy and therefore by Lemma 1.1 of [5] there exists $\{y_m\}$ in $\mathbb{R}^n$ such that for any $\varepsilon > 0$, there is $t = t(\varepsilon) > 0$ such that

$$\int_{|x-y_m|>t} (|\nabla u_m|^n + |u_m|^n) < \varepsilon. \tag{3.29}$$

Claim 6: $\{y_m\}$ is bounded.

If not, then without loss of generality suppose $y_m \xrightarrow{m} \infty$. Now

$$I(u_m) = I^{\infty}(u_m) + \int_{|x-y_m|<t+1} [\bar{F}(u_m) - F(x, u_m)] + \int_{|x-y_m|\geq t+1} [\bar{F}(u_m) - F(x, u_m)]. \tag{3.30}$$

Let η_m be cut-off functions such that $0 \leq \eta_m \leq 1$, $\eta_m \equiv 0$ for $|x - y_m| \leq t$; $\eta_m \equiv 1$ for $|x - y_m| \geq t+1$, $|\nabla \eta_m| \leq 2n$. Then for $\varepsilon < 1/\alpha_n (4n)^n \beta e$ and m large,

$$\int |\nabla(\eta_m u_m)|^n < \frac{1}{\alpha_n \beta e}.$$

Then by Lemma 2.3 and (3.29)

$$\begin{aligned} &\left| \int_{|x-y_m|\geq t+1} F(x, u_m) \right| \\ &\leq C \int_{|x-y_m|\geq t+1} |u_m|^n \\ &\quad + C \int_{|x-y_m|\geq t+1} |\eta_m u_n|^{n-1} (\exp(\alpha_n |\eta_m u_m|^{n/(n-1)}) - 1 - \alpha_n |\eta_m u_m|^{n/(n-1)}) \\ &< O(\varepsilon). \end{aligned} \tag{3.31}$$

Similarly

$$\left| \int_{|x_m|\geq t+1} \bar{F}(u_m) \mathrm{d}x \right| \leq O(\varepsilon). \tag{3.32}$$

Again as in the proof of (3.21), using the assumption $y_m \xrightarrow{m} \infty$ we obtain

$$\int_{|x_m|\geq t+1} (F(x, u_m) - \bar{F}(u_m)) \leq o(1) \quad \text{as } m \to \infty. \tag{3.33}$$

Thus $I(u_m) \geq I^{\infty}(u_m) - O(\varepsilon) - o(1)$. Again as earlier we can choose σ_m such that $\sigma_m = 1 - O(\varepsilon) + o(1)$, $\tilde{u}_m(x) = u_m(\sigma_m x)$ is in M^{∞} and

$$I(u_m) \geq I^{\infty}(\tilde{u}_m) - O(\varepsilon) + o(1) \geq C^{\infty} - O(\varepsilon) + o(1).$$

Taking $m \to \infty$ and then $\varepsilon \to 0$ we obtain $c \geqslant C^{\infty}$, a contradiction which proves the claim.

Therefore, for any $\varepsilon > 0$, there exists $t = t(\varepsilon) > 0$ such that

$$\int_{|x|>t} (|\nabla u_m|^n + |u_m|^n) dx < \varepsilon. \tag{3.34}$$

To use Strauss' lemma as in [3] we set $P(s) = s^{n-1} f(x, s)$, $Q(s) = \exp((2\alpha_n/(1+r))|s|^{n/(n-1)}) - \sum_{m=0}^{n-2} (1/m!)(2\alpha_n/(1+r))^m |s|^{n/(n-1)} + |s|^n$, so that $\lim_{|s|\to\infty} P(s)/Q(s) = 0$. Also by Lemma 2.1, $\int Q(u_m) \leqslant C$ for some constant $C > 0$. Therefore by Strauss' lemma, for any bounded Borel set Ω

$$\lim_{n\to\infty} \int_{\Omega} u_m^{n-1} f(x, u_m) = \int_{\Omega} u^{n-1} f(x, u).$$

In particular

$$\lim_{n\to\infty} \int_{|x|\leqslant t} u_m^{n-1} f(x, u_m) = \int_{|x|\leqslant t} u^{n-1} f(x, u). \tag{3.35}$$

Again, as in the proof of (3.21) we obtain

$$\left| \int_{|x|>t} u_m^{n-1} f(x, u_m) \right| \leqslant O(\varepsilon). \tag{3.36}$$

Thus

$$\lim_{m\to\infty} \int u_m^{n-1} f(x, u_m) = \int u^{n-1} f(x, u). \tag{3.37}$$

Claim 7: $u_m \to u$ in $W^{1,n}(\mathbb{R}^n)$.

Since $u_m \underset{m}{\longrightarrow} u$ weakly in $W^{1,n}(\mathbb{R}^n)$ we have by Rellich's lemma $u_m \underset{m}{\longrightarrow} u$ strongly in $L^n(\Omega)$ for any bounded smooth Ω. In particular,

$$\int_{|x|\leqslant t} |u_m|^n \underset{m}{\longrightarrow} \int_{|x|\leqslant t} |u|^n.$$

Thus using (3.29) we get

$$\int |u_m|^n \underset{m}{\longrightarrow} \int |u|^n. \tag{3.38}$$

As in (3.35), we have for any $\varphi \in C_b^0(\mathbb{R}^n)$,

$$\int u_m^{n-1} f(x, u_m)\varphi \underset{m}{\longrightarrow} \int u^{n-1} f(x, u)\varphi. \tag{3.39}$$

Now, for any $\varphi \in C_b^{\infty}(\mathbb{R}^n)$ we have, by (3.5), (3.38) and (3.39)

$$0 = \lim_{m\to\infty} \langle I'(u_m), \varphi \rangle$$

$$= \int T \cdot \nabla\varphi + \int |u|^{n-2} u\varphi - \int f(x, u) u^{n-2} \varphi, \tag{3.40}$$

$$0 = \lim_{m\to\infty} \langle I'(u_m), u_m\varphi\rangle$$

$$= \int \varphi d\mu + \int uT\cdot\nabla\varphi - \int f(x,u)u^{n-1}\varphi, \tag{3.41}$$

$$0 = \lim_{m\to\infty} \langle I'(u_m), u\varphi\rangle$$

$$= \int uT\cdot\nabla\varphi + \int \varphi T\cdot\nabla u + \int |u|^n\varphi - \int f(x,u)u^{n-1}\varphi. \tag{3.42}$$

Thus

$$\int uT\cdot\nabla\varphi = \int f(x,u)u^{n-1}\varphi - \int \varphi T\cdot\nabla u - \int |u|^n\varphi,$$

and substituting in (3.41) we get

$$\int \varphi d\mu = \int \varphi T\cdot\nabla u + \int |u|^n\varphi.$$

In view of (3.5) we get $\int \varphi|\nabla u_m|^n \xrightarrow[m]{} \int \varphi T\cdot\nabla u$ and hence $\int_{|x|\leqslant t}|\nabla u_m|^n \longrightarrow \int_{|x|\leqslant t} T\cdot\nabla u$. This implies, using (3.29), $\int|\nabla u_m|^n \xrightarrow[m]{} \int T\cdot\nabla u$. That is,

$$\lim_{m\to\infty}\int |\nabla u_m|^n = \lim_{m\to\infty}\int |\nabla u_m|^{n-2}\nabla u_m\cdot\nabla u.$$

Then

$$\lim_{m\to\infty}\int (|\nabla u_m|^{n-2}\nabla u_m - |\nabla u|^{n-2}\nabla u)\cdot(\nabla u_m - \nabla u) = 0,$$

which implies

$$\lim_{m\to\infty}\int |\nabla u_m - \nabla u|^n = 0,$$

by using an inequality

$$|a-b|^p \leqslant 2^{p-1}(|a|^{p-2}a - |b|^{p-2}b)\cdot(a-b)$$

for any $a, b\in\mathbb{R}^n$, $p\geqslant 2$. Therefore $u_m \xrightarrow[m]{} u$ strongly in $W^{1,n}(\mathbb{R}^n)$ as desired. □

Proof of Theorem 1.1: By the definition of S_p, for any $\varepsilon>0$ there exists $u_\varepsilon\in W^{1,n}(\mathbb{R}^n)$ such that $(\|u_\varepsilon\|/|u_\varepsilon|_p) < S_p+\varepsilon$. Let $v_\varepsilon = ((1-n\theta)^{1/n}/\|u_\varepsilon\|)\,|u_\varepsilon|_p$. Then $\|v_\varepsilon\|^n = 1-n\theta$, $|v_\varepsilon|_p = ((1-n\theta)^{1/n}/\|u_\varepsilon\|)\,|u_\varepsilon|_p$ and so $\|v_\varepsilon\|/|v_\varepsilon|_p = \|u_\varepsilon\|/|u_\varepsilon|_p$.

Claim: $\int \bar{F}(v_\varepsilon) > (1/n)|v_\varepsilon|_n^n$.

Choose ε small enough so that $C_p > (p/n)(S_p+2\varepsilon)^p(1-n\theta)^{1-(p/n)}$. Now $S_p+\varepsilon > \|v_\varepsilon\|/|v_\varepsilon|_p$ and so

$$\int \bar{F}(v_\varepsilon) \geqslant \frac{C_p}{p}\int |v_\varepsilon|^p > \frac{C_p}{p}\frac{\|v_\varepsilon\|^p}{(S_p+\varepsilon)^p}$$

$$> \frac{1-n\theta}{n}\left(\frac{S_p+2\varepsilon}{S_p+\varepsilon}\right)^p > \frac{1}{n}\int (|\nabla v_\varepsilon|^n + |v_\varepsilon|^n)$$

and this proves the claim.

Therefore by Lemma 2.2 I_0^∞ is achieved by some u_0 and $I_0^\infty \leqslant 1 - n\theta$. Then

$$-\Delta_n u_0 = \lambda(\bar{f}(u_0)u_0^{n-2} - |u_0|^{n-2}u_0)$$

for some Lagrange multiplier $\lambda \in \mathbb{R}$. By (f_3) we have

$$\int \bar{F}(u_0) = \frac{1}{n}\int |u_0|^n \leqslant \theta \int u_0^{n-1}\bar{f}(u_0).$$

Also we know that $I_0^\infty > 0$. Thus $\lambda > 0$. Let $u(x) = u_0(\lambda^{-1/n}x)$. Then u satisfies (1.1), $\int \bar{F}(u) = (1/n)\int |u|^n$ and $I(u) = (1/n)\int |\nabla u|^n = (1/n)\int |\nabla u_0|^n < (1/n) - \theta$. This proves the theorem. □

Proof of Theorem 1.2: By the assumptions we see that $\bar{f}(t)$ satisfies the conditions of Theorem 1.1. Thus there exists $\bar{u} \in W^{1,n}(\mathbb{R}^n)$ satisfying

$$-\Delta_n \bar{u} + |\bar{u}|^{n-2}\bar{u} = \bar{f}(\bar{u})\bar{u}^{n-2} \quad \text{in } \mathbb{R}^n. \tag{3.43}$$

Moreover, $I^\infty(\bar{u}) < (1/n) - \theta$. Let

$$h(t) = I^\infty(t\bar{u}) = \frac{t^n}{n}\int [|\nabla \bar{u}|^n + |\bar{u}|^n] - \int \bar{F}(t\bar{u}).$$

By (f_2) and (3.43) we have

$$h'(t) \geqslant 0 \quad \text{for } 0 \leqslant t < 1;\ h'(1) = 0$$

$$h'(t) \leqslant 0 \quad \text{for } t > 1.$$

Hence $I^\infty(\bar{u}) = \max_{t \geqslant 0} I^\infty(t\bar{u})$. Further, since $I(t\bar{u}) \to -\infty$ as $t \to \infty$, there exists $t_0 \in (0, \infty)$ such that $I(t_0\bar{u}) = \max_{t \geqslant 0} I(t\bar{u})$. Now, by (f_5) and the hypothesis that $f(x,t) \not\equiv \bar{f}(t)$ we have

$$I(t_0\bar{u}) < I^\infty(t_0\bar{u}) \leqslant \max_{t \geqslant 0} I^\infty(t\bar{u}) = I^\infty(\bar{u}). \tag{3.44}$$

We claim that $C^\infty = I^\infty(\bar{u})$. Clearly $C^\infty \leqslant I_0^\infty = I^\infty(\bar{u})$. Further, given $\varepsilon > 0$, we can find $u \in M^\infty$ such that $I^\infty(u) < C^\infty + \varepsilon$. Using (f_2) we can find $t \in \mathbb{R}^+$ such that $\int F(x, tu) = (1/n)|tu|_n^n$. Again as above we can show that $I^\infty(u) = \max_{t \geqslant 0} I^\infty(tu)$. Thus

$$I^\infty(\bar{u}) = I_0^\infty \leqslant I(tu) \leqslant I^\infty(tu) \leqslant I^\infty(u) < C^\infty + \varepsilon,$$

which gives the other inequality, since $\varepsilon > 0$ was arbitrary.

Therefore from (3.44) we get

$$I(t_0\bar{u}) < I^\infty(\bar{u}) = C^\infty < \frac{1}{n} - \theta. \tag{3.45}$$

It is easy to see, using (f_1), (f_2) and Lemma 2.3, that there exist $\rho, \alpha > 0$ with

$$I(u) > \alpha \quad \text{for all } u \text{ satisfying } \|u\| = \rho.$$

Choose $t_1 > t_0$ sufficiently large so that $I(t\bar{u}) < 0$ for $t > t_1$. Let Γ be the set of all continuous paths connecting 0 and $t_1\bar{u}$. Define

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} I(u). \tag{3.46}$$

Then $c > \alpha$. Also

$$c \leqslant \max_{0 \leqslant t \leqslant t_1} I(t\bar{u}) < C^{\infty} = \min\left\{C^{\infty}, \frac{1}{n} - \theta\right\}.$$

By Mountain Pass lemma (see [4]), there exists a sequence $\{u_m\}$ in $W^{1,n}(\mathbb{R}^n)$ such that

$$I(u_m) \underset{m}{\longrightarrow} c, \quad I'(u_m) \underset{m}{\longrightarrow} 0 \quad \text{in } (W^{1,n}(\mathbb{R}^n))^*.$$

By Lemma 3.1, for a subsequence $u_m \underset{m}{\longrightarrow} u$ strongly in $W^{1,n}(\mathbb{R}^n)$. Thus $I(u) = c$, $I'(u) = 0$, which implies that $u \not\equiv 0$ and u is a nontrivial solution of (1.1). This completes the proof of the theorem. □

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An axisymmetric steady-state thermoelastic problem of an external circular crack in an isotropic thick plate

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Abstract. A steady state thermoelastic mixed boundary value problem for an isotropic thick plate is considered in this paper. The faces of an external circular crack situated in the mid-plane of the plate are opened up by the application of temperature while the bounding surface of the plate are maintained at a constant zero temperature. Solution valid for large values of the ratio of the plate thickness to the diameter of the crack has been obtained. Expressions for various quantities of physical interest are derived by finding iterative solutions of the equations and the results are shown graphically.

Keywords. Axisymmetric; steady-state; external circular crack; stress-intensity factor.

1. Introduction

The strength of a material with cracks is an interesting problem in fracture as well as structural mechanics and the knowledge of the elastic stress field is potentially useful for strength estimation based upon brittle fracture theory.

Several papers have appeared which treat distributions of stress in an infinite solid due to the application of temperature or normal pressure on the faces of a flat internal circular crack (Das and Ghosh [2], Lowengrub [5], Bandyopadhyay and Das [1]). The problem of an infinite body containing an external circular crack covering the outside of a circle, due to the application of normal pressure has been considered by Uflyand [12] using toroidal coordinates and by Lowengrub and Sneddon [6] from the dual integral equation point of view. Lowengrub [7] has also solved the two-dimensional plane strain problem for an external crack $y = 0, |x| > 1$ opened up by normal pressure, using dual trigonometric equations. Distribution of stress in a thick plate containing an external circular crack opened up by the application of pressure has been considered by Dhawan [4].

This paper determines the thermoelastic stress distribution in the vicinity of an external circular crack situated in the mid-plane of an isotropic elastic plate of finite thickness and infinite radius. The temperature, the shear component of stress tensor and the normal component of displacement vector vanish over the plane boundaries while the crack is opened up by the application of a prescribed axially symmetric temperature to its faces. The method of solution is to seek suitable representations of the potential of thermoelastic displacements and the Love function and then to reduce the problem to the solution of two pairs of dual integral equations. Finally, these dual integral equations have been further reduced to Fredholm integral equations of the second kind which are solved in terms of power series. The results are illustrated by a number of diagrams (figures 2–7).

2. Basic equations of thermoelasticity

We consider the temperature and displacement fields in an isotropic elastic solid which is conducting heat. If we assume that there is symmetry about an axis, which we take to be the z-axis, then the position of a typical point of the solid may conveniently be expressed by the cylindrical polar coordinates (r, θ, z) and the displacement vector will have the components $(u_r, 0, u_z)$. The non-vanishing components of the stress tensor will be $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}$.

In the absence of body forces or heat sources within the solid, the steady-state equations of thermoelasticity with symmetry about z-axis are (Sneddon and Berry [10], p. 125)

$$\left.\begin{aligned} 2(1-\nu)\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)+(1-2\nu)\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial^2 u_z}{\partial r\,\partial z}&=2(1+\nu)\alpha\frac{\partial T}{\partial r}\\ (1-2\nu)\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+2(1-\nu)\frac{\partial^2 u_z}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)&=2(1+\nu)\alpha\frac{\partial T}{\partial z}\end{aligned}\right] \tag{1}$$

and

$$\nabla^2 T=0, \tag{2}$$

where $T=T(r,z)$ is the deviation of the absolute temperature of the solid from that in a state of zero stress and strain, α is the co-efficient of linear thermal expansion of the solid, ν is its Poisson ratio and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \tag{3}$$

3. Boundary conditions

With a suitable choice of our unit of length we can assume that the faces of the crack are described by the relations $z=0\pm, r\geqslant 1$. The thickness of the plate is assumed to be δ-times the diameter of the crack. We suppose that there is no external force acting on the crack-faces and that the face $z=0+, r\geqslant 1$ is heated (or cooled) exactly in the same way as the face $z=0-$, $r\geqslant 1$. Then following Sneddon [9] we reduce the crack problem for the thick plate $r\geqslant 0, |z|\leqslant\delta$ to the mixed boundary value problem for the layer $r\geqslant 0, 0\leqslant z\leqslant\delta$ for which the thermal and elastic conditions are:

on $z=0$:

$$\frac{\partial T}{\partial z}(r,0)=0, \qquad 0\leqslant r<1 \tag{4}$$

$$T(r,0)=f(r), \quad 1<r<\infty \tag{5}$$

$$\sigma_{rz}(r,0)=0, \qquad 0\leqslant r<\infty \tag{6}$$

$$u_z(r,0)=0, \qquad 0\leqslant r<1 \tag{7}$$

$$\sigma_{zz}(r,0)=0, \qquad 1<r<\infty \tag{8}$$

on $z=\delta$:

$$T(r,\delta)=0, \qquad 0\leqslant r<\infty \tag{9}$$

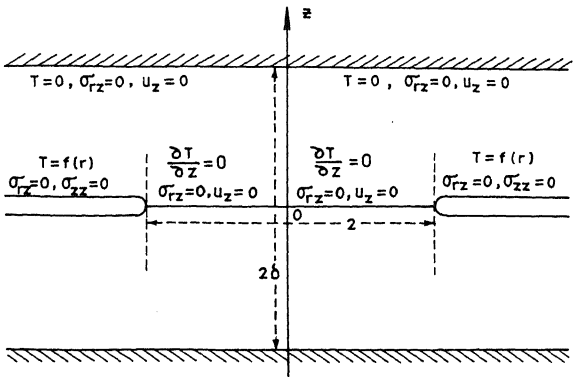

Figure 1.

$$\sigma_{rz}(r,\delta)=0, \qquad 0\leqslant r<\infty \tag{10}$$

$$u_z(r,\delta)=0, \qquad 0\leqslant r<\infty, \tag{11}$$

where $f(r)$ is prescribed.

We further assume that the disturbance is localized i.e. the temperature and the components of stress and displacement all vanish as $\sqrt{(r^2+z^2)}\to\infty$. Position of the crack and the boundary conditions for the plate are indicated in figure 1.

4. The heat conduction problem

A suitable Hankel integral representation of the temperature field satisfying the Laplace's equations (2) and (9) and vanishing at infinity is taken in the form

$$T(r,z)=\int_0^\infty \xi^{-1}B(\xi)\sinh\xi(\delta-z)\operatorname{sech}(\xi\delta)J_0(\xi r)\mathrm{d}\xi, \tag{12}$$

where $B(\xi)$ is an unknown function to be determined from the boundary conditions.

Conditions (4) and (5) are fulfilled if the function $B(\xi)$ is a solution of the set of dual integral equations

$$\int_0^\infty B(\xi)J_0(\xi r)\mathrm{d}\xi=0, \quad 0\leqslant r<1 \tag{13}$$

$$\int_0^\infty \xi^{-1}B(\xi)[1-H_1(\xi\delta)]J_0(\xi r)\mathrm{d}\xi=f(r), \quad 1<r<\infty, \tag{14}$$

where

$$H_1(\xi\delta)=1-\tanh(\xi\delta). \tag{15}$$

To reduce the above equations to a single integral equation, we apply Sneddon's method [11] and put

$$B(\xi)=\xi\int_1^\infty \psi_1(t)\sin(\xi t)\mathrm{d}t, \tag{16}$$

where, for the convergence of the integral, we assume that

$$\lim_{t\to\infty}\psi_1(t)=0. \tag{17}$$

Integrating by parts and making use of (17) we rewrite (16) in the form

$$B(\xi) = \psi_1(1)\cos\xi + \int_1^\infty \psi_1'(t)\cos(\xi t)\mathrm{d}t, \tag{18}$$

where the prime (′) denotes differentiation.

Substituting from (18) and making use of the result ([13] p. 405)

$$\int_0^\infty J_0(\xi r)\cos(\xi t)\mathrm{d}\xi = \begin{cases} 0, & r < t \\ \dfrac{1}{\sqrt{(r^2 - t^2)}}, & r > t \end{cases} \tag{19}$$

we can show that

$$\int_0^\infty B(\xi)J_0(\xi r)\mathrm{d}\xi = \begin{cases} 0, & 0 \leqslant r < 1 \\ \dfrac{\psi_1(1)}{\sqrt{(r^2-1)}} + \displaystyle\int_1^r \frac{\psi_1'(t)}{\sqrt{(r^2-t^2)}}\mathrm{d}t, & 1 < r < \infty. \end{cases} \tag{20}$$

It is clear from (20) that the form (16) satisfies (13). Now, from (14) we have

$$\int_1^\infty \psi_1(t)\mathrm{d}t \int_0^\infty \sin(\xi t)J_0(\xi r)\mathrm{d}\xi - \int_1^\infty \psi_1(t)\mathrm{d}t \int_0^\infty H_1(\xi\delta)J_0(\xi r)\sin(\xi t)\mathrm{d}\xi$$
$$= f(r), \quad 1 < r < \infty. \tag{21}$$

Making use of the result [13], p. 405

$$\int_0^\infty J_0(\xi r)\sin(\xi t)\mathrm{d}\xi = \begin{cases} \dfrac{1}{\sqrt{(t^2 - r^2)}}, & r < t \\ 0, & t < r \end{cases} \tag{22}$$

we find from (21)

$$\int_r^\infty \frac{\psi_1(t)\mathrm{d}t}{\sqrt{(t^2 - r^2)}} - \int_1^\infty \psi_1(t)\mathrm{d}t \int_0^\infty H_1(\xi\delta)J_0(\xi r)\sin(\xi t)\mathrm{d}\xi = f(r), \quad 1 < r < \infty. \tag{23}$$

If we replace $J_0(\xi r)$ by its integral representation,

$$\frac{2}{\pi}\int_r^\infty \frac{\sin(\xi u)\mathrm{d}u}{\sqrt{(u^2 - r^2)}}$$

we find that the second term on LHS is equal to

$$\frac{2}{\pi}\int_1^\infty \psi_1(t)\mathrm{d}t \int_0^\infty H_1(\xi\delta)\sin(\xi t)\mathrm{d}\xi \int_r^\infty \frac{\sin(\xi u)\mathrm{d}u}{\sqrt{(u^2 - r^2)}}.$$

Simplifying and interchanging the order of integrations the second term on LHS becomes

$$\frac{1}{\pi\delta}\int_r^\infty \frac{\mathrm{d}t}{\sqrt{(t^2 - r^2)}} \int_1^\infty \psi_1(u)\{H_1^*(t-u) - H_1^*(t+u)\}\mathrm{d}u$$

where

$$H_1^*(\omega) = \int_0^\infty H_1(u)\cos\left(\frac{u\omega}{\delta}\right)\mathrm{d}u. \tag{24}$$

Then from (23) we have

$$\int_r^\infty \frac{\psi_1(t)\mathrm{d}t}{\sqrt{(t^2-r^2)}} - \frac{1}{\pi\delta}\int_r^\infty \frac{\mathrm{d}t}{\sqrt{(t^2-r^2)}}\int_1^\infty \psi_1(u)\{H_1^*(t-u) - H_1^*(t+u)\}\mathrm{d}u$$
$$= f(r), \quad 1 < r < \infty$$

or,

$$\int_r^\infty \frac{\mathrm{d}t}{\sqrt{(t^2-r^2)}}\left[\psi_1(t) - \frac{1}{\pi\delta}\int_1^\infty \psi_1(u)\{H_1^*(t-u) - H_1^*(t+u)\}\mathrm{d}u\right]$$
$$= f(r), \quad 1 < r < \infty,$$

which on inversion gives

$$\psi_1(t) - \frac{1}{\pi\delta}\int_1^\infty \psi_1(u)\{H_1^*(t-u) - H_1^*(t+u)\}\mathrm{d}u$$
$$= -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{rf(r)}{\sqrt{(r^2-t^2)}}\mathrm{d}r,$$

or,

$$\psi_1(t) - \frac{1}{\pi\delta}\int_1^\infty \psi_1(u)K_1(u,t)\mathrm{d}u = -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{rf(r)}{\sqrt{(r^2-t^2)}}\mathrm{d}r, \quad 1 < t < \infty, \tag{25}$$

where

$$K_1(u,t) = H_1^*(t-u) - H_1^*(t+u). \tag{26}$$

5. The thermoelastic problem

The potential Φ of thermoelastic displacement satisfying the Poisson equation (Nowacki [8], p. 12) $\nabla^2\Phi = mT$, where $m = (1+\nu)\alpha/(1-\nu)$, is

$$\Phi(r,z) = -\frac{1}{2}m\int_0^\infty \xi^{-3}B(\xi)\operatorname{sech}(\xi\delta)[\sinh\xi(\delta-z)$$
$$+ z\xi\cosh\xi(\delta-z)]J_0(\xi r)\mathrm{d}\xi. \tag{27}$$

The Love function Ψ satisfying the biharmonic equation (Nowacki [8], p. 17) $\nabla^4\Psi = 0$, is sought in the form of the Hankel integral

$$\Psi(r,z) = -\int_0^\infty \xi^{-2}C(\xi)\operatorname{cosech}(\xi\delta)[2\nu\sinh\xi(\delta-z)$$
$$+ z\xi\cosh\xi(\delta-z) - \delta\xi\sinh(\xi z)\operatorname{cosech}(\xi\delta)]J_0(\xi r)\mathrm{d}\xi \tag{28}$$

which vanishes at infinity.

Using basic equations, we have

$$u_z = \frac{1}{2}mz\int_0^\infty \xi^{-1}B(\xi)\operatorname{sech}(\xi\delta)\sinh\xi(\delta-z)J_0(\xi r)\mathrm{d}\xi$$

$$+\int_0^\infty C(\xi)\operatorname{cosech}(\xi\delta)[2(1-\nu)\sinh\xi(\delta-z)+z\xi\cosh\xi(\delta-z)$$
$$-\delta\xi\sinh(\xi z)\operatorname{cosech}(\xi\delta)]J_0(\xi r)\mathrm{d}\xi \tag{29}$$

$$\sigma_{zz}=-m\mu\int_0^\infty \xi^{-1}B(\xi)\operatorname{sech}(\xi\delta)[\sinh\xi(\delta-z)+z\xi\cosh\xi(\delta-z)]J_0(\xi r)\mathrm{d}\xi$$
$$-2\mu\int_0^\infty \xi C(\xi)\operatorname{cosech}(\xi\delta)[\cosh\xi(\delta-z)+z\xi\sinh\xi(\delta-z)$$
$$+\delta\xi\cosh(\xi z)\operatorname{cosech}(\xi\delta)]J_0(\xi r)\mathrm{d}\xi \tag{30}$$

$$\sigma_{rz}=-m\mu z\int_0^\infty B(\xi)\operatorname{sech}(\xi\delta)\sinh\xi(\delta-z)J_1(\xi r)\mathrm{d}\xi$$
$$-2\mu\int_0^\infty \xi C(\xi)\operatorname{cosech}(\xi\delta)[z\xi\cosh\xi(\delta-z)$$
$$-\delta\xi\sinh(\xi z)\operatorname{cosech}(\xi\delta)]J_1(\xi r)\mathrm{d}\xi. \tag{31}$$

Equations (6), (10) and (11) are automatically satisfied. Using boundary conditions (7) and (8) we get,

$$\int_0^\infty C(\xi)J_0(\xi r)\mathrm{d}\xi=0,\quad 0\leqslant r<1 \tag{32}$$

$$\int_0^\infty \xi C(\xi)[1+H_2(\xi\delta)]J_0(\xi r)\mathrm{d}\xi=-\frac{m}{2}f(r),\quad 1<r<\infty \tag{33}$$

where

$$1+H_2(\xi\delta)=\frac{\cosh(\xi\delta)\sinh(\xi\delta)+\xi\delta}{\sinh^2(\xi\delta)}. \tag{34}$$

Following Lowengrub and Sneddon [6] we put

$$C(\xi)=\int_1^\infty \psi_2(t)\cos(\xi t)\mathrm{d}t, \tag{35}$$

where we assume that

$$\lim_{t\to\infty}\psi_2(t)=0. \tag{36}$$

Integrating by parts and making use of (36) we rewrite (35) in the form

$$C(\xi)=-\frac{\psi_2(1)\sin\xi}{\xi}-\int_1^\infty \psi_2'(t)\frac{\sin(\xi t)}{\xi}\mathrm{d}t, \tag{37}$$

where the prime ($'$) denotes differentiation. Substituting (35) and making use of the result (19) we have

$$\int_0^\infty C(\xi)J_0(\xi r)\mathrm{d}\xi=\begin{cases}0, & 0\leqslant r<1\\ \displaystyle\int_1^r\frac{\psi_2(t)\mathrm{d}t}{\sqrt{(r^2-t^2)}}, & 1<r<\infty.\end{cases} \tag{38}$$

It is clear from (38) that the form (35) satisfies (32). From (33) we have

$$\int_0^\infty \xi J_0(\xi r)\mathrm{d}\xi \int_1^\infty \psi_2(t)\cos(\xi t)\mathrm{d}t + \int_0^\infty \xi H_2(\xi\delta)J_0(\xi r)\mathrm{d}\xi \times \int_1^\infty \psi_2(t)\cos(\xi t)\mathrm{d}t = -\frac{m}{2}f(r), \quad 1<r<\infty. \tag{39}$$

The first term on the LHS of the above integral equation (39) becomes

$$-\int_r^\infty \frac{\psi_2'(t)\mathrm{d}t}{\sqrt{(t^2-r^2)}}.$$

Replacing $J_0(\xi r)$ by its integral representation

$$\frac{2}{\pi}\int_r^\infty \frac{\sin(\xi t)\mathrm{d}t}{\sqrt{(t^2-r^2)}}$$

the second term on the LHS of the above integral equation (39) becomes

$$\frac{2}{\pi}\int_0^\infty \xi H_2(\xi\delta)\mathrm{d}\xi \int_r^\infty \frac{\sin(\xi t)\mathrm{d}t}{\sqrt{(t^2-r^2)}} \int_1^\infty \psi_2(u)\cos(\xi u)\mathrm{d}u.$$

Interchanging the order of integration and simplifying the above term becomes

$$\frac{1}{\pi\delta}\int_r^\infty \frac{\mathrm{d}t}{\sqrt{(t^2-r^2)}} \int_1^\infty \left[H_2^*(t+u)+H_2^*(t-u)\right]\psi_2(u)\mathrm{d}u,$$

where

$$H_2^*(\omega) = \int_0^\infty \frac{\xi}{\delta}H_2(\xi)\sin\left(\frac{\xi\omega}{\delta}\right)\mathrm{d}\xi. \tag{40}$$

Thus (39) becomes

$$-\int_r^\infty \frac{\psi_2'(t)\mathrm{d}t}{\sqrt{(t^2-r^2)}} + \frac{1}{\pi\delta}\int_r^\infty \frac{\mathrm{d}t}{\sqrt{(t^2-r^2)}} \int_1^\infty [H_2^*(t+u)+H_2^*(t-u)]\psi_2(u)\mathrm{d}u = -\frac{m}{2}f(r), \quad 1<r<\infty.$$

or,

$$\int_r^\infty \frac{\mathrm{d}t}{\sqrt{(t^2-r^2)}}\left[-\psi_2'(t)+\frac{1}{\pi\delta}\int_1^\infty [H_2^*(t+u)+H_2^*(t-u)]\psi_2(u)\mathrm{d}u\right] = -\frac{m}{2}f(r), \quad 1<r<\infty,$$

which on inversion gives

$$-\psi_2'(t)+\frac{1}{\pi\delta}\int_1^\infty [H_2^*(t+u)+H_2^*(t-u)]\psi_2(u)\mathrm{d}u = \frac{m}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{rf(r)\mathrm{d}r}{\sqrt{(r^2-t^2)}}, \quad 1<t<\infty. \tag{41}$$

Assuming that $f(r)$ is continuous differentiable in $(1,\infty)$, we integrate (41) between the limits t to ∞ and on making use of (36), we obtain the following Fredholm integral

equation of the second kind

$$\psi_2(t)+\frac{1}{\pi\delta}\int_1^{\infty} K_2(u,t)\psi_2(u)\mathrm{d}u=-\frac{m}{\pi}\int_t^{\infty}\frac{rf(r)\mathrm{d}r}{\sqrt{(r^2-t^2)}} \tag{42}$$

where

$$K_2(u,t)=\int_t^{\infty}[H_2^*(t_1+u)+H_2^*(t_1-u)]\mathrm{d}t_1. \tag{43}$$

6. Method of solution

Assuming that $\delta \gg 1$, we can write (15) and (34) as

$$H_1(\xi\delta)=2\sum_1^{\infty}(-1)^{n-1}\mathrm{e}^{-2n\xi\delta} \tag{44}$$

and

$$H_2(\xi\delta)=2\sum_1^{\infty}(1+2n\xi\delta)\mathrm{e}^{-2n\xi\delta}. \tag{45}$$

Using (24) and (44) we have from (26)

$$K_1(u,t)=2\left[\frac{ut}{\delta^2}H_{12}-\frac{ut^3+u^3t}{6\delta^4}H_{14}+\frac{3ut^5+10u^3t^3+3u^5t}{360\delta^6}H_{16}+\cdots\right] \tag{46}$$

where

$$H_{1n}=\int_0^{\infty}H_1(\omega)\omega^n\mathrm{d}\omega. \tag{47}$$

To solve the Fredholm integral equation (25) we assume a series solution in the form

$$\psi_1(t)=\psi_{10}(t)+\frac{1}{\delta}\psi_{11}(t)+\frac{1}{\delta^2}\psi_{12}(t)+\frac{1}{\delta^3}\psi_{13}(t)+\cdots \tag{48}$$

Then from the Fredholm integral equation (25), we have

$$\left.\begin{aligned}
\psi_{10}(t)&=-\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{\infty}\frac{rf(r)\mathrm{d}r}{\sqrt{(r^2-t^2)}}\\
\psi_{11}(t)&=0\\
\psi_{12}(t)&=0\\
\psi_{13}(t)&=\frac{2}{\pi}H_{12}\int_1^{\infty}ut\psi_{10}(u)\mathrm{d}u\\
\psi_{14}(t)&=0\\
\psi_{15}(t)&=-\frac{1}{3\pi}H_{14}\int_1^{\infty}(ut^3+u^3t)\psi_{10}(u)\mathrm{d}u\\
\psi_{16}(t)&=\frac{2}{\pi}H_{12}\int_1^{\infty}ut\psi_{13}(u)\mathrm{d}u
\end{aligned}\right\} \tag{49}$$

etc.

Similarly for the Fredholm integral equation (42), we assume a series solution in the form

$$\psi_2(t) = \psi_{20}(t) + \frac{1}{\delta}\psi_{21}(t) + \frac{1}{\delta^2}\psi_{22}(t) + \frac{1}{\delta^3}\psi_{23}(t) + \cdots \tag{50}$$

and we obtain a set of equations of the form (49).

7. Solution for a particular type of temperature distribution: Quantities of physical interest

In this section we solve the integral equations (25) and (42) for large values of δ, by giving a particular value of $f(r)$ which is important from the physical point of view.

Let $f(r)$ be defined as

$$f(r) = -f_0 H(a-r), \quad a > 1 \tag{51}$$

where $H(t)$ is the Heaviside unit function.

Then

$$\psi_{10}(t) = \begin{cases} -\dfrac{2f_0}{\pi}\dfrac{t}{\sqrt{(a^2-t^2)}} & t < a \\ 0 & t > a. \end{cases} \tag{52}$$

Substituting this value in (25) we get

i) For $t > a$:

$$\psi_1(t) - \frac{1}{\pi\delta}\int_1^\infty \psi_1(u) K_1(u,t)\,\mathrm{d}u = 0. \tag{53}$$

It can be shown that its trivial solution is

$$\psi_1(t) = 0. \tag{54}$$

ii) For $t < a$:

In this case integral equation (25) becomes

$$\psi_1(t) - \frac{1}{\pi\delta}\int_1^\infty \psi_1(u) K_1(u,t)\,\mathrm{d}u = -\frac{2f_0}{\pi}\frac{t}{\sqrt{(a^2-t^2)}}, \tag{55}$$

which on considering terms up to δ^{-6} gives

$$\psi_{10}(t) = -\frac{2f_0}{\pi}\frac{t}{\sqrt{(a^2-t^2)}}$$

$$\psi_{11}(t) = 0$$

$$\psi_{12}(t) = 0$$

$$\psi_{13}(t) = -\frac{4f_0 H_{12}}{\pi^2}\left(\frac{a^2}{2}\cos^{-1}\frac{1}{a} + \frac{\sqrt{(a^2-1)}}{2}\right)t$$

$$\psi_{14}(t) = 0$$

$$\psi_{15}(t)=\frac{2f_0H_{14}}{3\pi^2}\left[t^3\left(\frac{a^2}{2}\cos^{-1}\frac{1}{a}+\frac{\sqrt{(a^2-1)}}{2}\right)\right.$$
$$\left.+t\left(\frac{3a^4}{8}\cos^{-1}\frac{1}{a}+\frac{(3a^2+2)\sqrt{(a^2-1)}}{8}\right)\right]$$
$$\psi_{16}(t)=-\frac{8f_0H_{12}^2}{3\pi^3}(a^3-1)\left(\frac{a^2}{2}\cos^{-1}\frac{1}{a}+\frac{\sqrt{(a^2-1)}}{2}\right)t.$$

Substituting the above values for $\psi_1(t)$ we have from (12)

$$T(r,0)=-\frac{2f_0}{\pi}\left[\sin^{-1}\sqrt{\left(\frac{a^2-1}{a^2-r^2}\right)}-\frac{H_{11}}{\delta^2}A_{11}+\frac{2H_{12}A_{11}Q(r)}{\pi\delta^3}\right.$$
$$+\frac{H_{13}}{6\delta^4}\left(A_{12}+\frac{3r^2}{2}A_{11}\right)-\frac{1}{\delta^5}\left\{\frac{2H_{12}H_{11}A_{11}(a^3-1)}{3\pi}\right.$$
$$\left.+\frac{H_{14}A_{11}}{3\pi}\left(Q(r)r^2+\frac{R(r)}{3}\right)+\frac{H_{14}A_{12}Q(r)}{3\pi}\right\}$$
$$+\frac{1}{\delta^6}\left\{\frac{4H_{12}^2A_{11}(a^3-1)Q(r)}{3\pi^2}\right.$$
$$\left.\left.-\frac{H_{15}}{120}\left(A_{13}+5r^2A_{12}+\frac{15}{8}r^4A_{11}\right)\right\}\right],\quad 0\leqslant r<1, \tag{56}$$

where

$$\left.\begin{aligned}A_{11}&=\frac{a^2}{2}\cos^{-1}\frac{1}{a}+\frac{\sqrt{(a^2-1)}}{2}\\A_{12}&=\frac{3a^4}{8}\cos^{-1}\frac{1}{a}+\frac{(3a^2+2)\sqrt{(a^2-1)}}{8}\\A_{13}&=\frac{5a^6}{16}\cos^{-1}\frac{1}{a}+\frac{(15a^4+10a^2+8)\sqrt{(a^2-1)}}{48}\end{aligned}\right\} \tag{57}$$

and

$$\left.\begin{aligned}Q(r)&=\sqrt{(a^2-r^2)}-\sqrt{(1-r^2)}\\R(r)&=(a^2-r^2)^{3/2}-(1-r^2)^{3/2}\end{aligned}\right\}. \tag{58}$$

Similarly we get a trivial solution $\psi_2(t)=0$ of the Fredholm integral equation (42), for $t>a$.

ii)′ For $t<a$:

In this case

$$K_2(u,t)=2\left[\frac{H_{22}(a^2-t^2)}{2\delta^2}-\frac{H_{24}}{6\delta^4}\left\{\frac{a^4-t^4}{4}+\frac{3u^2}{2}(a^2-t^2)\right\}\right.$$
$$\left.+\frac{H_{26}}{120\delta^6}\left\{\frac{a^6-t^6}{6}-\frac{10u^2}{4}(a^4-t^4)+\frac{5u^4}{2}(a^2-t^2)\right\}+\cdots\right] \tag{59}$$

where

$$H_{2n}=\int_0^\infty p^nH_2(p)\mathrm{d}p. \tag{60}$$

Using the method used earlier an iterative solution for $\psi_2(t)$ is obtained in the form

$$\psi_2(t)=\frac{mf_0}{\pi}\left[\sqrt{(a^2-t^2)}-\frac{H_{22}}{\pi\delta^3}B_{11}(a^2-t^2)+\frac{H_{24}}{3\pi\delta^5}\left\{\frac{a^4-t^4}{4}B_{11}+\frac{3(a^2-t^2)}{2}B_{12}\right\}+\frac{H_{22}^2}{3\pi^2\delta^6}B_{11}(a-1)(2a^2-a-1)(a^2-t^2)\right], \quad (61)$$

where

$$B_{11}=\frac{a^2}{2}\cos^{-1}\frac{1}{a}-\frac{\sqrt{(a^2-1)}}{2}, \quad B_{12}=\frac{a^4}{8}\cos^{-1}\frac{1}{a}+\frac{(a^2-2)\sqrt{(a^2-1)}}{8}. \quad (62a)$$

Now we derive expressions for quantities of physical interest.

Using (35) in (29) we have on the crack plane $z=0$,

$$u_z(r,0)=2(1-\nu)\int_1^a\psi_2(t)\mathrm{d}t\int_0^\infty J_0(\xi r)\cos(\xi t)\mathrm{d}\xi. \quad (63)$$

Now substituting the value of $\psi_2(t)$ from (61), we can easily find that

$$u_z(r,0)=\begin{cases}\dfrac{2(1-\nu)mf_0}{\pi}\left[a\left\{E\left(\dfrac{r}{a}\right)-E\left(\alpha,\dfrac{r}{a}\right)\right\}-\dfrac{H_{22}B_{11}}{\pi\delta^3}P(r)\right.\\ \quad+\dfrac{H_{24}}{3\pi\delta^5}\left\{\dfrac{B_{11}}{4}S(r)+\dfrac{3B_{12}}{2}P(r)\right\}\\ \quad\left.+\dfrac{H_{22}^2B_{11}(a-1)(2a^2-a-1)}{3\pi^2\delta^6}P(r)\right], \quad 1<r<a\\ \dfrac{2(1-\nu)mf_0}{\pi}\left[r\left\{E\left(\dfrac{a}{r}\right)-E\left(\beta,\dfrac{a}{r}\right)\right\}-\dfrac{r^2-a^2}{r}\left\{K\left(\dfrac{a}{r}\right)\right.\right.\\ \quad-F\left(\beta,\dfrac{a}{r}\right)\Big\}-\dfrac{H_{22}B_{11}}{\pi\delta^3}M(r)+\dfrac{H_{24}}{3\pi\delta^5}\left\{\dfrac{B_{11}}{4}N(r)\right.\\ \quad\left.\left.+\dfrac{3B_{12}}{2}M(r)\right\}+\dfrac{H_{22}^2B_{11}}{3\pi^2\delta^6}(a-1)(2a^2-a-1)M(r)\right], \quad r>a,\end{cases} \quad (64)$$

where

$$\alpha=\sin^{-1}\frac{1}{r}, \quad \beta=\sin^{-1}\frac{1}{a} \quad (65)$$

$$\left.\begin{aligned}P(r)&=\left(a^2-\frac{r^2}{2}\right)\cos^{-1}\frac{1}{r}-\frac{\sqrt{(r^2-1)}}{2}\\ S(r)&=\left(a^4-\frac{3r^4}{8}\right)\cos^{-1}\frac{1}{r}-\frac{(3r^2+2)\sqrt{(r^2-1)}}{8}\\ M(r)&=\left(a^2-\frac{r^2}{2}\right)\left(\sin^{-1}\frac{a}{r}-\sin^{-1}\frac{1}{r}\right)+\frac{1}{2}(a\sqrt{(r^2-a^2)}-\sqrt{(r^2-1)})\\ N(r)&=\left(a^4-\frac{3r^4}{8}\right)\left(\sin^{-1}\frac{a}{r}-\sin^{-1}\frac{1}{r}\right)\\ &\quad+\frac{a(3r^2+2a^2)\sqrt{(r^2-a^2)}}{8}-\frac{(3r^2+2)\sqrt{(r^2-1)}}{8}\end{aligned}\right\} \quad (66)$$

and

$E(a/r)$, $E(\beta, a/r)$, $K(a/r)$, $F(\beta, a/r)$ are elliptic integrals.

The normal component of stress on $z = 0$ is given by

$$\sigma_{zz}(r,0) = -m\mu\, T(r,0) + \frac{2m\mu f_0}{\pi\sqrt{(1-r^2)}}\left[\sqrt{(a^2-1)} - \frac{H_{22}B_{11}}{\pi\delta^3}(a^2-1)\right.$$
$$+ \frac{H_{24}}{3\pi\delta^5}\left\{\frac{a^4-1}{4}B_{11} + \frac{3(a^2-1)}{2}B_{12}\right\}$$
$$\left. + \frac{H_{22}^2 B_{11}}{3\pi^2\delta^6}(a-1)^2(a+1)(2a^2-a-1)\right]$$
$$- \frac{2m\mu f_0}{\pi}\left[\sin^{-1}\sqrt{\left(\frac{a^2-1}{a^2-r^2}\right)} - \frac{2H_{22}B_{11}}{\pi\delta^3}Q(r)\right.$$
$$+ \frac{H_{24}}{3\pi\delta^5}\left\{B_{11}\left(r^2Q(r) + \frac{1}{3}R(r)\right) + 3B_{12}Q(r)\right\}$$
$$\left. + \frac{2H_{22}^2 B_{11}(a-1)(2a^2-a-1)}{3\pi^2\delta^6}Q(r)\right] - \frac{2m\mu f_0}{\pi\delta^2}\left[H_{21}B_{11}\right.$$
$$- \frac{H_{23}}{2\delta^2}\left(B_{12} + \frac{r^2}{2}B_{11}\right) + \frac{H_{25}}{24\delta^4}\left(B_{13} + 3r^2B_{12} + \frac{3r^4}{8}B_{11}\right)$$
$$\left. - \frac{H_{21}H_{22}B_{11}}{3\pi\delta^3}(a-1)(2a^2-a-1)\right], \quad 0 \leqslant r < 1, \tag{67}$$

where

$$B_{13} = \frac{a^6}{16}\cos^{-1}\frac{1}{a} + \frac{(3a^4+2a^2-8)}{48}\sqrt{(a^2-1)}. \tag{62b}$$

The stress intensity factor is given by

$$N = \lim_{r\to 1-} \sqrt{(1-r)}\,\sigma_{zz}(r,0). \tag{68}$$

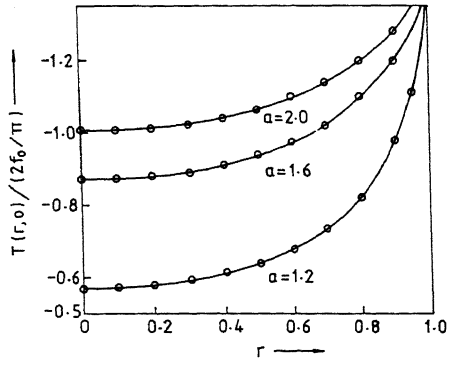

Figure 2. Variation of $T(r,0)/(2f_0/\pi)$ with r for $a = 1{\cdot}2$, $1{\cdot}6$, $2{\cdot}0$ and $\delta = 5$.

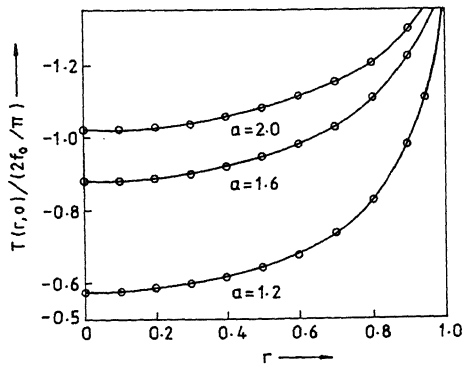

Figure 3. Variation of $T(r,0)/(2f_0/\pi)$ with r for $a = 1.2$, 1.6, 2.0 and $\delta = 7$.

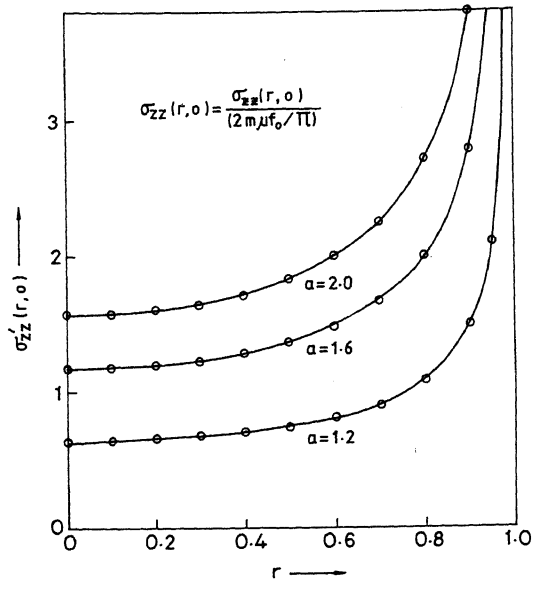

Figure 4. Variation of $\sigma_{zz}(r,0)/(2\,m\mu f_0/\pi)$ with r for $a = 1.2$, 1.6, 2.0 and $\delta = 5$.

Using (67) we have from (68)

$$N = \frac{\sqrt{2}m\mu f_0}{\pi}\left[\sqrt{(a^2-1)} - \frac{H_{22}B_{11}}{\pi\delta^3}(a^2-1) + \frac{H_{24}}{3\pi\delta^5}\left\{\frac{a^4-1}{4}B_{11} + \frac{3}{2}(a^2-1)B_{12}\right\} + \frac{H_{22}^2 B_{11}}{3\pi^2\delta^6}(a-1)^2(a+1)(2a^2-a-1)\right]. \tag{69}$$

Quantities of physical interest namely, the temperature and the normal components of stress and displacement on the crack plane $z = 0$ have been calculated for $a = 1.2$, 1.6, 2.0 and $\delta = 5, 7$. Variations of $T(r,0)$, $\sigma_{zz}(r,0)$ and $u_z(r,0)$ with r are shown graphically in figures 2–7 respectively.

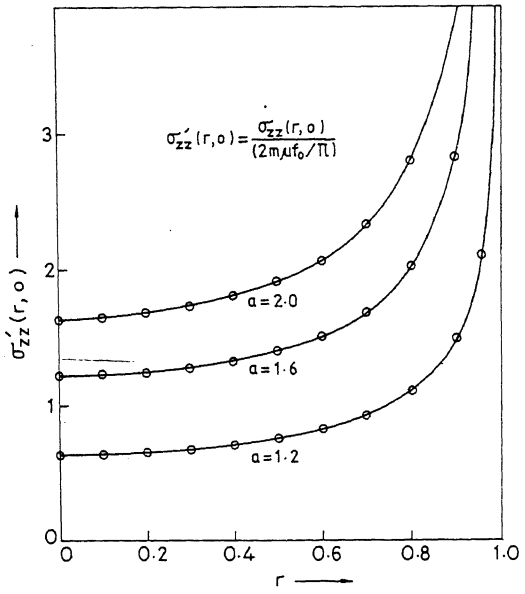

Figure 5. Variation of $\sigma_{zz}(r,0)/(2m\mu f_0/\pi)$ with r for $a = 1.2$, 1.6, 2.0 and $\delta = 7$.

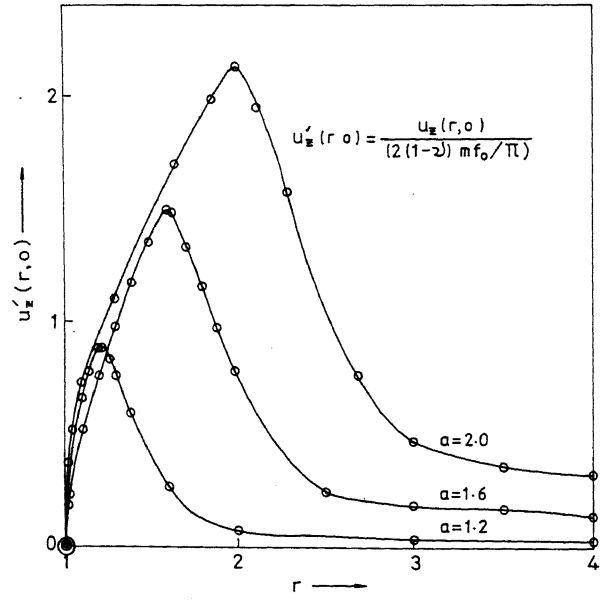

Figure 6. Variation of $u_z(r,0)/(2(1-\nu)mf_0/\pi)$ with r for $a = 1.2$, 1.6, 2.0 and $\delta = 5$.

8. Conclusions

When $\delta \to \infty$, the problem reduces to that of an infinite medium containing an external circular crack which has been solved by Das [3]. It is found that the limiting values as $\delta \to \infty$ of the temperature, stress intensity factor and the normal components of stress

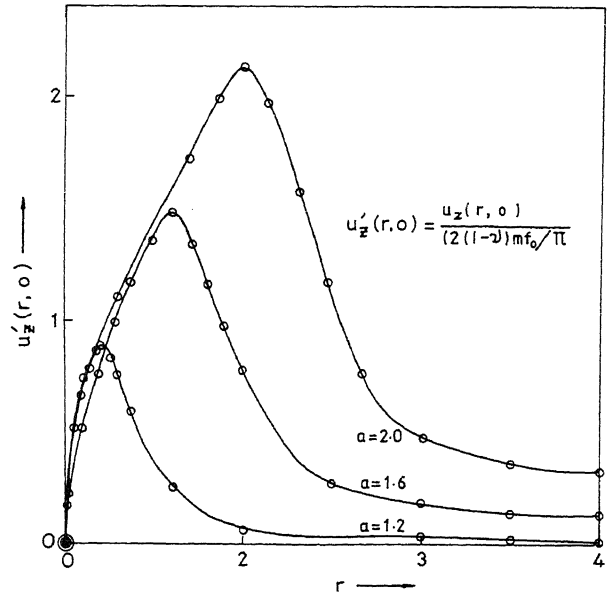

Figure 7. Variation of $u_z(r,0)/(2(1-\nu)mf_0/\pi)$ with r for $a = 1.2$, 1.6, 2.0 and $\delta = 7$.

and displacement given by (56), (69), (67) and (64) are the same as those obtained by Das.

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Some characterization theorems in rotatory magneto thermohaline convection

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Abstract. The present paper extends the results of Banerjee *et al* [2] for the hydromagnetic thermohaline convection problems of Veronis' [9] and Stern's [8] types to include the effect of a uniform vertical rotation.

Keywords. Hydromagnetic thermohaline convection; uniform vertical rotation.

1. Introduction

The establishment of non-occurrence of any slow oscillatory motions which may be neutral or unstable imply the validity of the principle of exchange of stabilities (PES). The validity of PES in a certain class of stability problems eliminates the unsteady terms from the linearized perturbation equations which results in notable mathematical simplicity since the transition from stability to instability occurs via a marginal state which is characterized by the vanishing of both real and imaginary parts of the complex time eigenvalue associated with the perturbation. Pellew and Southwell [5] proved the validity of PES for the classical Rayleigh–Bénard convection problem (RBCP). Chandrasekhar [3] in his investigations of hydromagnetic RBCP conjectured that if the total kinetic energy associated with a perturbation exceeds the total magnetic energy associated with it, then PES is valid. Sherman and Ostrach [7] established the above conjecture of Chandrasekhar for a more general problem when the fluid is confined in an arbitrary region and the uniform magnetic field is applied in an arbitrary direction. However, the result of Sherman and Ostrach is of limited value since one cannot *a priori* be certain when their criterion will be satisfied. Banerjee *et al* [1] established that for the hydromagnetic RBCP if $Q\sigma_1/\pi^2 \leqslant 1$, where Q is the Chandrasekhar number and σ_1 is the magnetic Prandtl number, then the total kinetic energy associated with an arbitrary perturbation which may be neutral or unstable is greater than the total magnetic energy associated with it and consequently PES is valid in this parameter regime. Banerjee *et al* [2] further extended these energy considerations to the hydromagnetic thermohaline convection problems of Veronis' [9] and Stern's [8] types. The aim of the present paper is to extend the results of Banerjee *et al* for the hydromagnetic thermohaline convection problems of Veronis' and Stern's types to include the effect of a uniform vertical rotation.

2. Basic equations and boundary conditions

The non-dimensional linearized perturbation equations governing thermohaline convection problem in the presence of a uniform vertical rotation and magnetic field are

given by (cf. Gupta *et al* [4]).

$$(D^2-a^2)(D^2-a^2-p/\sigma)w = Ra^2\theta - R_s a^2\phi - QD(D^2-a^2)h_z + TDZ \quad (1)$$

$$(D^2-a^2-p)\theta = -w \quad (2)$$

$$(D^2-a^2-p/\tau)\phi = -w/\tau \quad (3)$$

$$(D^2-a^2-p\sigma_1/\sigma)h_z = -Dw \quad (4)$$

$$(D^2-a^2-p/\sigma)Z = -Dw - QDX \quad (5)$$

$$(D^2-a^2-p\sigma_1/\sigma)X = -DZ \quad (6)$$

together with the boundary conditions

$$w = 0 = \theta = \phi = Dw = Z = DX = h_z \quad \text{at } z = 0, 1. \quad (7)$$

The various symbols occurring in the above equations are defined as follows:

z is real independent variable such that $0 \leqslant z \leqslant 1$ and stands for vertical coordinate, $D = \mathrm{d}/\mathrm{d}z$ denotes the derivatives with respect to z. a^2 is the square of the wave number, σ is the thermal Prandtl number, τ is the Lewis number, σ_1 is the magnetic Prandtl number, R is the thermal Rayleigh number, R_s is the thermohaline concentration Rayleigh number, Q is the Chandrasekhar number, T is the Taylor number and $p = p_r + ip_i$ is a complex constant in general representing the complex growth rate. Further w, θ, ϕ, Z, X and h_z are complex valued functions of z and stand respectively for the vertical velocity, temperature, concentration, vertical vorticity, vertical current density and vertical magnetic field. We note that $R > 0$ and $R_s > 0$ for Veronis' configuration whereas for Stern's configuration, we have $R < 0$ and $R_s < 0$.

System of eqs (1)–(7), constitute an eigenvalue problem for p for given values of $a^2, R, R_s, Q, T, \sigma$ and σ_1 and a given state of the system is stable, neutral or unstable according to $p_r < 0$ or $p_r = 0$ or $p_r > 0$. Further, if $p_r = 0$ implies $p_i = 0$ for all wave numbers a^2, then the principle of exchange of stabilities (PES) is valid, otherwise we will have overstability at least when instability sets in certain modes.

3. Mathematical analysis

We prove the following theorems:

Theorem 1. *A necessary condition for the existence of a nontrivial solutions* $(p, w, \theta, \phi, h_z, X, Z)$ *of eqs* (1)–(7) *with* $R > 0, R_s > 0$ *and* $p = p_r + ip_i, p_i \neq 0$ *is that*

$$J_1 < (J_2 + J_3 + J_4), \quad (8)$$

where

$$J_1 = \int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z,$$

$$J_2 = Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\mathrm{d}z, \quad (9)$$

$$J_3 = R_s a^2 \sigma \int_0^1 |\phi|^2 \mathrm{d}z, \quad (10)$$

and

$$J_4 = T\int_0^1 |Z|^2 \mathrm{d}z. \tag{11}$$

Proof. Multiplying eq. (1) by w^* (the complex conjugate of w) integrating the resulting equation over the range of z, we have

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-p/\sigma)w\,\mathrm{d}z = Ra^2\int_0^1 w^*\theta\,\mathrm{d}z - R_s a^2\int_0^1 w^*\phi\,\mathrm{d}z + T\int_0^1 w^*DZ\,\mathrm{d}z - Q\int_0^1 w^*D(D^2-a^2)h_z\,\mathrm{d}z. \tag{12}$$

Using eqs (2)–(6) and boundary conditions (7), we can write

$$Ra^2\int_0^1 w^*\theta\,\mathrm{d}z = -Ra^2\int_0^1 \theta(D^2-a^2-p^*)\theta^*\,\mathrm{d}z, \tag{13}$$

$$-R_s a^2\int_0^1 w^*\phi\,\mathrm{d}z = R_s a^2\int_0^1 \phi(D^2-a^2-p^*/\tau)\phi^*\mathrm{d}z, \tag{14}$$

$$-\int_0^1 w^*D(D^2-a^2)h_z = \int_0^1 Dw^*(D^2-a^2)h_z\,\mathrm{d}z = -\int_0^1 h_z(D^2-a^2)(D^2-a^2-p^*\sigma_1/\sigma)h_z^*\,\mathrm{d}z, \tag{15}$$

$$\int_0^1 w^*DZ\,\mathrm{d}z = -\int_0^1 Dw^*Z\,\mathrm{d}z = \int_0^1 Z(D^2-a^2-p^*/\sigma)Z^*\mathrm{d}z + Q\int_0^1 ZDX^*\mathrm{d}z = \int_0^1 Z(D^2-a^2-p^*/\sigma)Z^*\mathrm{d}z - Q\int_0^1 DZX^*\mathrm{d}z = \int_0^1 Z(D^2-a^2-p^*/\sigma)Z^*\mathrm{d}z + Q\int_0^1 X(D^2-a^2-p\sigma_1/\sigma)X^*\mathrm{d}z. \tag{16}$$

It follows from eqs (12)–(16) that

$$\int_0^1 w^*(D^2-a^2)(D^2-a^2-p/\sigma)w\,\mathrm{d}z = -Ra^2\int_0^1 \theta(D^2-a^2-p^*)\theta^*\mathrm{d}z + R_s a^2\tau\int_0^1 \phi(D^2-a^2-p^*/\tau)\phi^*\mathrm{d}z - Q\int_0^1 h_z(D^2-a^2) \times (D^2-a^2-p^*\sigma_1/\sigma)h_z^*\,\mathrm{d}z + T\int_0^1 Z(D^2-a^2-p^*/\sigma)Z^*\mathrm{d}z + QT\int_0^1 X(D^2-a^2-p\sigma_1/\sigma)X^*\mathrm{d}z. \tag{17}$$

Integrating various terms of eq. (17) by parts for an appropriate number of times and making use of boundary conditions (7), we have

$$I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0, \tag{18}$$

where

$$I_1 = \int_0^1 (|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\mathrm{d}z + p/\sigma \int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z,$$

$$I_2 = -Ra^2 \int_0^1 (|D\theta|^2 + a^2|\theta|^2 + p^*|\theta|^2)\mathrm{d}z,$$

$$I_3 = R_s a^2 \tau \int_0^1 (|D\phi|^2 + a^2|\phi|^2 + p^*/\tau|\phi|^2)\mathrm{d}z,$$

$$I_4 = T\int_0^1 (|DZ|^2 + a^2|Z|^2 + p^*/\sigma|Z|^2)\mathrm{d}z,$$

$$I_5 = TQ \int_0^1 (|DX|^2 + a^2|X|^2 + p\sigma_1/\sigma|X|^2)\mathrm{d}z,$$

and

$$I_6 = Q\left[\int_0^1 (|D^2 h_z|^2 + 2a^2|Dh_z|^2 + a^4|h_z|^2)\mathrm{d}z\right] + Qp^*\sigma_1/\sigma\left[\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\mathrm{d}z\right].$$

Equating the imaginary parts of both sides of eq. (18) and cancelling $p_i(\neq 0)$ throughout, we have

$$\int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z + Ra^2\sigma \int_0^1 |\theta|^2\mathrm{d}z + QT\sigma_1 \int_0^1 |X|^2\mathrm{d}z - \left\{R_s a^2\sigma \int_0^1 |\phi|^2\mathrm{d}z + T\int_0^1 |Z|^2\mathrm{d}z + Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\mathrm{d}z\right\} = 0, \tag{19}$$

or

$$[J_1 - \{J_2 + J_3 + J_4\}] + Ra^2\sigma \int_0^1 |\theta|^2\mathrm{d}z + QT\sigma_1 \int_0^1 |X|^2\mathrm{d}z = 0. \tag{20}$$

Equation (20) clearly implies that

$$J_1 < (J_2 + J_3 + J_4).$$

This completes the proof of the theorem.

We note that expressions for J_1, J_2, J_3 and J_4 as given by eqs (8)–(11) respectively, represent the total kinetic energy, magnetic energy, concentration energy and rotational energy. In view of this, Theorem 1 can be restated as follows:

A necessary condition for the existence of oscillatory motion which may be stable, neutral or unstable for Veronis' thermohaline convection problem in the presence of a uniform vertical rotation and magnetic field is that the sum total of magnetic, concentration and rotational energies must exceed the total kinetic energy or, equivalently, if the total kinetic energy exceeds the sum total of magnetic, concentration and rotational energies, then the oscillatory motions are not allowed.

The above result, no doubt yields us a condition in terms of energies of the system for the non-occurrence of oscillatory motions, however, it is of limited value, since one can

not *a priori* be certain when this condition will be satisfied as it involves the unknown eigen functions of the problem. It will therefore be more useful to express this condition in terms of the parameters of the problem prescribed by the fluid properties. We establish this in the following theorem.

Theorem 2. *If* $(p, w, \theta, \phi, h_z, X, Z)$, $p = p_r + ip_i, p_i \neq 0$, $p_r \geqslant 0$, $R > 0$ *and* $R_s > 0$ *is a solution of eqs* (1)–(7) *and* $\left[\frac{Q\sigma_1}{\pi^2} + \frac{R_s\sigma}{2\tau^2\pi^4} + \frac{T}{\pi^4}\right] \leqslant 1$, *then,* $J_1 > (J_2 + J_3 + J_4)$.

Proof. Multiplying eq. (3) by its complex conjugate, integrating over the range of z by parts a suitable number of times and making use of boundary conditions (7), we have

$$\int_0^1 (|D^2\phi|^2 + 2a^2|D\phi|^2 + a^4|\phi|^2)\mathrm{d}z + 2p_r/\tau \int_0^1 (|D\phi|^2 + a^2|\phi|^2)\mathrm{d}z$$

$$+ |p|^2/\tau^2 \int_0^1 |\phi|^2\mathrm{d}z = \frac{1}{\tau^2}\int_0^1 |w|^2\mathrm{d}z. \tag{21}$$

Since, $p_r \geqslant 0$, therefore eq. (21) gives

$$2a^2 \int_0^1 |D\phi|^2\mathrm{d}z < \frac{1}{\tau^2}\int_0^1 |w|^2\mathrm{d}z$$

which upon using Poincare inequality [6]

$$\pi^2 \int_0^1 |\phi|^2\mathrm{d}z \leqslant \int_0^1 |D\phi|^2\mathrm{d}z \quad (\text{since } \phi(0) = 0 = \phi(1))$$

yields that

$$a^2 \int_0^1 |\phi|^2\mathrm{d}z < \frac{1}{2\tau^2}\pi^2 \int_0^1 |w|^2\mathrm{d}z. \tag{22}$$

Further, since $w(0) = 0 = w(1)$ also, therefore

$$\int_0^1 |w|^2\mathrm{d}z \leqslant \frac{1}{\pi^2}\int_0^1 |Dw|^2\mathrm{d}z. \tag{23}$$

Combining inequalities (22)–(23), we have

$$a^2 \int_0^1 |\phi|^2\mathrm{d}z < \frac{1}{2\tau^2}\pi^4 \int_0^1 |Dw|^2\mathrm{d}z$$

$$< \frac{1}{2\tau^2}\pi^4 \int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z. \tag{24}$$

Multiplying eq. (4) by h_z^* (the complex conjugate of h_z), integrating the resulting equation by parts a suitable number of times in the range of z, making use of boundary conditions (7) and then equating the real parts from both sides of the resulting equation, we have

$$\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\mathrm{d}z + p_r\sigma_1/\sigma \int_0^1 |h_z|^2\mathrm{d}z$$

$$= \text{real part of} \left[\int_0^1 wDh_z^* \mathrm{d}z \right]$$

$$\leqslant \left| \int_0^1 wDh_z^* \mathrm{d}z \right|$$

$$\leqslant \int_0^1 |w||Dh_z| \mathrm{d}z$$

$$\leqslant \left[\int_0^1 |w|^2 \mathrm{d}z \right]^{1/2} \left[\int_0^1 |Dh_z|^2 \mathrm{d}z \right]^{1/2}$$

(by Schwartz inequality). (25)

Since $p_r \geqslant 0$, therefore inequality (25) implies that

$$\int_0^1 |Dh_z|^2 \mathrm{d}z < \left[\int_0^1 |w|^2 \mathrm{d}z \right]. \tag{26}$$

Combining inequalities (23), (25) and (26), we get

$$\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\mathrm{d}z < \frac{1}{\pi^2} \int_0^1 |Dw|^2 \mathrm{d}z$$

$$< \frac{1}{\pi^2} \int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z. \tag{27}$$

Now, multiplying eq. (5) by Z^* (the complex conjugate of Z), integrating by parts a suitable number of times, using boundary conditions (7) and equating the real parts of the resulting equation, we have

$$\int_0^1 (|DZ|^2 + a^2|Z|^2 + p_r/\sigma|Z|^2)\mathrm{d}z + Q \int_0^1 (|DX|^2 + a^2|X|^2 + p_r\sigma_1/\sigma|X|^2)\mathrm{d}z$$

$$= \text{real part of} \left(\int_0^1 Z^* Dw \mathrm{d}z \right)$$

$$= \text{real part of} \left(- \int_0^1 wDZ^* \mathrm{d}z \right)$$

$$\leqslant \left| - \int_0^1 wDZ^* \mathrm{d}z \right| = \left| \int_0^1 wDZ^* \mathrm{d}z \right|$$

$$\leqslant \int_0^1 |w||DZ| \mathrm{d}z$$

$$\leqslant \left[\int_0^1 |w|^2 \mathrm{d}z \right]^{1/2} \left[\int_0^1 |DZ|^2 \mathrm{d}z \right]^{1/2},$$

(by Schwartz inequality)

which by virtue of inequality (23) and the fact that $p_r \geqslant 0$ gives

$$\int_0^1 |DZ|^2 < \frac{1}{\pi^2} \int_0^1 |Dw|^2 \mathrm{d}z$$

$$< \frac{1}{\pi^2} \int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z. \tag{28}$$

Inequality (28) together with the Poincare inequality

$$\int_0^1 |Z|^2 \mathrm{d}z \leqslant \frac{1}{\pi^2} \int_0^1 |DZ|^2 \mathrm{d}z$$

leads to the inequality

$$\int_0^1 |Z|^2 \mathrm{d}z < \frac{1}{\pi^4} \int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z. \tag{29}$$

Combining inequalities (24), (27) and (29), we have

$$(J_2 + J_3 + J_4) < \left[\frac{Q\sigma_1}{\pi^2} + \frac{R_s\sigma}{2\tau^2\pi^4} + \frac{T}{\pi^4}\right] J_1. \tag{30}$$

Inequality (30) clearly implies that if

$$\left[\frac{Q\sigma_1}{\pi^2} + \frac{R_s\sigma}{2\tau^2\pi^4} + \frac{T}{\pi^4}\right] \leqslant 1,$$

then

$$J_1 > (J_2 + J_3 + J_4).$$

This completes the proof of the theorem.

Theorem 2 implies that if $\left[\frac{Q\sigma_1}{\pi^2} + \frac{R_s\sigma}{2\tau^2\pi^4} + \frac{T}{\pi^4}\right] \leqslant 1$, then the total kinetic energy associated with an arbitrary oscillatory ($p_i \neq 0$) perturbation which may be neutral ($p_r = 0$) or unstable ($p_r > 0$) exceeds the sum total of its magnetic, concentration and rotational energies. In particular, it follows that, in the parameter regime $\left[\frac{Q\sigma_1}{\pi^2} + \frac{R_s\sigma}{2\tau^2\pi^4} + \frac{T}{\pi^4}\right] \leqslant 1$, the principle of exchange of stabilities is valid for the problem under consideration.

Theorem 3. *A necessary condition for the existence of a nontrivial solutions* $(p, w, \theta, \phi, h_z, X, Z)$ *of eqs* (1)–(7) *with* $R < 0$, $R_s < 0$ *and* $p = p_r + ip_i$, $p_i \neq 0$ *is that*

$$J_1 < (J_2 + J_4 + J_5),$$

where J_1, J_2 *and* J_4 *are as given by eqs* (8), (9), *and* (11) *and*

$$J_5 = |R|a^2\sigma \int_0^1 |\theta|^2 \mathrm{d}z. \tag{31}$$

Proof. Putting $R = -|R|$ and $R_s = -|R_s|$ in eq. (18) and proceeding exactly as in Theorem 1, we get the desired result. Keeping in view the fact that J_5 represents the thermal energy, Theorem 3 can be restated as follows:

A necessary condition for the existence of oscillatory motions which may be stable, neutral or unstable for Stern's thermohaline convection problem in the presence of a uniform vertical rotation and magnetic field is that the sum total of magnetic, thermal and rotational energies must exceed that total kinetic energy, or, equivalently, if the

total kinetic energy exceeds the sum total of magnetic, thermal and rotational energies then the oscillatory motions are not allowed. Further, Theorem 3 is qualitatively of the same form as Theorem 1 and possesses the same drawback. We remedy this in the following theorem analogous to Theorem 2.

Theorem 4. *If* $(p, w, \theta, \phi, h_z, X, Z), p = p_r + ip_i, p_i \neq 0, p_r \geqslant 0, R < 0$ *and* $R_s < 0$ *is the solution of the eqs* (1)–(7) *and* $\left[\frac{Q\sigma_1}{\pi^2} + \frac{|R|\sigma}{2\pi^4} + \frac{T}{\pi^4}\right] \leqslant 1$, *then*

$$J_1 > (J_2 + J_4 + J_5).$$

Proof. Multiplying eq. (2) by its complex conjugate, integrating by parts a suitable number of times over the range of z, using boundary condition (7) and equating the real parts of the resulting equation, we have

$$\int_0^1 (|D^2\theta|^2 + 2a^2|D\theta|^2 + a^4|\theta|^2)\mathrm{d}z + 2p_r \int_0^1 (|D\theta|^2 + a^2|\theta|^2)\mathrm{d}z + |p|^2 \int_0^1 |\theta|^2 \mathrm{d}z = \int_0^1 |w|^2 \mathrm{d}z. \tag{32}$$

Since, $p_r \geqslant 0$, it follows from eq. (32) that

$$2a^2 \int_0^1 |D\theta|^2 \mathrm{d}z < \int_0^1 |w|^2 \mathrm{d}z,$$

which upon using the Poincare inequality

$$\int_0^1 |\theta|^2 \mathrm{d}z \leqslant \frac{1}{\pi^2} \int_0^1 |D\theta|^2 \mathrm{d}z$$

and inequality (23) gives

$$a^2 \int_0^1 |\theta|^2 \mathrm{d}z < \frac{1}{2\pi^4} \int_0^1 |Dw|^2 \mathrm{d}z < \frac{1}{2\pi^4} \int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z. \tag{33}$$

It follows from inequalities (24), (29) and (33) that

$$(J_2 + J_4 + J_5) < \left[\frac{Q\sigma_1}{\pi^2} + \frac{|R|\sigma}{2\pi^4} + \frac{T}{\pi^4}\right] J_1 \tag{34}$$

Inequality (34) clearly implies that if

$$\left[\frac{Q\sigma_1}{\pi^2} + \frac{|R|\sigma}{2\pi^4} + \frac{T}{\pi^4}\right] \leqslant 1,$$

then

$$J_1 > (J_2 + J_4 + J_5).$$

This completes the proof of the theorem.

Theorem 4 implies that if $\left[\frac{Q\sigma_1}{\pi^2}+\frac{|R|\sigma}{2\pi^4}+\frac{T}{\pi^4}\right]\leqslant 1$, then the total kinetic energy associated with an arbitrary oscillatory perturbation which may be neutral or unstable exceeds the sum total of its magnetic, thermal and rotational energies. In particular it follows that in the parameter regime $\left[\frac{Q\sigma_1}{\pi^2}+\frac{|R|\sigma}{2\pi^4}+\frac{T}{\pi^4}\right]\leqslant 1$, the PES is valid for the problem under consideration. Theorems 1–4 clearly provide a natural extension of the results of Banerjee *et al* [12] as could be easily seen by putting $T=0$.

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